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MATRIX FEJÉR AND LEVIN-STEČKIN INEQUALITIES

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Abstract. Fejér and Levin-Stečkin inequalities treat integrals of the product of convex functions with symmetric functions. The main goal of this article is to present possible matrix versions of these inequalities. In particular, majorization results are shown of Fejér type for both convex and log-convex functions. For the matrix Levin-Stečkin type, we present more rigorous results involving the partial Löewner ordering for Hermitian matrices. Further related results involving synchronous functions are presented, too.

1. Introduction

The theory of convex functions has played a major role in the study of mathematical inequalities. Related to convex-type inequalities, the Levin-Stečkin's inequality states that if the function $p:[0,1]\to\mathbb{R}$ is symmetric about $t=\frac{1}{2}$, namely $p(1-t)=p(t)$, and non-decreasing on $\left[0,\frac{1}{2}\right]$, then for every convex function f on $[0,1]$, the inequality

$$\int_0^1 p(t)\,f(t)\,dt\le\int_0^1 p(t)\,dt\int_0^1 f(t)\,dt$$

holds true [6]. If p is symmetric non-negative (without any knowledge about its monotonicity) and $f:[a,b]\to\mathbb{R}$ is convex, Fejér inequality states that [4]

$$f\left(\frac{a+b}{2}\right)\int_0^1 p(t)\,dt\le\int_0^1 p(t)\,f((1-t)\,a+tb)\,dt\le\frac{f(a)+f(b)}{2}\int_0^1 p(t)\,dt.$$

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We notice that Fejér inequality reduces to the Hermite-Hadamard inequality [5] when $p(t) = 1$. In mathematical inequalities, it is of interest to extend known inequalities from the setting of scalars to other objects, such as matrices. In this article, we will be interested in extending both the Levin-Stečkin and Fejér inequalities to the matrices setting.

In the sequel, $\mathcal{M}_n$ will denote the algebra of all $n \times n$ complex matrices. The conjugate transpose (or adjoint) of $A \in \mathcal{M}_n$ is denoted by A^*, and then the matrix A will be called Hermitian if $A^* = A$. When $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$, A is said to be positive semi-definite and is denoted as $A \geq 0$. If $A \geq 0$ and A is invertible, then A is said to be positive (strictly positive or positive definite). When $A, B \in \mathcal{M}_n$ are Hermitian, we say that $A \leq B$ if $B - A \geq 0$. This provides a partial ordering on the class of Hermitian matrices. The eigenvalues of a Hermitian matrix A will be denoted by $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$, repeated according to their multiplicity and arranged decreasingly. That is $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$.

The relation $A \leq B$ implies $\lambda_i(A) \leq \lambda_i(B)$ for any such Hermitian matrices $A, B \in \mathcal{M}_n$. However, the converse is not true. This urges the need to discuss, in some cases, the latter order. For convenience, we will write $\lambda(A) \leq \lambda(B)$ to mean that $\lambda_i(A) \leq \lambda_i(B)$, $i = 1, 2, \ldots, n$.

Another weaker ordering among matrices is the so-called weak majorization $\prec_w$, defined for the Hermitian matrices A, B as

$$A \prec_w B \quad \text{if and only if} \quad \sum_{i=1}^{k} \lambda_i(A) \leq \sum_{i=1}^{k} \lambda_i(B), \quad k = 1, \ldots, n.$$

It is clear that (see [1]) $A \leq B$ implies $\lambda(A) \leq \lambda(B)$, which implies $A \prec_w B$. It is customary to obtain one of these orders when extending a scalar inequality to a matrix inequality. For example, in this article, we obtain

$$\lambda\left(\left(\int_0^1 p(t)\,dt\right) f\left(\frac{A+B}{2}\right)\right) \leq \lambda\left(\int_0^1 p(t)\, f\left((1-t)\,A + tB\right) dt\right),$$

as an extension of Fejér inequality, to the Hermitian matrices A, B with spectra in the domain of f.

Further, if f is monotone, then

$$\lambda\left(\int_0^1 p(t)\, f\left((1-t)\,A + tB\right) dt\right) \leq \lambda\left(\left(\int_0^1 p(t)\,dt\right) \frac{f(A) + f(B)}{2}\right),$$

as matrix inequalities of the Fejér inequality. We remark that integral inequalities have played a key role in advancing matrix inequalities, as seen in [8, 9, 12], and the references therein.

In the next section, we study the possible matrix versions of Fejér inequality, which implies certain versions of the Hermite-Hadamard matrix inequality [10]. Then log-convex functions will be deployed to obtain new matrix Fejér inequalities for this type of functions, and we conclude with the discussion of the matrix Levin-Stečkin inequality.

2. Fejér Matrix Inequalities for Convex Functions

We begin with the following weak majorization of Fejér-type inequality.

Theorem 2.1. *Let $f : J \to \mathbb{R}$ be convex and let $p : [0,1] \to [0,+\infty)$ be symmetric about $t = \frac{1}{2}$. If $A, B \in \mathcal{M}_n$ are Hermitian with spectra in the interval J, then*

$$\lambda\left(\left(\int_0^1 p(t)\,dt\right) f\left(\frac{A+B}{2}\right)\right) \prec_w \lambda\left(\int_0^1 p(t)\, f\left((1-t)A + tB\right) dt\right).$$

Proof. If f is a convex function, then for any $0 \le t \le 1$, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{(1-t)\,a + tb + (1-t)\,b + ta}{2}\right) \\ &\le \frac{f\left((1-t)\,a + tb\right) + f\left((1-t)\,b + ta\right)}{2}. \end{aligned}$$

Thus,

$$f\left(\frac{a+b}{2}\right) \le \frac{f\left((1-t)\,a + tb\right) + f\left((1-t)\,b + ta\right)}{2}. \tag{2.1}$$

If the function p is non-negative, we get from (2.1),

$$p(t)\, f\left(\frac{a+b}{2}\right) \le p(t)\left(\frac{f\left((1-t)\,a + tb\right) + f\left((1-t)\,b + ta\right)}{2}\right).$$

Integrating on $t \in [0,1]$, and using symmetry assumption on p, we get

$$\left(\int_0^1 p(t)dt\right) f\left(\frac{a+b}{2}\right) \le \int_0^1 p(t)\, f\left((1-t)\,a + tb\right) dt. \tag{2.2}$$

If we replace a, b by $\langle Ax, x\rangle$, $\langle Bx, x\rangle$ respectively, in (2.2), we get

$$\left(\int_0^1 p(t)\,dt\right) f\left(\frac{\langle Ax, x\rangle + \langle Bx, x\rangle}{2}\right) \le \int_0^1 p(t)\, f\left((1-t)\langle Ax, x\rangle + t\langle Bx, x\rangle\right) dt. \tag{2.3}$$

On the other hand, it follows from Jensen's inequality [11, Theorem 1.2],

$$f\left(\langle\left((1-t)\,A + tB\right)x, x\rangle\right) \le \langle f\left((1-t)\,A + tB\right)x, x\rangle.$$

By multiplying both sides by $p(t)$, we get

$$p(t)\, f\left(\langle\left((1-t)\,A + tB\right)x, x\rangle\right) \le p(t)\,\langle f\left((1-t)\,A + tB\right)x, x\rangle.$$

Therefore,

$$\int_0^1 p(t) f(\langle((1-t)A+tB)x,x\rangle)\,dt \le \left\langle \left(\int_0^1 p(t) f((1-t)A+tB)\,dt\right)x,x\right\rangle. \tag{2.4}$$

Combining inequalities (2.3) with (2.4), we obtain

$$\left(\int_0^1 p(t)\,dt\right) f\left(\frac{\langle Ax,x\rangle+\langle Bx,x\rangle}{2}\right) \le \left\langle \left(\int_0^1 p(t) f((1-t)A+tB)\,dt\right)x,x\right\rangle. \tag{2.5}$$

Suppose that $\lambda_1,\dots,\lambda_n$ are the eigenvalues of $\frac{A+B}{2}$ with $x_1,\dots,x_n$ as an orthonormal system of corresponding eigenvectors arranged such that $f(\lambda_1)\ge\cdots\ge f(\lambda_n)$. We have, for $1\le k\le n$,

$$\begin{aligned}
\sum_{j=1}^k \lambda_j\left(\left(\int_0^1 p(t)\,dt\right) f\left(\frac{A+B}{2}\right)\right) &= \sum_{j=1}^k \left(\int_0^1 p(t)\,dt\right) f\left(\left\langle\left(\frac{A+B}{2}\right)x_j,x_j\right\rangle\right)\\
&= \sum_{j=1}^k \left(\int_0^1 p(t)\,dt\right) f\left(\frac{\langle Ax_j,x_j\rangle+\langle Bx_j,x_j\rangle}{2}\right)\\
&\le \sum_{j=1}^k \left\langle\left(\int_0^1 p(t) f((1-t)A+tB)\,dt\right)x_j,x_j\right\rangle\\
&\qquad \text{(by the inequality (2.5))}\\
&\le \sum_{j=1}^k \lambda_j\left(\int_0^1 p(t) f((1-t)A+tB)\,dt\right).
\end{aligned}$$

Namely, for $1\le k\le n$,

$$\sum_{j=1}^k \lambda_j\left(\left(\int_0^1 p(t)\,dt\right) f\left(\frac{A+B}{2}\right)\right) \le \sum_{j=1}^k \lambda_j\left(\int_0^1 p(t) f((1-t)A+tB)\,dt\right).$$

Therefore,

$$\lambda\left(\left(\int_0^1 p(t)\,dt\right) f\left(\frac{A+B}{2}\right)\right) \prec_w \lambda\left(\int_0^1 p(t) f((1-t)A+tB)\,dt\right)..$$

□

3. Fejér Inequalities via Log-convex Functions

In this part of the paper, we show a matrix Fejér inequality for log-convex functions.

Theorem 3.1. *Let $f:(0,+\infty)\to(0,+\infty)$ be log-convex and $p:[0,1]\to(0,+\infty)$ be symmetric and normalized in the sense that $\int_0^1 p(t)dt=1$. If $A,B\in\mathcal{M}_n$ are positive, then*

$$\lambda\left(\log f\left(\frac{A+B}{2}\right)\right) \prec_w \lambda\left(\log\int_0^1 p(t)f((1-t)A+tB)dt\right).$$

Proof. When f is convex, by (2.3), we have

$$f\left(\left\langle \frac{A+B}{2}x, x\right\rangle\right) \le \int_0^1 p(t) f\left(\langle ((1-t)A + tB)x, x\rangle\right) dt,$$

for any unit vector x. Since f is log-convex and A, B are positive, it follows that

$$\log f\left(\left\langle \frac{A+B}{2}x, x\right\rangle\right) \le \int_0^1 p(t) \log f \left\langle ((1-t)A + tB)x, x\right\rangle dt.$$

Noting that log is a concave function and that $d\mu(t) := p(t)dt$ is a probability measure, we have

$$\begin{aligned}
\log f\left(\left\langle \frac{A+B}{2}x, x\right\rangle\right) &\le \int_0^1 p(t) \log f\left(\langle ((1-t)A + tB)x, x\rangle\right) dt \\
&= \int_0^1 \log f\left(\langle ((1-t)A + tB)x, x\rangle\right) d\mu(t) \\
&\le \log \int_0^1 f\left(\langle ((1-t)A + tB)x, x\rangle\right) d\mu(t) \\
&= \log \int_0^1 p(t) f\left(\langle ((1-t)A + tB)x, x\rangle\right) dt,
\end{aligned}$$

for any unit vector x. Now, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\frac{A+B}{2}$ with orthonormal eigenvectors $x_1, x_2, \dots, x_n$, so that $f(\lambda_1) \ge \cdots \ge f(\lambda_n)$. Then, for $1 \le k \le n$,

$$\begin{aligned}
\sum_{j=1}^k \lambda_j \left(\log f\left(\frac{A+B}{2}\right)\right) &= \sum_{j=1}^k \log f(\lambda_j) \\
&= \sum_{j=1}^k \log f\left(\left\langle \frac{A+B}{2}x_j, x_j\right\rangle\right) \\
&\le \sum_{j=1}^k \log \int_0^1 p(t) f\left(\langle ((1-t)A + tB)x_j, x_j\rangle\right) dt \\
&\le \sum_{j=1}^k \lambda_j \left(\log \int_0^1 p(t) f((1-t)A + tB))dt\right).
\end{aligned}$$

This completes the proof. ☐

As a consequence, we have the following.

Corollary 3.1. *Let $f : (0, +\infty) \to (0, +\infty)$ be log-convex and $p : [0, 1] \to (0, +\infty)$ be symmetric and normalized. Then*

$$\prod_{j=1}^k \lambda_j \left(f\left(\frac{A+B}{2}\right)\right) \le \prod_{j=1}^k \lambda_j \left(\int_0^1 p(t) f((1-t)A + tB)dt\right), \quad k = 1, \dots, n,$$

for any positive matrices $A, B \in \mathcal{M}_n$.

Proof. From Theorem 3.1, we have

$$\sum_{j=1}^{k}\lambda_j\left(\log f\left(\frac{A+B}{2}\right)\right)\le\sum_{j=1}^{k}\lambda_j\left(\log\int_0^1 p(t)f((1-t)A+tB))dt\right),$$

which is equivalent to

$$\sum_{j=1}^{k}\log\lambda_j\left(f\left(\frac{A+B}{2}\right)\right)\le\sum_{j=1}^{k}\log\lambda_j\left(\int_0^1 p(t)f((1-t)A+tB))dt\right).$$

Consequently,

$$\log\prod_{j=1}^{k}\lambda_j\left(f\left(\frac{A+B}{2}\right)\right)\le\log\prod_{j=1}^{k}\lambda_j\left(\int_0^1 p(t)f((1-t)A+tB)dt\right),$$

which implies the desired inequality. □

4. Levin-Stečkin Matrix Inequalities

We present a new inequality of Levin-Stečkin type. The significance of this inequality is its validity for any positive function p without imposing any conditions on its symmetry or monotony.

Theorem 4.1. *Let $f:[0,1]\to\mathbb{R}$ be convex differentiable and let $p:[0,1]\to[0,+\infty)$ be continuous. Then*

$$\int_0^1 f(t)dt\int_0^1 p(t)dt+\left(\int_0^1 f'(t)dt\int_0^1 tp(t)dt-\int_0^1 tf'(t)dt\int_0^1 p(t)dt\right)\le\int_0^1 f(t)p(t)dt.$$

Further,

$$\int_0^1 p(t)\,f(t)\,dt+\frac{1}{2}\int_0^1 p(t)\,f'(t)\,dt-\int_0^1 p(t)\,tf'(t)\,dt\le\int_0^1 p(t)\,dt\int_0^1 f(t)\,dt.$$

Proof. For the convex differentiable function f and $s,t\in[0,1]$ we have

$$f(s)+f'(s)(t-s)\le f(t). \tag{4.1}$$

Since $p(t)\ge 0$, it follows that

$$p(t)f(s)+p(t)f'(s)(t-s)\le p(t)f(t),\quad s,t\in[0,1].$$

Integrating this inequality over $t\in[0,1]$ then over $s\in[0,1]$ implies

$$\int_0^1 f(s)ds\int_0^1 p(t)dt+\left(\int_0^1 f'(s)ds\int_0^1 tp(t)dt-\int_0^1 sf'(s)ds\int_0^1 p(t)dt\right)\le\int_0^1 f(t)p(t)dt,$$

which is equivalent to the first desired inequality.

For the second inequality, integrating (4.1) over $t\in[0,1]$, we obtain

$$f(s)+f'(s)\left(\frac{1}{2}-s\right)\le\int_0^1 f(t)\,dt.$$

If we put $s = t$, we have

$$f(t) + f'(t)\left(\frac{1}{2} - t\right) \le \int_0^1 f(t)\,dt.$$

Multiplying both sides by $p(t)$, we get

$$p(t)\,f(t) + p(t)\,f'(t)\left(\frac{1}{2} - t\right) \le p(t)\int_0^1 f(t)\,dt.$$

Again, if we take integral over $t \in [0,1]$, we infer that

$$\int_0^1 p(t)\,f(t)\,dt + \frac{1}{2}\int_0^1 p(t)\,f'(t)\,dt - \int_0^1 p(t)\,tf'(t)\,dt \le \int_0^1 p(t)\,dt \int_0^1 f(t)\,dt.$$

This completes the proof. □

Corollary 4.1. *Let $f : [0,1] \to \mathbb{R}$ be convex differentiable and let $p : [0,1] \to [0,+\infty)$ be symmetric about $\frac{1}{2}$ and non-decreasing on $\left[0, \frac{1}{2}\right]$. Then*

$$\int_0^1 f'(t)dt \int_0^1 tp(t)dt \le \int_0^1 tf'(t)dt \int_0^1 p(t)dt.$$

Proof. This follows from the first inequality in Theorem 4.1 because when p is symmetric about $\frac{1}{2}$ and non-decreasing on $\left[0, \frac{1}{2}\right]$, we have

$$\int_0^1 f(t)p(t)dt \le \int_0^1 f(t)dt \int_0^1 p(t)dt.$$ □

Assume that τ_t and σ_t are two arbitrary weighted symmetric operator means with $0 \le t \le 1$. A real-valued continuous function $f : J \subseteq \mathbb{R} \to \mathbb{R}$ is operator τ_t-σ_t-convex if

$$f(A\tau_t B) \le f(A)\,\sigma_t f(B),$$

for Hermitian $A, B \in \mathcal{M}_n$ whose spectra are contained in J. For $t = \frac{1}{2}$, we say f is operator τ-σ, and we write

$$f(A\tau B) \le f(A)\,\sigma f(B). \tag{4.2}$$

An important example of operator mean is the arithmetic mean, which is denoted by ∇_t, as the weighted version, for $0 \le t \le 1$.

To prove the next lemma, we need the following important property of the weighted operator means:

$$(A\tau_\alpha B)\,\tau_\gamma\,(A\tau_\beta B) = A\tau_{(1-\gamma)\alpha+\gamma\beta}B, \quad \alpha, \beta, \gamma \in [0,1]. \tag{4.3}$$

Lemma 4.1. *Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be an operator τ_t-σ_t-convex and let $x \in \mathbb{C}^n$. Then*

$$F(t) = \langle f(A\tau_t B)\,x, x\rangle$$

is convex on $0 \le t \le 1$.

Proof. Indeed,

$$\begin{aligned} F\left(\frac{t+s}{2}\right) &= \left\langle f\left(A\tau_{\frac{t+s}{2}}B\right)x,x\right\rangle \\ &= \left\langle f\left(\left(A\tau_tB\right)\tau\left(A\tau_sB\right)\right)x,x\right\rangle \quad \text{(by (4.3))} \\ &\le \left\langle f\left(A\tau_tB\right)\sigma f\left(A\tau_sB\right)x,x\right\rangle \quad \text{(by (4.2))} \\ &\le \left\langle f\left(A\tau_tB\right)\nabla f\left(A\tau_sB\right)x,x\right\rangle \\ &\qquad \text{(since arithmetic mean is the biggest one among symmetric means)} \\ &= \frac{\left\langle f\left(A\tau_tB\right)x,x\right\rangle + \left\langle f\left(A\tau_sB\right)x,x\right\rangle}{2} \\ &= \frac{F\left(t\right)+F\left(s\right)}{2}. \end{aligned}$$

This completes the proof. □

Theorem 4.2. *Let $A,B\in\mathcal{M}_n$ be two Hermitian matrices with spectra contained in J, let f be operator τ_t-σ_t-convex and $p:[0,1]\to[0,+\infty)$ be symmetric about $t=\frac{1}{2}$ and non-decreasing on $\left[0,\frac{1}{2}\right]$. Then*

$$\int_0^1 p\left(t\right)f\left(A\tau_tB\right)dt \le \int_0^1 p\left(t\right)dt\int_0^1 f\left(A\tau_tB\right)dt.$$

Proof. Let $x\in\mathcal{M}_n$ be a unit vector. Then

$$\begin{aligned} \left\langle\left(\int_0^1 p\left(t\right)f\left(A\tau_tB\right)dt\right)x,x\right\rangle &= \int_0^1 p\left(t\right)\left\langle f\left(A\tau_tB\right)x,x\right\rangle dt \\ &\le \int_0^1 p\left(t\right)dt\int_0^1 \left\langle f\left(A\tau_tB\right)x,x\right\rangle dt \\ &= \left\langle\left(\int_0^1 p\left(t\right)dt\int_0^1 f\left(A\tau_tB\right)dt\right)x,x\right\rangle \end{aligned}$$

where we have employed Lemma 4.1. This completes the proof. □

The case $\tau_t=\sigma_t=\nabla_t$, in Theorem 4.2, reduces to

$$\int_0^1 p\left(t\right)f\left(\left(1-t\right)A+tB\right)dt \le \int_0^1 p\left(t\right)dt\int_0^1 f\left(\left(1-t\right)A+tB\right)dt. \tag{4.4}$$

The following theorem gives a reverse for the inequality (4.4) by employing the Mond-Pečarić method [11].

Theorem 4.3. *Let $f:[m,M]\to\mathbb{R}$ be convex and let $p:[0,1]\to[0,+\infty)$ be symmetric about $t=\frac{1}{2}$. If $A,B\in\mathcal{M}_n$ are Hermitian with spectra in the interval $[m,M]$, then for* *any* $\alpha \geq 0$

$$\int_0^1 p(t)\,dt \int_0^1 f((1-t)A+tB)\,dt \leq \beta \int_0^1 p(t)\,dt I + \alpha \int_0^1 p(t) f((1-t)A+tB)\,dt,$$

where $\beta = \max_{m \leq x \leq M} \{a_f x + b_f - \alpha f(x)\}$, $a_f = \frac{f(M)-f(m)}{M-m}$ *and* $b_f = \frac{Mf(m)-mf(M)}{M-m}$.

Proof. Since $f(x) \leq a_f x + b_f$, we get by the functional calculus

$$f((1-t)A+tB) \leq a_f((1-t)A+tB) + b_f I.$$

By taking integral over $0 \leq t \leq 1$, we reach

$$\int_0^1 f((1-t)A+tB)\,dt \leq a_f\left(\frac{A+B}{2}\right) + b_f I.$$

This implies

$$\int_0^1 p(t)\,dt \int_0^1 f((1-t)A+tB)\,dt \leq a_f\left(\int_0^1 p(t)\,dt\right)\frac{A+B}{2} + b_f\left(\int_0^1 p(t)\,dt\right)I.$$

Hence for any vector y,

$$\left\langle\left(\int_0^1 p(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\right)y,y\right\rangle \leq a_f\left(\int_0^1 p(t)\,dt\right)\left\langle\left(\frac{A+B}{2}\right)y,y\right\rangle + \int_0^1 p(t)\,dt b_f.$$

Now, by (2.3), we can write

$$\begin{aligned}
&\left\langle\left(\int_0^1 p(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\right)y,y\right\rangle - \alpha\int_0^1 p(t) f(\langle((1-t)A+tB)y,y\rangle)\,dt\\
\leq& a_f\left(\int_0^1 p(t)\,dt\right)\left\langle\left(\frac{A+B}{2}\right)y,y\right\rangle + b_f\int_0^1 p(t)\,dt - \alpha\int_0^1 p(t) f(\langle((1-t)A+tB)y,y\rangle)\,dt\\
\leq& a_f\left(\int_0^1 p(t)\,dt\right)\left\langle\left(\frac{A+B}{2}\right)y,y\right\rangle + b_f\int_0^1 p(t)\,dt - \alpha\left(\int_0^1 p(t)\,dt\right) f\left(\left\langle\left(\frac{A+B}{2}\right)y,y\right\rangle\right)\\
=&\left(\int_0^1 p(t)\,dt\right)\left(a_f\left\langle\left(\frac{A+B}{2}\right)y,y\right\rangle + b_f - \alpha f\left(\left\langle\left(\frac{A+B}{2}\right)y,y\right\rangle\right)\right)\\
\leq&\left(\int_0^1 p(t)\,dt\right)\max_{m\leq x\leq M}\{a_f x + b_f - \alpha f(x)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left\langle\left(\int_0^1 p(t)\,dt \int_0^1 f((1-t)A+tB)\,dt\right)y,y\right\rangle\\
\leq&\beta\int_0^1 p(t)\,dt+\alpha\int_0^1 p(t)\,f(\langle((1-t)A+tB)\,y,y\rangle)\,dt\\
\leq&\beta\int_0^1 p(t)\,dt+\alpha\int_0^1 p(t)\,\langle f((1-t)A+tB)\,y,y\rangle\,dt\\
&\text{(by [11, Theorem 1.2])}\\
=&\left\langle\left(\beta\left(\int_0^1 p(t)\,dt\right)I+\alpha\int_0^1 p(t)\,f((1-t)A+tB)\,dt\right)y,y\right\rangle
\end{aligned}$$

as desired. □

5. Further inequalities via synchronous functions

We say that the functions $f,g:J\to\mathbb{R}$ are synchronous (asynchronous) on the interval J if they satisfy the following condition, for all $s,t\in J$,

$$(f(t)-f(s))(g(t)-g(s))\geq(\leq)\,0.$$

It is obvious that if f,g are monotonic and have the same monotonicity on the interval J, then they are synchronous on J while if they have opposite monotonicity, they are asynchronous.

Related to the Levin-Stečkin inequality, the celebrated Čebyšev inequality [2] states that if f and g are two functions having the same monotonicity on $[0,1]$, then

$$\int_0^1 f(t)\,dt\int_0^1 g(t)\,dt\leq\int_0^1 f(t)\,g(t)\,dt.$$

For some Čebyšev type inequalities for Hilbert space operators, see [7].

The following result provides a refinement and a reverse of this inequality via synchronous functions.

Theorem 5.1. *Let $f,g:[a,b]\to\mathbb{R}$ be synchronous functions on the interval $[a,b]$. Then*

$$\begin{aligned}
&\min\left\{\frac{1}{b-a}\int_a^b f^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)^2,\frac{1}{b-a}\int_a^b g^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)^2\right\}\\
\leq&\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)
\end{aligned}$$

$$\leq \max\left\{\frac{1}{b-a}\int_a^b f^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)^2,\frac{1}{b-a}\int_a^b g^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)^2\right\}.$$

If f and g have opposite monotonicity then

$$\min\left\{\frac{1}{b-a}\int_a^b f^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)^2,\frac{1}{b-a}\int_a^b g^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)^2\right\}$$

$$\leq \left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)-\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt$$

$$\leq \max\left\{\frac{1}{b-a}\int_a^b f^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)^2,\frac{1}{b-a}\int_a^b g^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)^2\right\}.$$

Proof. We prove the first inequality. The second inequality goes likewise, and we omit the details. We have

$$\begin{aligned}
&f(t)\,g(t)+f(s)\,g(s)-(f(t)\,g(s)+f(s)\,g(t))\\
&=(f(t)-f(s))\,(g(t)-g(s))\\
&=|(f(t)-f(s))\,(g(t)-g(s))|\\
&=|f(t)-f(s)|\,|g(t)-g(s)|\\
&\geq \min\left\{(f(t)-f(s))^2,(g(t)-g(s))^2\right\}\\
&=\min\left\{f^2(t)+f^2(s)-2f(t)\,f(s)\,,g^2(t)+g^2(s)-2g(t)\,g(s)\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\min\left\{f^2(s)+f^2(t)-2f(s)\,f(t)\,,g^2(t)+g^2(s)-2g(t)\,g(s)\right\}\\
&\leq f(t)\,g(t)+f(s)\,g(s)-(f(t)\,g(s)+f(s)\,g(t))\,.
\end{aligned}$$

Consequently,

$$\min\left\{(b-a)\,f^2(s)+\int_a^b f^2(t)\,dt-2f(s)\int_a^b f(t)\,dt,\int_a^b g^2(t)\,dt+(b-a)\,g^2(s)-2g(s)\int_a^b g(t)\,dt\right\}$$

$$\leq \int_a^b f(t)\,g(t)\,dt+(b-a)\,f(s)\,g(s)-g(s)\int_a^b f(t)\,dt-f(s)\int_a^b g(t)\,dt.$$

Upon integration, this implies

$$\min\left\{2(b-a)\int_a^b f^2(t)\,dt-2\left(\int_a^b f(t)\,dt\right)^2,2(b-a)\int_a^b g^2(t)\,dt-2\left(\int_a^b g(t)\,dt\right)^2\right\}$$

$$\leq 2(b-a)\int_a^b f(t)\,g(t)\,dt-2\int_a^b f(t)\,dt\int_a^b g(t)\,dt.$$

Multiplying both sides by $\frac{1}{2(b-a)^2}$, we obtain,

$$\begin{aligned}&\min\left\{\frac{1}{b-a}\int_a^b f^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)^2,\frac{1}{b-a}\int_a^b g^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)^2\right\}\\&\le\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right).\end{aligned}$$

The second inequality is obtained from the same arguments and the following relation

$$\begin{aligned}&\max\left\{f^2(s)+f^2(t)-2f(s)\,f(t)\,,g^2(t)+g^2(s)-2g(t)\,g(s)\right\}\\&\ge f(t)\,g(t)+f(s)\,g(s)-(f(t)\,g(s)+f(s)\,g(t))\,.\end{aligned}$$

□

In the following result, we establish a refinement and a reverse for the Levin-Stečkin inequality.

Theorem 5.2. *Let* $p:[0,1]\to\mathbb{R}$ *be a symmetric about* $t=\frac{1}{2}$, *namely* $p(1-t)=p(t)$, *and non-decreasing on* $\left[0,\frac{1}{2}\right]$, *then for every convex function* f *on* $[0,1]$,

$$\begin{aligned}&\int_0^1 p(t)\,f(t)\,dt\le\int_0^1 p(t)\,dt\int_0^1 f(t)\,dt\\&-\min\left\{2\int_0^{1/2} p^2(t)\,dt-\left(\int_0^1 p(t)\,dt\right)^2,\frac{1}{2}\int_0^{1/2}(f(t)+f(1-t))^2dt-\left(\frac{1}{2}\int_0^1(f(t)+f(1-t))dt\right)^2\right\}.\end{aligned}$$

A similar but reversed inequality holds if we replace min *with* max.

Proof. If f is symmetric and convex, by Theorem 5.1, we have

$$\begin{aligned}&\int_0^1 p(t)\,dt\int_0^1 f(t)\,dt\\&=\left(\int_0^{1/2} p(t)\,dt+\int_{1/2}^1 p(t)\,dt\right)\left(\int_0^{1/2} f(t)\,dt+\int_{1/2}^1 f(t)\,dt\right)=4\int_0^{1/2} p(t)\,dt\int_0^{1/2} f(t)\,dt\\&\ge 2\int_0^{1/2} p(t)\,f(t)\,dt+\min\left\{2\int_0^{1/2} p^2(t)\,dt-\left(2\int_0^{1/2} p(t)\,dt\right)^2,2\int_0^{1/2} f^2(t)\,dt-\left(2\int_0^{1/2} f(t)\,dt\right)^2\right\}\\&=\int_0^1 p(t)\,f(t)\,dt+\min\left\{2\int_0^{1/2} p^2(t)\,dt-\left(\int_0^1 p(t)\,dt\right)^2,2\int_0^{1/2} f^2(t)\,dt-\left(\int_0^1 f(t)\,dt\right)^2\right\}.\end{aligned}$$

Namely,

$$\int_0^1 p(t) f(t)\, dt + \min\left\{2\int_0^{1/2} p^2(t)\, dt - \left(\int_0^1 p(t)\, dt\right)^2, 2\int_0^{1/2} f^2(t)\, dt - \left(\int_0^1 f(t)\, dt\right)^2\right\}$$

$$\leq \int_0^1 p(t)\, dt \int_0^1 f(t)\, dt.$$

We shall now consider an arbitrary f. For convex f, the function $\frac{f(x)+f(1-x)}{2}$ is convex and symmetric so that we can use the above inequality. Hence,

$$\int_0^1 p(t) f(t)\, dt$$

$$= \frac{\int_0^1 p(t) f(t)\, dt + \int_0^1 p(1-t) f(1-t)\, dt}{2} = \int_0^1 p(t) \frac{f(t) + f(1-t)}{2} dt$$

$$\leq \int_0^1 p(t)\, dt \int_0^1 \frac{f(t) + f(1-t)}{2} dt$$

$$- \min\left\{2\int_0^{1/2} p^2(t)\, dt - \left(\int_0^1 p(t)\, dt\right)^2, \frac{1}{2}\int_0^{1/2} (f(t) + f(1-t))^2 dt - \left(\frac{1}{2}\int_0^1 (f(t) + f(1-t)) dt\right)^2\right\}$$

$$= \int_0^1 p(t)\, dt \int_0^1 f(t)\, dt$$

$$- \min\left\{2\int_0^{1/2} p^2(t)\, dt - \left(\int_0^1 p(t)\, dt\right)^2, \frac{1}{2}\int_0^{1/2} (f(t) + f(1-t))^2 dt - \left(\frac{1}{2}\int_0^1 (f(t) + f(1-t)) dt\right)^2\right\},$$

which yields the desired inequality. □

We can improve the second inequality in Theorem 5.1 in the following way.

Theorem 5.3. *Let $f, g : J \to \mathbb{R}$ be synchronous functions on the interval* $[0, 1]$. *Then*

$$\int_0^1 f(t) g(t)\, dt - \int_0^1 f(t)\, dt \int_0^1 g(t)\, dt$$

$$\leq \frac{1}{2}\left(\int_0^1 f^2(t)\, dt - \left(\int_0^1 f(t)\, dt\right)^2 + \int_0^1 g^2(t)\, dt - \left(\int_0^1 g(t)\, dt\right)^2\right).$$

Proof. We have

$$\begin{aligned}
&f(t)\,g(t)+f(s)\,g(s)-(f(t)\,g(s)+f(s)\,g(t))\\
=&(f(t)-f(s))\,(g(t)-g(s))\\
=&|(f(t)-f(s))\,(g(t)-g(s))|\\
=&|f(t)-f(s)|\,|g(t)-g(s)|\\
\leq&\frac{1}{2}\left((f(t)-f(s))^2+(g(t)-g(s))^2\right)\\
=&\frac{1}{2}\left(f^2(t)+f^2(s)+g^2(t)+g^2(s)-2\,(g(t)\,g(s)+f(t)\,f(s))\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&f(t)\,g(t)+f(s)\,g(s)-(f(t)\,g(s)+f(s)\,g(t))\\
\leq&\frac{1}{2}\left(f^2(t)+f^2(s)+g^2(t)+g^2(s)-2\,(g(t)\,g(s)+f(t)\,f(s))\right).
\end{aligned}$$

The remaining part of the proof is similar to the proof of Theorem 5.1, so we omit the details. □

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References

[1] R. Bhatia, *Matrix Analysis*, Springer Verlag, New York, 1997.

[2] P. L. Chebyshev, *O približennyh vyraženijah odnih integralov čerez drugie*, in: *Ššit soobšćenija i protokoly zasedani Matemmatičeskogo občestva pri Imperatorskom Har'kovskom Universitete*, No. 2, 93–98; P. L. Chebyshev, *Polnoe sobranie sočineni*, Moskva, Leningrad **1948a** (1882), 128–131, (in Russian).

[3] S. S. Dragomir, *Hermite-Hadamard's type inequalities for operator convex functions*, Appl. Math. Comput. **218**(3) (2011), 766–772. `https://doi.org/10.1016/j.amc.2011.01.056`

[4] L. Fejér, *Über die fourierreihen, II*, Math. Naturwiss Anz. Ungar. Akad. Wiss **24** (1906), 369–390.

[5] J. Hadamard, *Étude sur les proprits des fonctions entiéres et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl. **58** (1893), 171–215, (in French).

[6] V. I. Levin and S. B. Stěckin, *Inequalities*, Trans. Amer. Math. Soc. **14** (1960), 1–29.

[7] H. R. Moradi, M. E. Omidvar and S. S. Dragomir, *An operator extension of Čebyšev inequality*, Analele Stiintifice ale Universitatii Ovidius Constanta **25**(2) (2017), 135–148. `https://doi.org/10.1515/auom-2017-0025`

[8] H. R. Moradi, M. Sababheh and S. Furuichi, *On the operator Hermite-Hadamard inequality*, Complex Anal. Oper. Theory **15**(122), (2021). `https://doi.org/10.1007/s11785-021-01172-w`

[9] H. R. Moradi and M. Sababheh, *More accurate numerical radius inequalities (II)*, Linear Multilinear Algebra **69** (2021), 921–933. `https://doi.org/10.1080/03081087.2019.1703886`

[10] M. S. Moslehian, *Matrix Hermite-Hadamard type inequalities*, Houston J. Math. **39**(1) (2013), 177–189.

[11] J. Pečarić, T. Furuta, J. Mićić Hot and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Element, Zagreb, 2005.

[12] M. Sababheh and H. R. Moradi, *More accurate numerical radius inequalities* (*I*), Linear Multilinear Algebra **69** (2021), 1964–1973. https://doi.org/10.1080/03081087.2019.1651815

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