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VECTOR VALUED HYPERSTRUCTURES

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Abstract. Vector valued hyperstructures, i.e., (n, m)-hyperstructures, where $n = m + k$, $k \geq 1$, as a generalization of vector valued structures and n-ary hyperstructures are introduced and supported by many examples. We have presented some initial properties about (n, m)-hypersemigroups and (n, m)-hypergroups. Moreover, by properly defining regular and strongly regular binary relations, from vector valued hypersemigroups (hypergroups) we obtain "ordinary" vector valued semigroups (groups) on quotients.

1. Introduction and Basic Definitions

An (n, m)-groupoid is a nonempty set G with one vector valued operation, i.e., an operation $[\ ] : G^n \to G^m$, where $n \geq m$. Such a structure $(G, [\ ])$ is called a vector valued groupoid as well. Vector valued groupoids were investigated in [15] and other special vector valued structures such as (n, m)-semigroups and (n, m)-groups were investigated in [1,3,4,10–13,16]. A good expository paper on vector valued structures is [2]. Compared with the papers devoted to n-ary structures, the number of the papers devoted to vector valued structures is smaller. Having in mind some recent works on n-ary hyperstructures such as [6–9], we define the notion of vector valued hypergroupoid and present some initial concepts, examples and results.

Let H be a nonempty set and let n, m be positive integers such that $n \geq m$. We denote by $\mathcal{P}^*(H)$ the set of all nonempty subsets of H and by H^n the n-th Cartesian product of H, $H \times \cdots \times H$, where H appears n times.

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Definition 1.1. Let $[\,]$ be a mapping $[\,] : H^n \to (\mathcal{P}^*(H))^m$ from the n-th Cartesian product of H to the m-th Cartesian product of $\mathcal{P}^*(H)$. Then $[\,]$ is called an (n,m)-*hyperoperation* on H or, if it is not necessary to emphasize the integers n and m, then we will say that $[\,]$ is a *vector valued hyperoperation* instead of (n,m)-hyperoperation.

In this sense, a $(2,1)$-hyperoperation on H means a binary hyperoperation on H and an $(n,1)$-hyperoperation means an n-ary hyperoperation on H.

Definition 1.2. A sequence of m n-ary hyperoperations $[\,]_s : H^n \to \mathcal{P}^*(H)$, $s \in \{1,2,\dots,m\}$, can be associated to $[\,]$ by putting

$$[a_1 \dots a_n]_s = B_s \;\Leftrightarrow\; [a_1 \dots a_n] = (B_1, \dots, B_m), \tag{1.1}$$

for all $a_1, \dots, a_n \in H$. Then, we call $[\,]_s$ the *s-th component hyperoperation* of $[\,]$ and write $[\,] = ([\,]_1, \dots, [\,]_m)$.

Note that there is a unique (n,m)-hyperoperation on H whose component hyperoperations are $[\,]_s$.

An (n,m)-hyperoperation $[\,]$ on H is extended to subsets $A_1, A_2, \dots, A_n$ of H in a natural way, i.e.,

$$[A_1A_2 \dots A_n] = ([A_1A_2 \dots A_n]_1, [A_1A_2 \dots A_n]_2, \dots, [A_1A_2 \dots A_n]_m),$$

where $[A_1A_2 \dots A_n]_s = \cup\{[a_1^n]_s \mid a_i \in A_i, i = 1,2,\dots,n\}$ and $s = 1,2,\dots,m$.

Note that $C_1^p \subseteq B_1^p$ if and only if $C_i \subseteq B_i$, for $i = 1,\dots,p$, and, $x_1^p \in C_1^p$ if and only if $x_i \in C_i$ for $i = 1,\dots,p$.

Definition 1.3. An algebraic structure $\boldsymbol{H} = (H,[\,])$, where $[\,]$ is an (n,m)-ary hyperoperation defined on a nonempty set H, is called an (n,m)-*hypergroupoid* or *vector valued hypergroupoid*. If $[\,] = ([\,]_1, \dots, [\,]_m)$, we denote by $(H; [\,]_1, \dots, [\,]_m)$ the *component hypergroupoid* of $\boldsymbol{H}$ and $(H,[\,]_j)$ is the *j-th component n-ary hypergroupoid* of $\boldsymbol{H}$.

Identifying the set $\{x\}$ with the element x, any (n,m)-groupoid is an (n,m)-hypergroupoid.

Throughout the paper, we use the following simplified notation. The elements of H^n, i.e., the sequences $(x_1,x_2,\dots,x_n)$ will be denoted by $x_1x_2\dots x_n$ or x_1^n. The symbol x_i^j will denote the sequence $x_ix_{i+1}\dots x_j$ of elements of H when $i \le j$ and the empty symbol when $i > j$. If $x_{i+1} = x_{i+2} = \cdots = x_{i+r} = x$, then the sequence x_{i+1}^{i+r} is denoted by $\overset{(r)}{x}$. Under this convention the sequence $x_1 \dots x_i \underbrace{x \dots x}_{r} x_{i+r+1} \dots x_n$ will be denoted by $x_1^i \overset{(r)}{x} x_{i+r+1}^n$.

In what follows we will assume that n and m are such that $n > m$, i.e., $n = m + k$, for $k \ge 1$.

Definition 1.4. An (n,m)-hyperoperation is said to be (i,j)-*associative* if for all $x_1, \dots, x_{n+k} \in H$

$$[x_1^i[x_{i+1}^{i+n}]x_{i+n+1}^{n+k}] = [x_1^j[x_{j+1}^{j+n}]x_{j+n+1}^{n+k}],$$

and *weakly* (i,j)*-associative* if for all $x_1,\ldots,x_{n+k}\in H$

$$[x_1^i[x_{i+1}^{i+n}]x_{i+n+1}^{n+k}]_s\cap[x_1^j[x_{j+1}^{j+n}]x_{j+n+1}^{n+k}]_s\neq\emptyset,$$

holds for fixed i and j, where $1\le i<j\le n$ and for every $s\in\{1,2,\ldots,m\}$.

If the above conditions are satisfied for all $i,j\in\{1,2,\ldots,n\}$, then we say that the operation $[\ ]$ is *associative* (*weakly associative*, respectively). An (n,m)-hypergroupoid with an associative operation (weakly associative operation) is called an (n,m)*-hypersemigroup* (*weak* (n,m)*-hypersemigroup*).

Definition 1.5. An (n,m)-hypergroupoid $(H,[\ ])$ is *partially* i*-cancellative* if there exists a sequence $a_1^k\in H^k$ such that

$$[a_1^ix_1^ma_{i+1}^k]=[a_1^iy_1^ma_{i+1}^k]\ \ \Rightarrow\ \ x_1^m=y_1^m,$$

for all $x_1^m,y_1^m\in H^m$ and some $i\in\{0,1,\ldots,k\}$. The sequence a_1^k is called i*-cancellable.* If this implication holds for all $i=0,1,\ldots,k$, then we say that $(H,[\ ])$ is *partially cancellative* and the sequence a_1^k is *cancellable.* An (n,m)-hypergroupoid in which this implication holds for some $i\in\{0,1,\ldots,k\}$ and all sequences $a_1^k\in H^k$ is said to be i*-cancellative.* For $i=0$ $(i=k)$ we say that $(H,[\ ])$ is *right cancellative* (*left cancellative*). If $(H,[\ ])$ is i-cancellative for every $i=0,1,\ldots,k$, then it is said to be *cancellative.* An (n,m)-hypergroupoid is *strongly* i*-cancellative* if for all $a_1^k\in H^k$ the following implication holds:

$$[a_1^iX_1^ma_{i+1}^k]=[a_1^iY_1^ma_{i+1}^k]\ \ \Rightarrow\ \ X_1^m=Y_1^m,$$

where $X_i,Y_i\subseteq H$ and some $i\in\{0,1,\ldots,k\}$. If this implication holds for all $i\in\{0,1,\ldots,k\}$ we say that $(H,[\ ])$ is *strongly cancellative* and the sequence a_1^k is *strongly cancellable.* If there exists a sequence $a_1^k\in H^k$ such that the above implication holds, then we say that the (n,m)-hypergroupoid is *partially strongly cancellative.*

Remark 1.1. The definition of partially i-cancellative (partially cancellative) (n,m)-hypergroupoid for $m=1$ corresponds to the definition of weakly i-cancellative (weakly cancellative) n-ary hypergroupoid. Note that, if an (n,m)-hypergroupoid is strongly i-cancellative (strongly cancellative), then it is partially i-cancellative (partially cancellative).

Definition 1.6. Let $(H,[\ ])$ be an (n,m)-hypergroupoid. A sequence $e_1^k\in H^k$ is called an i*-neutral polyad* if

$$[x_1^ie_1^kx_{i+1}^m]=(\{x_1\},\{x_2\}\ldots,\{x_m\}).$$

We write this identity in the form $[x_1^ie_1^kx_{i+1}^m]=x_1^m$, for all $x_1^m\in H^m$. A 0-neutral polyad is also called *left neutral* and an m-neutral polyad is also called *right neutral.* A polyad that is i-neutral for each $i\in\{0,1,\ldots,m\}$ is called a *neutral polyad.*

If there exits $e\in H$ such that for any sequence $x_1^m\in H^m$ the relation

$$x_1^m\in[x_1^i\ \overset{(k)}{e}\ x_{i+1}^m]$$

holds for all $i = 0, \dots, m$, then we say that e is a *weak neutral element* in $\boldsymbol{H}$. If

$$[x_1^i \overset{(k)}{e} x_{i+1}^m] = (\{x_1\}, \{x_2\} \dots, \{x_m\}) \tag{1.2}$$

holds for any $x_1^m \in H^m$ and fixed i, where $i \in \{0, 1, \dots, m\}$, then we say that e is *i-neutral element* in H and it is called a *neutral element* in H if the equation (1.2) holds for every $i = 0, 1, \dots, m$. We write the identity (1.2) in the form $[x_1^i \overset{(k)}{e} x_{i+1}^m] = x_1^m$.

Definition 1.7. An (n, m)-hypergroupoid $(H, [\ ])$ is called an *(n, m)-hyperquasigroup* if for every $a_1^n \in H^n$ there exists $x_1^m \in H^m$ such that

$$a_{k+1}^n \in [a_1^i x_1^m a_{i+1}^k], \tag{1.3}$$

for every $i = 0, 1, \dots, k$.

Definition 1.8. An (n, m)-hyperquasigroup that is an (n, m)-hypersemigroup (weak (n, m)-hypersemigroup) is called *(n, m)-hypergroup* (*weak (n, m)-hypergroup*).

Definition 1.9. An (n, m)-hypergroupoid $(H, [\ ])$ is called *(i, j)-commutative* if the equality $[a_1^n] = [a_1^i a_j a_{i+2}^j a_i a_{j+2}^n]$ holds for fixed i, j such that $0 \le i < j \le n-1$ and for every sequence $a_1^n \in H^n$. If this equation holds for every i, j and for every sequence $a_1^n \in H^n$, then $(H, [\ ])$ is called *commutative* (n, m)-hypergroupoid. In that case $[a_1^n] = [a_{\sigma(1)}^{\sigma(n)}]$, where $\sigma \in S_n$ and for every sequence $a_1^n \in H^n$. An (n, m)-hypergroupoid $(H, [\ ])$ is called *weakly commutative* if $\bigcap_{\sigma \in S_n} [a_{\sigma(1)}^{\sigma(n)}]_s \neq \emptyset$ for every sequence $a_1^n \in H^n$ and every $s = 1, 2, \dots, m$.

Definition 1.10. Let $(H, [\ ])$ and $(H', [\ ]')$ be (n, m)-hypergroupoids. A mapping $\varphi : H \to H'$ is:

a) a *strong homomorphism* if and only if $\varphi([a_1^n]_s) = [\varphi(a_1) \dots \varphi(a_n)]'_s$;
b) an *inclusion homomorphism* if and only if $\varphi([a_1^n]_s) \subseteq [\varphi(a_1) \dots \varphi(a_n)]'_s$;
c) a *weak homomorphism* if and only if $\varphi([a_1^n]_s) \cap [\varphi(a_1) \dots \varphi(a_n)]'_s \neq \varnothing$, for every $s = 1, 2, \dots, n$.

If φ is a bijective mapping and a strong homomorphism, then it is called an *isomorphism*, and it is called an *automorphism* if φ is defined on the same (n, m)-hypergroupoid.

2. Examples

Example 2.1. Let H be the set $\mathbb{Z}$ of integers and let $[\ ]$ be defined as follows: $[x_1^4] = (\{x_1, x_3\}, \{x_2, x_4\})$. By a direct verification one can show that the component operations are:

$$\begin{aligned}
[[x_1^4] x_5^6]_1 &= [\{x_1, x_3\}\{x_2, x_4\} x_5 x_6]_1 \\
&= [x_1 x_2 x_5 x_6]_1 \cup [x_1 x_4 x_5 x_6]_1 \cup [x_3 x_2 x_5 x_6]_1 \cup [x_3 x_4 x_5 x_6]_1 \\
&= \{x_1, x_5\} \cup \{x_1, x_5\} \cup \{x_3, x_5\} \cup \{x_3, x_5\} = \{x_1, x_3, x_5\}, \\
[[x_1^4] x_5^6]_2 &= [\{x_1, x_3\}\{x_2, x_4\} x_5 x_6]_2
\end{aligned}$$

$$= [x_1x_2x_5x_6]_2 \cup [x_1x_4x_5x_6]_2 \cup [x_3x_2x_5x_6]_2 \cup [x_3x_4x_5x_6]_2$$
$$= \{x_2, x_6\} \cup \{x_4, x_6\} \cup \{x_2, x_6\} \cup \{x_4, x_6\} = \{x_2, x_4, x_6\}.$$

So,

$$\begin{aligned}
[[x_1^4]x_5^6] &= (\{x_1, x_3, x_5\}, \{x_2, x_4, x_6\}),\\
[x_1[x_2^5]x_6]_1 &= [x_1\{x_2, x_4\}\{x_3, x_5\}x_6]_1\\
&= [x_1x_2x_3x_6]_1 \cup [x_1x_2x_5x_6]_1 \cup [x_1x_4x_3x_6]_1 \cup [x_1x_4x_5x_6]_1\\
&= \{x_1, x_3\} \cup \{x_1, x_5\} = \{x_1, x_3, x_5\},\\
[x_1[x_2^5]x_6]_2 &= [x_1\{x_2, x_4\}\{x_3, x_5\}x_6]_2\\
&= [x_1x_2x_3x_6]_2 \cup [x_1x_2x_5x_6]_2 \cup [x_1x_4x_3x_6]_2 \cup [x_1x_4x_5x_6]_2\\
&= \{x_2, x_6\} \cup \{x_4, x_6\} = \{x_2, x_4, x_6\}.
\end{aligned}$$

So,

$$\begin{aligned}
[x_1[x_2^5]x_6] &= (\{x_1, x_3, x_5\}, \{x_2, x_4, x_6\}),\\
[x_1^2[x_3^6]]_1 &= [x_1x_2\{x_3, x_5\}\{x_4x_6\}]_1\\
&= [x_1x_2x_3x_4]_1 \cup [x_1x_2x_3x_6]_1 \cup [x_1x_2x_5x_4]_1 \cup [x_1x_2x_5x_6]_1\\
&= \{x_1, x_3\} \cup \{x_1, x_3\} \cup \{x_1, x_5\} \cup \{x_1, x_5\} = \{x_1, x_3, x_5\},\\
[x_1^2[x_3^6]]_2 &= [x_1x_2\{x_3, x_5\}\{x_4x_6\}]_2\\
&= [x_1x_2x_3x_4]_2 \cup [x_1x_2x_3x_6]_2 \cup [x_1x_2x_5x_4]_2 \cup [x_1x_2x_5x_6]_2\\
&= \{x_2, x_4\} \cup \{x_2, x_6\} \cup \{x_2, x_4\} \cup \{x_2, x_6\} = \{x_2, x_4, x_6\}.
\end{aligned}$$

So, $[x_1^2[x_3^6]] = (\{x_1, x_3, x_5\}, \{x_2, x_4, x_6\})$ and, obviously, $[[x_1^4]x_5^6] = [x_1[x_2^5]x_6] = [x_1^2[x_3^6]]$, i.e., $(H, [\,])$ is a $(4, 2)$-hypersemigroup.

Remark 2.1. Note that the set H with any of the component 4-ary hyperoperations does not have to be a 4-hypersemigroup.

Remark 2.2. If $(H, [\,]_1)$ and $(H, [\,]_2)$ are, for example, ternary hypersemigroups, then $(H, [\,])$, where $[\,] = ([\,]_1, [\,]_2)$, does not necessarily have to be a $(3, 2)$-hypersemigroup. For instance, let $H = \{a, b, c\}$ and $[\,]_1$ be the ternary hyperoperation defined as in Example 2.4 in [9] and $[\,]_2$ be the ternary hyperoperation defined as in Example 4 in [7]. Both $(H, [\,]_1)$ and $(H, [\,]_2)$ are ternary hypersemigroups as it is shown in [9] and [7]. However, $(H; [\,]_1, [\,]_2)$ is not a $(3, 2)$-hypersemigroup, since $[[baa]a]_1 = [[baa]_1[baa]_2a]_1 = [bba]_1 = \{a, c\} \neq [b[aaa]]_1 = [b[aaa]_1[aaa]_2]_1 = [baa]_1 = b$.

The next example presents a weak $(4, 2)$-hypersemigroup that is not a $(4, 2)$-hypersemigroup.

Example 2.2. Let H be the set $\mathbb{Z}$ of integers, $(\mathbb{Z}, +)$ be the additive group of integers and let $[\,]$ be defined as follows:

$$[x_1^4] = (\{x_1, x_1 + x_3\}, \{x_2, x_2 + x_4\}).$$

By a direct verification one can show that the component operations are:

$$\begin{aligned}
[[x_1^4]x_5^6]_1 =& [\{x_1, x_1+x_3\}\{x_2, x_2+x_4\}x_5x_6\}]_1 \\
=& [x_1x_2x_5x_6]_1 \cup [x_1(x_2+x_4)x_5x_6]_1 \\
& \cup [(x_1+x_3)x_2x_5x_6]_1 \cup [(x_1+x_3)(x_2+x_4)x_5x_6]_1 \\
=& \{x_1, x_1+x_5\} \cup \{x_1+x_3, x_1+x_3+x_5\} \\
=& \{x_1, x_1+x_3, x_1+x_5, x_1+x_3+x_5\}, \\
[[x_1^4]x_5^6]_2 =& [\{x_1, x_1+x_3\}\{x_2, x_2+x_4\}x_5x_6\}]_2 \\
=& [x_1x_2x_5x_6]_2 \cup [x_1(x_2+x_4)x_5x_6]_2 \\
& \cup [(x_1+x_3)x_2x_5x_6]_2 \cup [(x_1+x_3)(x_2+x_4)x_5x_6]_2 \\
=& \{x_2, x_2+x_6\} \cup \{x_2+x_4, x_2+x_4+x_6\} \\
=& \{x_2, x_2+x_4, x_2+x_6, x_2+x_4+x_6\}.
\end{aligned}$$

From here we obtain that

$$[[x_1^4]x_5^6] = (\{x_1, x_1+x_3, x_1+x_5, x_1+x_3+x_5\}, \{x_2, x_2+x_4, x_2+x_6, x_2+x_4+x_6\}).$$

In a similar way as in the previous step, one can show that

$$[x_1[x_2^5]x_6] = (\{x_1, x_1+x_3, x_1+x_3+x_5\}, \{x_2, x_2+x_4, x_2+x_6, x_2+x_4+x_6\})$$

and that

$$[x_1^2[x_3^6]] = (\{x_1, x_1+x_3, x_1+x_3+x_5\}, \{x_2, x_2+x_4, x_2+x_4+x_6\}).$$

Note that $[[x_1^4]x_5^6] \neq [x_1[x_2^5]x_6]$ and therefore $(H, [\ ])$ is not a $(4,2)$-hypersemigroup. However, for $i = 1, 2$, $[[x_1^4]x_5^6]_i \cap [x_1[x_2^5]x_6]_i \neq \emptyset$, $[[x_1^4]x_5^6]_i \cap [x_1^2[x_3^6]]_i \neq \emptyset$ and $[x_1[x_2^5]x_6]_i \cap [x_1^2[x_3^6]]_i \neq \emptyset$. Thus $(H, [\ ])$ is a weak $(4,2)$-hypersemigroup.

Example 2.3. Let $H = \mathbb{Z}_3$ and let $[\ ]$ be a $(3,2)$-hyperoperation on H defined by $[x_1^3] = (\max\{x_1, x_3\}, x_2)$. Then $(H, [\ ])$ is partially left cancellative, since there is an element $0 \in \mathbb{Z}_3$ such that $[0x_1^2] = [0y_1^2] \Rightarrow (\max\{0, x_2\}, x_1) = (\max\{0, y_2\}, y_1) \Rightarrow x_1 = y_1, x_2 = y_2$. It can be shown in a similar way that $(H, [\ ])$ is partially right cancellative as well.

Example 2.4. The $(4,2)$-hypersemigroup defined in the Example 2.1 is a cancellative $(4,2)$-hypergroupoid. Namely, let $a_1^2 \in H^2$. Then

$$\begin{aligned}
[a_1^2x_1^2] = [a_1^2y_1^2] \ &\Rightarrow\ (\{a_1, x_1\}, \{a_2, x_2\}) = (\{a_1, y_1\}, \{a_2, y_2\}) \\
&\Rightarrow\ \{a_1, x_1\} = \{a_1, y_1\}, \{a_2, x_2\} = \{a_2, y_2\} \\
&\Rightarrow\ x_1 = y_1, x_2 = y_2 \\
&\Rightarrow\ x_1^2 = y_1^2, \\
[x_1a_1^2x_2] = [y_1a_1^2y_2] \ &\Rightarrow\ (\{x_1, a_2\}, \{a_1, x_2\}) = (\{y_1, a_2\}, \{a_1, y_2\}) \\
&\Rightarrow\ \{x_1, a_2\} = \{y_1, a_2\}, \{a_1, x_2\} = \{a_1, y_2\} \\
&\Rightarrow\ x_1 = y_1, x_2 = y_2
\end{aligned}$$

$$\Rightarrow \quad x_1^2 = y_1^2.$$

In a similar way one can show that $[x_1^2 a_1^2] = [y_1^2 a_1^2] \;\Rightarrow\; x_1^2 = y_1^2$.

Example 2.5. Let $H = \mathbb{Z}$, $(\mathbb{Z}, +)$, be the additive group of integers and $[\,]$ be a $(4,2)$-hyperoperation defined by $[x_1^4] = (x_1+x_3, x_2+x_4)$. It is strongly cancellative. Namely, let $[x_1x_2AB] = [x_1x_2CD]$. Then, $(x_1 + A,\, x_2 + B) = (x_1 + C, x_2 + D)$ implies that $A = C$, $B = D$. If $[x_1ABx_4] = [x_1CDx_4]$, then $(x_1{+}B,\, A{+}x_4) = (x_1{+}D,\, C{+}x_4)$, and thus, $B = D$, $A = C$. If $[ABx_3x_4] = [CDx_3x_4]$, then $(A{+}x_3, B{+}x_4) = (C{+}x_3, D{+}x_4)$, and thus, $A = C$, $B = D$.

Note that if $(H, [\,])$ is strongly cancellative, then it is cancellative as well. The converse is not true. For instance, if $(H, [\,])$ is defined as in Example 2.4 then $(H, [\,])$ is cancellative but it is not strongly cancellative, since $[12\{1,2\}\{2,4\}] = [1224] \not\Rightarrow \{1,2\} = \{2\}$ and $\{2,4\} = \{4\}$.

Example 2.6. Let $H = \mathbb{Z}_4$, $(\mathbb{Z}_4, +)$, be the additive group of integers modulo 4 and $[\,]$ be a $(4,2)$-hyperoperation on H defined by

$$[x_1^4] = (\{x_1 + x_3, \max\{x_1, x_3\}\}, \{x_2 + x_4, \max\{x_2, x_4\}\}).$$

Since $[00x_3x_4] = (\{x_3, x_3\}, \{x_4, x_4\}) = (x_3, x_4)$, $[x_100x_4] = (\{x_1, x_1\}, \{x_4, x_4\}) = (x_1, x_4)$ and $[x_1x_200] = (\{x_1, x_1\}, \{x_2, x_2\}) = (x_1, x_2)$, it follows that 0 is a neutral element in $\boldsymbol{H}$.

Example 2.7. Let $H = \mathbb{Z}$, where $(\mathbb{Z}, +)$ is the additive group of integers, and $[\,]$ be a $(4,2)$-hyperoperation on H defined by

$$[x_1^4] = (\{x_1 + x_3, x_3\}, \{x_2 + x_4, x_4\}).$$

Since $[00x_3x_4] = (x_3, x_4) \ni (x_3, x_4)$, $[x_100x_4] = (\{x_1, 0\}, x_4) \ni (x_1, x_4)$ and $[x_1x_200] = (\{x_1, 0\}, \{x_2, 0\}) \ni (x_1, x_2)$, it follows that 0 is a weak neutral element in $\boldsymbol{H}$.

Example 2.8. Let $H = \mathbb{Z}_2$ and $[\,]$ be a (3,2)-hyperoperation on H defined by $[x_1^3] = (\{x_1, x_2\}, \{x_2, x_3\})$. By a direct verification for 8 sequences from elements of H, one can show that the relation (1.3) has a solution for every $(x, y) \in H^2$ and thus it is (3,2)-hyperquasigroup. Since it is a hypersemigroup as well (it can be easily verified that $[[x_1^3]x_4] = (\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}) = [x_1[x_2^4]]$) it follows that $\boldsymbol{H}$ is a (3,2)-hypergroup.

Example 2.9. Let $H = \mathbb{Z}$ and $[\,]$ is a $(4,2)$-hyperoperation on H defined by $[x_1^4] = (\{x_1, x_3\}, \{x_2, x_4\})$. Clearly, $(H, [\,])$ is $(0,2)$-commutative and $(1,3)$-commutative $(4,2)$-hypergroupoid that is not $(0,1)$-commutative.

Example 2.10. Let $H = \mathbb{Z}$, where $(\mathbb{Z}, +, \cdot)$ is the ring of integers and $[\,]$, be a $(3,2)$-hyperoperation on H defined by

$$[x_1^3] = (\{x_1 + x_2 + x_3, 0\}, \{x_1x_2x_3, 1\}).$$

Clearly, $(H, [\,])$ is a commutative $(3,2)$-hypergroupoid.

Example 2.11. Let $H = \mathbb{Z}$, where $(\mathbb{Z}, +, \cdot)$ is the ring of integers, and $[\,]$ be a $(3,2)$-hyperoperation on H defined by $[x_1^3] = (\{x_1 + x_2 + x_3, x_1\}, \{x_1x_2x_3, x_2\})$. Then

$$[x_1x_2x_3]_1 = \{x_1 + x_2 + x_3, x_1\} = [x_1x_3x_2]_1,$$
$$[x_2x_1x_3]_1 = \{x_1 + x_2 + x_3, x_2\} = [x_2x_3x_1]_1,$$
$$[x_3x_1x_2]_1 = \{x_1 + x_2 + x_3, x_3\} = [x_3x_2x_1]_1,$$

and thus $\bigcap_{\sigma\in S_n}[x_{\sigma(1)}^{\sigma(3)}]_1 = \bigcap_{j=1}^{3}\{x_1 + x_2 + x_3, x_j\} = \{x_1 + x_2 + x_3\} \neq \emptyset$.

In a similar way, we obtain that

$$[x_1x_2x_3]_2 = \{x_1x_2x_3, x_2\} = [x_3x_2x_1]_2,$$
$$[x_2x_1x_3]_2 = \{x_1x_2x_3, x_1\} = [x_3x_1x_2]_2,$$
$$[x_1x_3x_2]_2 = \{x_1x_2x_3, x_3\} = [x_2x_3x_1]_2,$$

and thus $\bigcap_{\sigma\in S_n}[x_{\sigma(1)}^{\sigma(3)}]_2 = \bigcap_{j=1}^{3}\{x_1x_2x_3, x_j\} = \{x_1x_2x_3\} \neq \emptyset$.

Therefore, $\boldsymbol{H}$ is a weakly commutative $(3,2)$-hypergroupoid.

3. Some Results on (n,m)-hyperstructures

Proposition 3.1. *Let $(H, [\,])$ be an (n,m)-hypersemigroup. The following conditions are equivalent:*

(i) *$(H, [\,])$ is strongly cancellative;*
(ii) *$(H, [\,])$ is strongly left and strongly right cancellative;*
(iii) *$(H, [\,])$ is strongly i-cancellative for some $0 < i < k$.*

Proof. Implications (i)⇒(ii) and (i)⇒(iii) are obvious.

(ii)⇒(i) Let $(H, [\,])$ be an i-cancellative hypersemigroup for $i = 0$ and $i = k$. If

$$[a_1^j x_1^m a_{j+1}^k] = [a_1^j y_1^m a_{j+1}^k],$$

for $1 \leq j \leq k-1$, then for all $b_1^k \in H^k$ we have

$$[b_1^{k-j}[a_1^j x_1^m a_{j+1}^k] b_{k-j+1}^k] = [b_1^{k-j}[a_1^j y_1^m a_{j+1}^k] b_{k-j+1}^k].$$

From the associativity it follows that

$$[b_1^{k-j}a_1^j[x_1^m a_{j+1}^k b_{k-j+1}^k]] = [b_1^{k-j}a_1^j[y_1^m a_{j+1}^k b_{k-j+1}^k]].$$

From the strong cancellativity it follows that

$$[x_1^m a_{j+1}^k b_{k-j+1}^k] = [y_1^m a_{j+1}^k b_{k-j+1}^k],$$

which implies that $x_1^m = y_1^m$.

(iii)⇒(i) Let $(H, [\,])$ be strongly i-cancellative hypersemigroup for some $0 < i < k$. First we will show that if $(H, [\,])$ is strongly i-cancellative, then $(H, [\,])$ is strongly $(i+1)$-cancellative hypersemigroup.

We have

$$[a_1^{i+1} x_1^m a_{i+2}^k] = [a_1^{i+1} y_1^m a_{i+2}^k]$$

$$\Rightarrow\ [b_1^{i-1}[a_1^{i+1}x_1^m a_{i+2}^k]b_i^k] = [b_1^{i-1}[a_1^{i+1}y_1^m a_{i+2}^k]b_i^k]$$
$$\Rightarrow\ [b_1^{i-1}a_1[a_2^{i+1}x_1^m a_{i+2}^k b_i]b_{i+}^k] = [b_1^{i-1}a_1[a_2^{i+1}y_1^m a_{i+2}^k b_i]b_{i+1}^k]$$
$$\Rightarrow\ [a_2^{i+1}x_1^m a_{i+2}^k b_i] = [a_2^{i+1}y_1^m a_{i+2}^k b_i].$$

From the strong i-cancellativity it follows that $x_1^m = y_1^m$. Now, we will show that if $(H, [\,])$ is strongly i-cancellative, then $(H, [\,])$ is strongly $(i-1)$-cancellative hypersemigroup

$$[a_1^{i-1}x_1^m a_i^k] = [a_1^{i-1}y_1^m a_i^k]$$
$$\Rightarrow\ [b_1^{i+1}[a_1^{i-1}x_1^m a_i^k]b_{i+2}^k] = [b_1^{i+1}[a_1^{i-1}y_1^m a_i^k]b_{i+2}^k]$$
$$\Rightarrow\ [b_1^i[b_{i+1}a_1^{i-1}x_1^m a_i^{k-1}]a_k b_{i+2}^k] = [b_1^i[b_{i+1}a_1^{i-1}y_1^m a_i^{k-1}]a_k b_{i+2}^k]$$
$$\Rightarrow\ [b_{i+1}a_1^{i-1}x_1^m a_i^{k-1}] = [b_{i+1}a_1^{i-1}y_1^m a_i^{k-1}]$$
$$\Rightarrow\ x_1^m = y_1^m.$$

Hence, $(H, [\,])$ is strongly cancellative hypersemigroup. □

Proposition 3.2. *If for some j such that $1 \le j \le i-1$ the (n,m)-hypergroupoid $(H, [\,])$ is $(i-1, i+j-1)$-associative and partially strongly $(i+j-1)$-cancellative, then it is $(i-j-1, i-1)$-associative.*

Proof. Let $(H, [\,])$ be an (n,m)-hypergroupoid that is $(i-1, i+j-1)$-associative and partially strongly $(i+j-1)$-cancellative. Then it follows

$$\begin{aligned}
&[a_1^{i+j-1}[x_1^{i-j-1}[x_{i-j}^{n+i-j-1}]x_{n+i-j}^{n+k}]a_{i+j}^k]\\
=&[a_1^{i-1}[a_i^{i+j-1}x_1^{i-j-1}[x_{i-j}^{n+i-j-1}]x_{n+i-j}^{n+k-j}]x_{n+k-j+1}^{n+k}a_{i+j}^k]\\
=&[a_1^{i-1}[a_i^{i+j-1}x_1^{i-j-1}x_{i-j}^{i-1}[x_i^{n+i-1}]x_{n+i}^{n+k-j}]x_{n+k-j+1}^{n+k}a_{i+j}^k]\\
=&[a_1^{i-1}a_i^{i+j-1}[x_1^{i-1}[x_i^{n+i-1}]x_{n+i}^{n+k-j}x_{n+k-j+1}^{n+k}]a_{i+j}^k]\\
=&[a_1^{i+j-1}[x_1^{i-1}[x_i^{n+i-1}]x_{n+i}^{n+k}]a_{i+j}^k].
\end{aligned}$$

Using the fact that $(H, [\,])$ is partially strongly $(i+j-1)$-cancellative we obtain that

$$[x_1^{i-j-1}[x_{i-j}^{n+i-j-1}]x_{n+i-j}^{n+k}] = [x_1^{i-1}[x_i^{n+i-1}]x_{n+i}^{n+k}],$$

i.e., $(i-j-1, i-1)$-associativity holds in $(H, [\,])$ for $1 \le j \le i-1$. □

Proposition 3.3. *If a partially strongly $(i-1)$-cancellative ($i \ge 1$) (n,m)-hypergroupoid $(H, [\,])$ is $(i-1, i+j-1)$-associative for some $j \ge 1$ and $2j \le k-i$, then it is $(i+j-1, i+2j-1)$-associative.*

Proof. Since $(H, [\,])$ is partially strongly $(i-1)$-cancellative, it follows that there is a sequence $a_1^k \in H^k$ such that $[a_1^{i-1}x_1^m a_i^k] = [a_1^{i-1}y_1^m a_i^k] \ \Rightarrow\ x_1^m = y_1^m$. Then, using the $(i-1, i+j-1)$-associativity for some $j \ge 1$ and $2j \le k-i$, and we obtain that

$$\begin{aligned}
&[a_1^{i-1}[x_1^{i+j-1}[x_{i+j}^{i+j+n+1}]x_{i+j+n}^{n+k}]a_i^k]\\
=&[a_1^{i-1}[x_1^{i+2j-1}[x_{i+2j}^{i+2j+n-1}]x_{i+2j+n}^{n+k}]a_i^k]
\end{aligned}$$

$$
\begin{aligned}
&=[a_1^{i-1}[x_1^j[x_{j+1}^{i+j-1}[x_{i+j}^{i+j+n-1}]x_{i+j+n}^{n+k}]a_i^{i+j-1}]a_{i+j}^k]\\
&=[a_1^{i-1}x_1^j[x_{j+1}^{i+2j-1}[x_{i+2j}^{i+2j+n-1}]x_{i+2j+n}^{n+k}a_i^{i+j-1}]a_{i+j}^k]\\
&=[a_1^{i-1}[x_1^j x_{j+1}^{i+2j-1}[x_{i+2j}^{i+2j+n-1}]x_{i+2j+n}^{n+k}]a_i^k]\\
&=[a_1^{i-1}[x_1^{i+2j-1}[x_{i+2j}^{i+2j+n-1}]x_{i+2j+n}^{n+k}]a_i^k].
\end{aligned}
$$

The (n,m)-hypergroupoid $(H,[\,])$ is partially strongly $(i-1)$-cancellative and therefore

$$[[x_{i+j}^{i+j+n+1}]x_{i+j+n}^{n+k}] = [x_1^{i+2j-1}[x_{i+2j}^{i+2j+n-1}]x_{i+2j+n}^{n+k}],$$

i.e., $(H,[\,])$ is $(i+j-1, i+2j-1)$-associative. □

Proposition 3.4. *Let $(H,[\,])$ be an (n,m)-hypersemigroup and let e be a neutral element in H. Then the following conditions are equivalent:*

(i) *the sequence $\overset{(k)}{e}$ is strongly cancellable;*

(ii) *the sequence $\overset{(k)}{e}$ is strongly i-cancellable for $i=0$ and $i=k$;*

(iii) *The sequence $\overset{(k)}{e}$ is strongly i-cancellable for some $0<i<k$.*

Proof. The proof is obvious from the proof of the Proposition 3.1. □

Proposition 3.5. *Let $(H,[\,])$ be an $(i,i+j)$-associative (n,m)-hypergroupoid for $1\le j\le i$. If there exists a sequence $e_1^k\in H^k$ such that the equality $[e_1^{i+j}x_1^m e_{i+j+1}^k]=x_1^m$ holds for every $x_1^m\in H^m$, then $(H,[\,])$ is an $(i-j,i)$-associative (n,m)-hypergroupoid.*

Proof. Using the equation given in the supposition we obtain that

$$[x_1^{i-j}[x_{i-j+1}^{i-j+m+k}]x_{i-j+m+k+1}^{m+2k}] = [e_1^{i+j}[x_1^{i-j}[x_{i-j+1}^{i-j+m+k}]x_{i-j+m+k+1}^{m+2k}]e_{i+j+1}^k].$$

Applying the $(i,i+j)$-associativity twice and using the given equation in the last step, one obtains that

$$
\begin{aligned}
&[e_1^{i+j}[x_1^{i-j}[x_{i-j+1}^{i-j+m+k}]x_{i-j+m+k+1}^{m+2k}]e_{i+j+1}^k]\\
=&[e_1^i[x_1^{i-j}[x_{i-j+1}^{i-j+m+k}]x_{i-j+m+k+1}^{m+2k-j}]x_{m+2k-j+1}^{m+2k}e_{i+j+1}^k]\\
=&[e_1^i[e_{i+1}^j x_1^i[x_{i+1}^{i+m+k}]x_{i+m+k+1}^{m+2k-j}]x_{m+2k-j+1}^{m+2k}e_{i+j+1}^k]\\
=&[e_1^{i+j}[x_1^i[x_{i+1}^{i+m+k}]x_{i+m+k+1}^{m+2k}]e_{i+j+1}^k]\\
=&[x_1^i[x_{i+1}^{i+m+k}]x_{i+m+k+1}^{m+2k}].
\end{aligned}
$$

Thus, $(H,[\,])$ is an $(i-j,i)$-associative hypergroupoid for $1\le j\le i$. □

Proposition 3.6. *Let $(H,[\,])$ be an $(i,i+j)$-associative (n,m)-hypergroupoid for $j\ge 1$. If there exists a sequence $e_1^k\in H^k$ such that the equality $[e_1^i x_1^m e_{i+1}^k]=x_1^m$ holds for every $x_1^m\in H^m$ and if $i+2j\le k$, then $(H,[\,])$ is an $(i+j,i+2j)$-associative (n,m)-hypergroupoid.*

Proof. Using the equation given in the supposition we obtain that

$$[x_1^{i+j}[x_{i+j+1}^{i+j+m+k}]x_{i+j+m+k+1}^{m+2k}] = [e_1^i[x_1^{i+j}[x_{i+j+1}^{i+j+m+k}]x_{i+j+m+k+1}^{m+2k}]e_{i+1}^k].$$

Applying the $(i, i+j)$-associativity three times and using the given equation in the last step, one obtains that

$$\begin{aligned} & [e_1^i[x_1^{i+j}[x_{i+j+1}^{i+j+m+k}]x_{i+j+m+k+1}^{m+2k}]e_{i+1}^k] \\ =& [e_1^i x_1^j[x_{j+1}^{i+j}[x_{i+j+1}^{i+j+m+k}]x_{i+j+m+k+1}^{m+2k} e_{i+1}^{i+j}]e_{i+j+1}^k] \\ =& [e_1^i x_1^j[x_{j+1}^{i+2j}[x_{i+2j+1}^{i+2j+m+k}]x_{i+2j+m+k+1}^{m+2k} e_{i+1}^{i+j}]e_{i+j+1}^k] \\ =& [e_1^i[x_1^{i+2j}[x_{i+2j+1}^{i+2j+m+k}]x_{i+2j+m+k+1}^{m+2k}]e_{i+1}^k] \\ =& [x_1^{i+2j}[x_{i+2j+1}^{i+2j+m+k}]x_{i+2j+m+k+1}^{m+2k}]. \end{aligned}$$

Hence, $(H, [\ ])$ is an $(i+j, i+2j)$-associative hypergroupoid for $i+2j \leq k$ and $j \geq 1$. □

Proposition 3.7. *Let $(H, [\,])$ be an (n, m)-hypersemigroup with an m-neutral element e that satisfies the identity $[ex_1^m \overset{(k-1)}{e}] = x_1^m$. Then*

a) $[\overset{(i)}{e}\, x_1^m \overset{(k-i)}{e}] = x_1^m$ *for* $2 \leq i \leq k$*;*

b) *e is a neutral element.*

Proof. a) First we will prove that $[\overset{(2)}{e}\, x_1^m \overset{(k-2)}{e}] = x_1^m$. Namely:

$$\begin{aligned} [\overset{(2)}{e}\, x_1^m \overset{(k-2)}{e}] =& [[\overset{(2)}{e}\, x_1^m \overset{(k-2)}{e}]\overset{(k)}{e}] = [e[ex_1^m \overset{(k-2)}{e}\, e]\overset{(k-1)}{e}] \\ =& [e[ex_1^m \overset{(k-1)}{e}]\overset{(k-1)}{e}] = [ex_1^m \overset{(k-1)}{e}] = x_1^m. \end{aligned}$$

Iterating this procedure for every $3 \leq i \leq k$, using the condition and every result obtained in the previous step, we obtain that

$$\begin{aligned} [\overset{(i)}{e}\, x_1^m \overset{(k-i)}{e}] =& [[\overset{(i)}{e}\, x_1^m \overset{(k-i)}{e}]\overset{(k)}{e}] = [\overset{(i-1)}{e}\, [ex_1^m \overset{(k-1)}{e}]\overset{(k-i+1)}{e}] \\ =& [\overset{(i-1)}{e}\, x_1^m \overset{(k-i+1)}{e}] = x_1^m. \end{aligned}$$

b) The proof follows from the supposition and a suitable application of a). Namely, $[x_1 \overset{(k)}{e}\, x_2^m] = [[x_1 \overset{(k)}{e}\, x_2^m]\overset{(k)}{e}] = [x_1[\overset{(k)}{e}\, x_2^m e]\overset{(k-1)}{e}] = [x_1 x_2^m e \overset{(k-1)}{e}] = [x_1^m \overset{(k)}{e}] = x_1^m$. □

The next proposition is symmetrical to Proposition 3.7.

Proposition 3.8. *Let $(H, [\,])$ be an (n, m)-hypersemigroup with a 0-neutral element e that satisfies the identity $[\overset{(k-1)}{e}\, x_1^m e] = x_1^m$. Then*

a) $[\overset{(i)}{e}\, x_1^m \overset{(k-i)}{e}] = x_1^m$ *for* $2 \leq i \leq k$*;*

b) *e is a neutral element.*

Proposition 3.9. *If $(H, [\ ])$ is an $(m+k, m)$-hypergroupoid such that $k < m$ and $(H, [\,])$ has a neutral element e, then $|\,H\,| = 1$.*

Proof. Let $x \in H$ be arbitrarily chosen element. Then, $[\overset{(k)}{e}\overset{(m-k)}{x}\overset{(k)}{e}] = \overset{(m-k)}{x}\overset{(k)}{e}$ and $[\overset{(k)}{e}\overset{(m-k)}{x}\overset{(k)}{e}] = \overset{(k)}{e}\overset{(m-k)}{x}$, which implies that $x = e$, i.e., $|\,H\,| = 1$. □

Theorem 3.1. *An (n, m)-hypersemigroup $(H, [\ ])$ is an (n, m)-hypergroup if and only if the relation* (1.3) *holds for*

a) $i = 0$ *and* $i = k$ *(i.e.,* $a_{k+1}^n \in [x_1^m a_1^k]$ *and* $a_{k+1}^n \in [a_1^k x_1^m]$*, for all* $a_1^n \in H^n$*);*
b) *some* i*,* $1 \leq i < k - 1$*.*

Proof. The direct statements are obvious.

a)⇒(1.3) For the converse, let the relation (1.3) holds for $i = 0$ and $i = k$. Then for $a_1^k \in H^k$ and $b_1^m \in H^m$, there is an $x_1^m \in H^m$ such that $b_1^m \in [x_1^m a_1^k]$ and for $a_1^k \in H^k$ and $x_1^m \in H^m$, there is an y_1^m such that $x_1^m \in [a_1^k y_1^m]$. From here we obtain that

$$b_1^m \in [[a_1^k y_1^m]a_1^k] = [a_1^i[a_{i+1}^k y_1^m a_1^i]a_{i+1}^k],$$

for $1 \leq i \leq k - 1$, so the relation (1.3) holds for every i, i.e., $(H, [\ ])$ is an (n, m)-hypergroup.

b)⇒(1.3) Let the relation (1.3) holds for some i, $1 \leq i < k - 1$. Then, for $a_1^k \in H^k$ and $b_1^m \in H^m$, there is $x_1^m \in H^m$ such that $b_1^m \in [a_1^i x_1^m a_{i+1}^k]$. For $\overset{(k)}{a_{i+1}} \in H^k$ and $x_1^m \in H^m$, there is y_1^m such that $x_1^m \in [\overset{(i)}{a_{i+1}}\, y_1^m\, \overset{(k-i)}{a_{i+1}}]$. Then

$$b_1^m \in [a_1^i[\overset{(i)}{a_{i+1}}\, y_1^m\, \overset{(k-i)}{a_{i+1}}]a_{i+1}^k] = [a_1^{i+1}[\overset{(i-1)}{a_{i+1}}\, y_1^m\, \overset{(k-i+1)}{a_{i+1}}]a_{i+2}^k].$$

So, the relation (1.3) is solvable in the $i+1$ place, for all $1 \leq i \leq k-1$ and consequently, it is solvable in the k place.

For $\overset{(k)}{a_i} \in H^k$ and $x_1^m \in H^m$, there is $z_1^m \in H^m$ such that $x_1^m \in [\overset{(i)}{a_i}\, z_1^m\, \overset{(k-i)}{a_i}]$. Then

$$b_1^m \in [a_1^i[\overset{(i)}{a_i}\, z_1^m\, \overset{(k-i)}{a_i}]a_{i+1}^k] = [a_1^{i-1}[\overset{(i+1)}{a_i}\, z_1^m\, \overset{(k-i-1)}{a_i}]a_i^k].$$

So, the relation (1.3) is solvable in the $i-1$ place, for all $1 \leq i \leq k-1$ and consequently, it is solvable in the 0 place.

Hence, $(H, [\ ])$ is an (n, m)-hypergroup. □

4. Relations on (n, m)-hyperstructures

Let $(H, [\ ])$ be an (n, m)-hypersemigroup. An equivalence relation ρ on H is said to be

(a) *regular* if $a_j \rho b_j$, $j \in \{1, 2, \ldots, n\}$ and $x \in [a_1^n]_s$, $s \in \{1, 2, \ldots, m\}$, implies that there exists $y \in [b_1^n]_s$, $s \in \{1, 2, \ldots, m\}$, such that $x\rho y$;
(b) *strongly regular* if $a_j \rho b_j$, $j \in \{1, 2, \ldots, n\}$ implies that $x\rho y$, for every $x \in [a_1^n]_s$, $y \in [b_1^n]_s$ and $s \in \{1, 2. \ldots, m\}$.

Theorem 4.1. *Let $(H, [\ ])$ be an (n, m)-hypersemigroup and ρ be an equivalence relation on H.*

(i) *If ρ is regular, then $(H/\rho, [\,]^\rho)$ is a (n,m)-hypersemigroup, where the operation $[\,]^\rho$ consists of m component hyperoperations, $[\,]^\rho = ([\,]^\rho_1, \dots, [\,]^\rho_m)$, each of which is defined as follows:*

$$[\rho(a_1)\rho(a_2)\dots\rho(a_n)]^\rho_s = \{\rho(x) \mid x \in [a_1^n]_s\},$$

for every $s \in \{1, 2, \dots, m\}$.

(ii) *If $(H/\rho, [\,]^\rho)$ is a (n,m)-hypersemigroup, then ρ is a regular relation.*

(iii) *The canonical projection $\pi : H \to H/\rho$ such that $\pi(a) = \rho(a)$ is a strong homomorphism on the (n,m)-hypersemigroups $(H, [\,])$ and $(H/\rho, [\,]^\rho)$.*

(iv) *If $(H, [\,])$ is a (n,m)-hypergroup, then $(H/\rho, [\,]^\rho)$ is a (n,m)-hypergroup as well.*

Proof. (i) First, we will show that $[\,]^\rho$ is a well defined operation on H/ρ. Let $\rho(a_i) = \rho(c_i)$, for $i = 1, 2, \dots, n$. Clearly, $a_i \rho c_i$, for $i = 1, 2, \dots, n$. Let $\rho(x) \in [\rho(a_1)\rho(a_2)\dots\rho(a_n)]^\rho_s$. Then $x \in [a_1^n]_s$. By the assumption, ρ is a regular relation and thus, for every $x \in [a_1^n]_s$ there exists $y \in [c_1^n]_s$, $s = 1, 2, \dots, m$, such that $x\rho y$. Since $\rho(x) = \rho(y)$ and $\rho(x) = \rho(y) \in [\rho(c_1)\rho(c_2)\dots\rho(a_n)]^\rho_s$ it follows that $[\rho(a_1)\rho(a_2)\dots\rho(a_n)]^\rho_s \subseteq [\rho(c_1)\rho(c_2)\dots\rho(c_n)]^\rho_s$, $s = 1, 2, \dots, m$. The converse inclusion can be shown in a similar way.

In order to prove the $(m+k,m)$-associativity, suppose that $\rho(a_i) \in H/\rho$, where $i = 1, 2, \dots, m+2k$ and let $\rho(z)$ belongs in

$$[[\rho(a_1)\dots\rho(a_{m+k})]^\rho_1 \dots\ [\rho(a_1)\dots\rho(a_{m+k})]^\rho_m\ \rho(a_{m+k+1})\dots\rho(a_{m+2k})]^\rho_s,$$

for $s = 1, 2.\dots, m$. Then, there exists $\rho(u_\lambda) \in [\rho(a_1)\rho(a_2)\dots\rho(a_{m+k})]^\rho_\lambda$, for $\lambda = 1, 2, \dots, m$ and $z \in [u_1^m a_{m+k+1}^{m+2k}]_s$, for $s = 1, 2, \dots, m$.

Clearly, $z \in [[a_1^{m+k}]_1 \dots, [a_1^{m+k}]_m\, a_{m+k+1}^{m+2k}]_s = [a_1^j\, [a_{j+1}^{j+m+k}]_1 \dots [a_{j+1}^{j+m+k}]_m\, a_{j+m+k+1}^{m+2k}]_s$. There exists $x_\lambda \in [a_{j+1}^{j+m+k}]_\lambda$, for $\lambda = 1, \dots, m$ and $j = 1, \dots, k$, such that $z \in [a_1^j x_1^m a_{j+m+k+1}^{m+2k}]_s$, for $s = 1, 2, \dots, m$. Therefore,

$$\begin{aligned}\rho(z) \in &[\rho(a_1)\dots\rho(a_j)[\rho(a_{j+1})\dots\rho(a_{j+m+k})]^\rho_1 \dots [\rho(a_{j+1})\dots\rho(a_{j+m+k})]^\rho_m \\ &\quad \rho(a_{j+m+k+1})\dots\rho(a_{m+2k})]^\rho_s,\end{aligned}$$

and thus

$$\begin{aligned}&[[\rho(a_1)\dots\rho(a_{m+k})]^\rho_1 \dots\ [\rho(a_1)\dots\rho(a_{m+k})]^\rho_m\ \rho(a_{m+k+1})\dots\rho(a_{m+2k})]^\rho_s \\ \subseteq &[\rho(a_1)\dots\rho(a_j)[\rho(a_{j+1})\dots\rho(a_{j+m+k})]^\rho_1 \dots [\rho(a_{j+1})\dots\rho(a_{j+m+k})]^\rho_m \\ &\rho(a_{j+m+k+1})\dots\rho(a_{m+2k})]^\rho_s,\end{aligned}$$

for $s = 1, 2, \dots, m$ and $j = 1, 2, \dots, k$.

The converse inclusion can be shown similarly.

(ii) Let $a_i \rho c_i$, for $i = 1, 2, \dots, n$. Then, $\rho(a_i) = \rho(c_i)$, $i = 1, 2, \dots, n$ and so $[\rho(a_1)\rho(a_2)\dots\rho(a_n)]^\rho_s = [\rho(c_1)\rho(c_2)\dots\rho(c_n)]^\rho_s$, for $s = 1, 2, \dots, m$. For every $x \in [a_1^n]_s$, $s = 1, 2, \dots, m$, we have that $\rho(x) \in [\rho(c_1)\rho(c_2)\dots\rho(c_n)]^\rho_s$, so there exists $y \in [c_1^n]_s$, such that $\rho(x) = \rho(y)$, and therefore $x\rho y$.

(iii) We claim that $\pi([a_1^n]_s) = [\pi(a_1)\pi(a_2)\dots\pi(a_n)]_s^\rho$, $s = 1, 2, \dots, m$. Let $\rho(a) \in \pi([a_1^n]_s)$, $s = 1, 2, \dots, m$. Then, there exists $a' \in [a_1^n]_s$, such that $\rho(a) = \pi(a')$. So, $\rho(a) = \rho(a') \in [\rho(a_1)\rho(a_2)\dots\rho(a_n)]_s^\rho = [\pi(a_1)\pi(a_2)\dots\pi(a_n)]_s^\rho$. Thus, $\pi([a_1^n]_s) \subseteq [\pi(a_1)\pi(a_2)\dots\pi(a_n)]_s^\rho$, $s = 1, 2, \dots, m$.

For the converse inclusion, let $\rho(a) \in [\pi(a_1)\pi(a_2)\dots\pi(a_n)]_s^\rho$, $s = 1, 2, \dots, m$. Since $\pi(a_i) = \rho(a_i)$, $i = 1, 2, \dots, n$, it follows that $\rho(a) \in [\rho(a_1)\rho(a_2)\dots\rho(a_n)]_s^\rho$, $s = 1, 2, \dots, m$. Then, there exists $a' \in \rho(a)$ such that $a' \in [a_1^n]_s$, $s = 1, 2, \dots, m$. Thus, $\rho(a) = \rho(a') = \pi(a') \in \pi([a_1^n]_s)$, so $[\pi(a_1)\pi(a_2)\dots\pi(a_n)]_s^\rho \subseteq \pi([a_1^n]_s)$, $s = 1, 2, \dots, m$.

(iv) Suppose that $(H, [\,])$ is a (n, m)-hypergroup, i.e., $(H, [\,])$ is a (n, m)-hypersemigroup such that for every $a_1^n \in H^n$, there is $x_1^m \in H^m$ such that $a_{k+1}^{m+k} \in [a_1^j x_1^m a_{j+1}^k]$, for every $j = 0, 1, \dots, k$. Let $\rho(a_i) \in H/\rho$, $i = 1, 2, \dots, n$. Then, there exists $t_1, t_2, \dots, t_m \in H$ such that $a_{k+1}^{m+k} \in [a_1^j t_1^m a_{j+1}^k]$, for every $j = 0, 1, \dots, k$. Thus, $\rho(a_{k+s}) \in [\rho(a_1)\dots\rho(a_j)\rho(t_1)\dots\rho(t_m)\rho(a_{j+1})\dots\rho(a_k)]_s^\rho$, for every $j = 0, 1, \dots, k$ and $s = 1, 2, \dots, m$. Therefore, $(H/\rho, [\,]^\rho)$ is a (n, m)-hypergroup. □

Theorem 4.2. *Let $(H, [\,])$ be an (n, m)-hypersemigroup and ρ be a strongly regular relation on H. Then*

(i) *$(H/\rho, [\,]^\rho)$ is an (n, m)-semigroup;*

(ii) *if $(H, [\,])$ is an (n, m)-hypergroup, then $(H/\rho, [\,]^\rho)$ is an (n, m)-group.*

Proof. (i) Let $\rho(x), \rho(y) \in [\rho(a_1)\rho(a_2)\dots\rho(a_n)]_s^\rho$, $s = 1, 2, \dots, m$. Then, $x, y \in [a_1^n]_s$. Since ρ is strongly regular, it follows that $x\rho y$, i.e., $\rho(x) = \rho(y)$. Thus, $|[\rho(a_1)\rho(a_2)\dots\rho(a_n)]_s^\rho| = 1$, for $s = 1, 2, \dots, m$. Therefore, $(H/\rho, [\,]^\rho)$ is an (n, m)-semigroup.

(ii) It follows from (i) and Theorem 4.1(iv). □

5. Future Work

We have started the investigation on vector valued hypersemigroups and vector valued hypergroups. As a future work we are planing to introduce a relation β on a vector valued hypersemigroup (hypergroup) and to investigate the fundamental equivalence relation β^* as the smallest equivalence relation on a vector valued hypersemigroup (hypergroup), such that H/β^* would be a vector valued semigroup (group).

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