Kragujevac Journal of Mathematics
Volume 51(7) (2027), Pages 1225–1244.

MITTAG-LEFFLER-HYERS-ULAM STABILITY FOR A FRACTIONAL DIFFERENTIAL EQUATION VIA LAPLACE TRANSFORM AND GRONWALL INEQUALITY

MESFIN TESHOME BEYENE[1], MITIKU DABA FIRDI[2], AND TAMIRAT TEMESGEN DUFERA[2]

Abstract. In this study, we examined the existence, uniqueness, and Mittag-Leffler-Hyers-Ulam stability of solutions to a Caputo-Hadamard fractional differential equation subject to initial conditions of the powers of $(t\frac{d}{dt})$ derivatives. Applying a modified Laplace transform to this equation yields an analytical solution with respect to Mittag-Leffler functions. Our main results were achieved by applying Banach, Boyd-Wong, and Schaefer's fixed-point theorems for the existence and uniqueness results; the Gronwall inequality related to the Hadamard integral; and a lemma for the Mittag-Leffler-Hyers-Ulam stability result. Numerical examples are provided to illustrate the applicability of our main findings.

1. Introduction

Since various fractional modeling, such as memory diffusion of water in sand, geomagnetic field, dynamics of particles, etc., can explain physical phenomena more realistically, fractional derivatives have been used extensively in differential equations, see [2, 11, 12, 24, 34, 40, 49] and the references therein. The majority of works on fractional differential equations rely on the fractional derivatives of Caputo and Riemann-Liouville. The fractional derivative owing to Hadamard, or Hadamard derivative [20], is another type of fractional operator that can be found in the literature. As compared to the Caputo and Riemann-Liouville derivatives, this derivative's kernel contains a logarithmic function $(\log t - \log \tau)$ of arbitrary order rather than $(t-\tau)$, and

Key words and phrases. Caputo-Hadamard fractional derivative, Laplace transform, Mittag-Leffler-Hylers-Ulam stability, Existence and uniqueness.
2020 *Mathematics Subject Classification.* Primary: 34A60. Secondary: XXXXX, XXXXX.
DOI
Received: June 08, 2025.
Accepted: May 06, 2026 .

its definition of the operator $t\frac{d}{dt}$ has proven to be invariant on the half-axis with regard to dilation [18]. The Caputo-Hadamard (CH) derivative [18], which was proposed recently, is a derivative that was produced by modifying the Hadamard derivative in the sense of the Caputo one. It is derived by reversing the order of differentiation and integration of the Hadamard derivative. Because the CH fractional derivative's integration kernel is determined by the power of the logarithmic function rather than the power function, it is better suited to describe the ultra-slow process. The regional gradient controllability for ultra slow diffusion processes and the COVID-19 epidemic brought on by the Omicron variant are modeled in studies [10] and [9] using the CH derivative.

The study of fixed point theory constitutes a fundamental part of nonlinear functional analysis and has profound implications in the theory of differential equations. Fixed point results provide powerful and unifying tools for establishing the existence and uniqueness (EU) of solutions to a wide variety of problems arising in ordinary, functional, and integro-differential equations. Among the most influential results in this area are the Banach contraction principle [15] and its nonlinear generalizations, notably the Boyd-Wong (BW) fixed point theorem [8].

By transforming a differential equation into an equivalent integral equation, one may define an operator on a suitable Banach space of continuous functions. Under a Lipschitz condition on the nonlinear term, this operator becomes a contraction, and the Banach theorem guarantees both the EU of solutions. However, the Banach contraction principle requires a strong linear contraction condition, which is often too restrictive for many nonlinear problems encountered in applications. To overcome this limitation, Boyd and Wong introduced a significant generalization of the contraction principle by replacing the linear contraction constant with a nonlinear control function. In the BW theorem, the contraction condition is governed by an upper semicontinuous function ϕ satisfying $\phi(t) < t$ for all $t > 0$. This relaxation allows for a broader class of mappings while still ensuring the existence and uniqueness of fixed points in complete metric spaces.

Although BW fixed point theory has been recognized as a significant generalization of the classical Banach contraction principle, the number of studies that specifically develop and apply Boyd-Wong type contractive conditions remains relatively limited. Compared with the extensive research on Banach-type mappings, fewer works systematically explore BW conditions in diverse differential equation settings [5, 27]. In particular, applications to nonlinear CH fractional-order problems are rarely investigated in the literature. This observed gap motivates the present study, which aims to expand the theoretical framework and demonstrate novel applications of Boyd-Wong type fixed point results to existence and uniqueness problems beyond the reach of existing techniques.

The concept of Hyers-Ulam (HU) stability originates from a problem posed by S. M. Ulam [45] in 1940 concerning the stability of functional equations. In 1941,

D. H. Hyers [23] gave the first affirmative answer for additive mappings. This phenomenon of stability was called HU stability. Since then, the concept has been extended to ordinary differential equations, integral equations, and fractional differential equations. This notion guarantees that approximate solutions remain close to true solutions under small perturbations. The HU stability of different types of fractional differential equations (FDEs) has become increasingly important and popular in the last few decades because of its wide range of applications in numerous diverse and widespread fields of science and engineering. Using a variety of mathematical techniques, researchers examined the EU and HU stability analysis of solutions to numerous FDEs [1, 3, 6, 7, 14, 19, 25, 28, 38, 42–44].

In the context of fractional differential equations, solutions often exhibit non-exponential behavior that is naturally described by the Mittag-Leffler function [21] To capture this behavior, the concept of HU stability has been refined to the Mittag-Leffler-Hyers-Ulam stability (MLHU). This form of stability reflects the memory and hereditary properties inherent in fractional-order systems.

The main difference between Ulam-Hyers stability and MLHU stability lies in the bounding function. HU stability typically uses constant or exponential bounds, whereas MLHU stability employs Mittag-Leffler functions, which generalize exponential functions and are better suited for fractional dynamics. Thus, MLHU stability can be viewed as a natural extension of Ulam-Hyers stability for fractional differential equations.

Several researchers have investigated MLHU stability for various classes of fractional problems. In [32] proved the EU and MLHU stability for ψ-Hilfer fractional delay differential equations using Picard operators and generalized Gronwall inequality involved in a ψ-Riemann-Liouville fractional integral which is not in a CH setting, where CH has a different kernel which is logarithmic $(\ln t - \ln \tau)$, were proved.

In [16] uniqueness and MLHU stability of a Φ-Caputo fractional derivative using generalized Laplace transforms and a fractional Gronwall inequality, which is not appropriate to CH operators were proved. The Ulam-Hyers-Mittag-Leffler stability analysis of a class of Caputo nabla fractional-order delay difference equation is also addressed in [25].

A coupled implicit (κ,θ)-fractional systems have been shown to admit existence and κMittag-Leffler-Ulam-Hyers stability under suitable conditions [38]. In the context of fractional partial differential equations, Ulam-Hyers-Rassias Mittag-Leffler stability has also been established using generalized Gronwall approaches [36].

Recently, [30] demonstrated the MLHU stability as well as the Ulam-Hyers-Rassias-Mittag-Leffler stability pertaining to ψ-Hilfer equations. By leveraging fixed-point theory and generalized Gronwall's inequality not related to CH settings, they developed a rigorous framework that guarantees the existence and stability of the solution.

The existence, uniqueness and MLHU stability of the solutions a nonlinear Caputo fractional-order delay differential equation has been investigated by [43]. The Henry-Gronwall inequality related to the Riemann-Liouville integral is utilized to establish the MLHU stability results, which is not appropriate to CH FDEs whose kernel is a logarithmic function. Using the fixed-point technique, [39] examined the MLHU stability of a class of nonlinear fractional reaction-diffusion equations on a confined interval. We can also refer to the following MLHU stability results explored by different researchers [1, 17, 22, 33, 46, 47].

Such results highlight the importance of Mittag-Leffler-type bounds in accurately describing stability behavior in fractional-order models. However, to the best of our knowledge, corresponding stability results for fractional differential equations governed by CH derivatives via an appropriate Gronwall inequality in relation to the Hadamard integral have not yet been explored.

Motivated by the above works, this present paper is devoted to developing sufficient conditions for the EU and MLHU stability of a class of Cauchy hybrid CH fractional differential equations subject to initial conditions involving the logarithmic operator, $(t\frac{d}{dt})$, which is given as follows

$$ {}^{CH}D^{\lambda}_{L_1^+}g(t)-\beta g(t)=f(t,g(t),I^{\delta}g(t)),\quad t\in K=[L_1,L_2], L_2>L_1>0,\beta\geq 0, \tag{1.1}$$

subject to the initial conditions:

$$\begin{cases} g(L_1)=0,\\ \xi g(L_1)=\sigma_1\in\mathbb{R},\\ \xi^2 g(L_1)=\sigma_2\in\mathbb{R}, \end{cases} \tag{1.2}$$

where $\xi=(t\frac{t}{dt})$ and the operator ${}^{CH}D^{\lambda}_{L_1^+}$ denotes the CH fractional derivatives of order $2<\lambda\leq 3$, whereas I^{δ} denotes the Hadamard integral of order $\delta>0$. $f:[L_1,+\infty]\times\mathbb{R}^2\to\mathbb{R}$ is also a given continuous function.

It is significant to notice that, under the following two conditions, our work logically extends and continues the findings presented in the previously mentioned studies.

- The functions $\tan^{-1}(x)$, $\frac{Hx}{H+1}$ and $\frac{Mx}{M+x}$ where H and M are defined in the articles, were used by the authors of [5, 27, 41], to transform the corresponding operators in the articles to the respective nonlinear contractions and developed conditions for the uniqueness of solutions to the respective problems. Inspired by the aforementioned works, we employed the function $G(x)=1-e^{-x}$ to get a new EU result for the problem (1.1)–(1.2).
- A novel finding on MLHU stability is also obtained by using the Gronwall inequality related with Hadamard integral (2.6) and Lemma 2.4, which are not utilized by any researchers that explore such related topics.

From both theoretical and applied perspectives, the proposed approach offers several advantages.

- It is mathematically consistent with the intrinsic structure of CH operators.
- It provides sharper stability results expressed via Mittag-Leffler functions.
- It yields explicit solution formulas that support numerical simulation and qualitative analysis.

- It extends the applicability of Hadamard fractional models to broader classes of nonlinear problems.

The remaining portion of the paper is divided into the following sections: A few important definitions, remarks, lemmas, and theorems that are essential to the study are covered in Section 2. Sufficient conditions for the stability, uniqueness, and existence of solutions are given in Sections 3 and 4. Examples illustrating how to apply our main conclusions are given in Section 5. The final conclusion is given in Section 6.

2. Preliminaries

Definitions, remarks, theorems and lemmas that are important for the sequel are introduced in this section.

Definition 2.1 ([29])**.** The Hadamard fractional integral of order $\delta > 0$ for a function $w \in L^p[a,b]$, $a \le t \le b \le +\infty$, is defined as

$$ {}^{H}I^{\delta}_{a^+}w(\tau) = \frac{1}{\Gamma(\delta)}\int_a^{\tau}\left(\ln\frac{\tau}{s}\right)^{\delta-1}\frac{w(s)}{s}ds, \quad \delta > 0, $$

Definition 2.2 ([29])**.** Let $[c,d] \subset \mathbb{R}$. The Hadamard fractional derivative of order $\eta > 0$ for a function $q \in AC^n_\delta[c,d]$ is defined as

$$ {}^{H}D^{\eta}_{c^+}q(\tau) = \frac{1}{\Gamma(n-\eta)}\delta^n\int_c^{\tau}\left(\ln\frac{\tau}{s}\right)^{n-\eta-1}\frac{q(s)}{s}ds, \quad n-1 < \eta \le n,\ n = [\eta]+1, $$

where $[\eta]$ denotes the integer part of the real number η, $\delta = t\frac{d}{dt}$ and $AC[c,d]$ be the space of functions that are absolutely continuous on $[c,d]$ and the space $AC^n_\delta[c,d]$ which consists of functions q by

$$ AC^m_\delta[c,d] = \{q : [c,d] \to \mathbb{R} \mid \delta^{m-1}q(t) \in AC[c,d]\}. $$

Lemma 2.1 ([26])**.** *Let $\Re(\beta) \ge 0$ and $n = [\Re(\beta)] + 1$. If $v(x) \in AC^n_\delta[c,d]$, where $0 < c < d < +\infty$, then the Caputo-Hadamard fractional derivative of order*

a) *$\beta \notin \mathbb{N}_0$ is defined as*

$$ {}^{CH}D^{\beta}_{c^+}v(x) = \frac{1}{\Gamma(n-\beta)}\int_c^{x}\left(\ln\frac{x}{s}\right)^{n-\beta-1}\delta^n\frac{v(s)}{s}ds, $$

b) *$\beta \in \mathbb{N}$ is defined as*

$$ {}^{CH}D^{\beta}_{c^+}v(x) = \delta^n v(x), $$

where $\delta = t\frac{d}{dt}$.

In particular, ${}^{CH}D^{0}_{c^+}v(x) = v(x)$.

Lemma 2.2 (Contraction Mapping Principle [15])**.** *Let F be a Banach space, $Q \subset F$ be closed and $M : Q \to Q$ a strict contraction, i.e.,*

$$ |My - Mz| \le r|y - z|, \tag{2.1} $$

for some $r \in (0,1)$ and for all $y, z \in Q$. Then, M has a unique fixed point.

Definition 2.3. A mapping $Q : X \to X$, where X is a Banach space, is said to be a nonlinear contraction if a continuous non-decreasing function $g : [0, +\infty) \to [0, +\infty)$ exists, such that $g(0) = 0$ and $g(m) < m$, for all $m > 0$ and that $\|Qu - Qv\| \leq g(\|w_1 - w_2\|)$ for all $w_1, w_2 \in X$.

Theorem 2.1 ([8])**.** *Let $\mathbb{G} : Y \to Y$ be a non-linear contraction where Y is a Banach space. Then, $\mathbb{G}$ has a unique fixed point in Y.*

Theorem 2.2 (Schaefer fixed-point theorem [13])**.** *Let Ω be a normed linear space, and let R be a convex subset of Ω such that $0 \in R$. Let $S : R \to R$ be a completely continuous operator, and let S be a bounded subset of V.*

$$G = \{v \in R : v = \lambda S v, \text{ for some } \lambda \in (0, 1)\}.$$

Then, either G is unbounded or S has a fixed point.

The Gronwall inequality has recently been investigated in relation to ψ-fractional operators by [4] and its result is given as follows.

Theorem 2.3. *Let $u, v : [a, b] \to \mathbb{R}$ be two integrable functions, $g : [a, b] \to \mathbb{R}$ be a continuous function and α a positive real number. Suppose that*

(a) *u and v are nonnegative;*
(b) *g is non-negative and nondecreasing.*

The inequality indicated below is established:

$$u(t) \leq v(t) + g(t) \int_a^t \psi'(t)(\psi(t) - \psi(\tau))^{\alpha-1} u(\tau) d\tau, \quad \text{for all } t \in [a, b], \tag{2.2}$$

Then, for all $t \in [a, b]$,

$$u(t) \leq v(t) + \int_a^t \sum_{k=1}^{+\infty} \binom{n}{k} \frac{((g(t)\Gamma(\alpha))^k}{\Gamma(k\alpha} \psi'(t)(\psi(t) - \psi(\tau))^{k\alpha-1} v(\tau) d\tau.$$

Moreover, if v is non-decreasing, then

$$u(t) \leq v(t) \mathbb{E}_\alpha(g(t)\Gamma(\alpha)(\psi(t) - \psi(a))^\alpha), \quad \text{for all } t \in [a, b]$$

The generalized Gronwall inequality for Hadamard fractional integrals is obtained by replacing the kernel function $\psi(t)$ of the previous Theorem 2.3 with the function $\ln t$ as follows.

Theorem 2.4. *Suppose $\alpha > 0$, $u(t)$ and $v(t)$ are integrable functions on $t \in [b_1, b_2]$ and $h(t)$ is a continuous function defined on $t \in [b_1, b_2]$. Assume that*

(a) *v and u are non-negative;*
(b) *h is non-negative and non-decreasing.*

If the following inequality

$$u(t) \leq v(t) + h(t) \int_{b_1}^t \left(\ln \frac{t}{s}\right)^{q-1} \frac{u(s)}{s} ds, \quad \text{for all } t \in [b_1, b_2], \tag{2.3}$$

holds, then

$$u(t) \leq v(t) + \sum_{n=1}^{+\infty} \frac{((h(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \int_{b_1}^{t} \left(\ln \frac{t}{s}\right)^{nq-1} \frac{v(s)}{s} ds, \quad \text{for all } t \in [b_1, b_2].$$

Moreover, if a is nondecreasing, then

$$u(t) \leq v(t)\mathbb{E}_\alpha \left(h(t)\Gamma(\alpha) \left(\ln \frac{t}{b_1}\right)^\alpha \right).$$

The standard Laplace transform cannot be used directly to a CH derivative because the derivative starts at an initial time $\eta_1 > 0$. For the case where the starting time is at $t = \eta_1 > 0$, a new definition is established as follows.

Definition 2.4 ([31])**.** Let a function $h(t)$ be defined on $[\eta_1, +\infty)$, $\eta_1 > 0$. Then, the modified Laplace transform of $h(t)$, $\mathbb{L}_m h(t)$, is defined as

$$\mathbb{L}_m(h(t)) = \overline{h}(s) = \int_{\eta_1}^{+\infty} e^{-s\log\frac{t}{\eta_1}} h(t) \frac{dt}{t}. \tag{2.4}$$

Definition 2.5 ([31])**.** As stated in Definition 2.4, the inverse modified Laplace transform of $\overline{h}(s)$, $\mathbb{L}_m^{-1}(\overline{h}(s))$, is provided by

$$h(t) = \mathbb{L}_m^{-1}(\overline{h}(s)) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha-+\infty} e^{s\log\frac{t}{\eta_1}} \overline{h}(s) ds, \quad \alpha > 0,\ i^2 = -1.$$

Theorem 2.5 ([31])**.** *If* $\mathbb{L}_m(h(t)) = \overline{h}(s)$, *then*

$$\mathbb{L}_m(\delta^n h(t)) = s^n \overline{h}(s) - \sum_{m=0}^{n-1} s^{n-m-1} \delta^m h(c), \quad t > \alpha > 0,\ n \in \mathbb{Z}^+. \tag{2.5}$$

Definition 2.6 ([31])**.** If $m_1(t)$ and $m_2(t)$ are given functions defined on $[T_1, +\infty)$, $T_1 > 0$, the convolution of $m_1(t)$ and $m_2(t)$, i.e.,

$$m_1(t) * m_2(t) = \int_{T_1}^{t} m_1(T_1 \frac{t}{s}) m_2(s) \frac{ds}{s}$$

Theorem 2.6 (Convolution theorem [31])**.** *If* $\mathbb{L}_m(m_1(\tau)) = M_1(s)$ *and* $\mathbb{L}_m(m_2(\tau)) = M_2(s)$, *then*

$$\mathbb{L}_m(m_1(\tau) * m_2(\tau)) = \mathbb{L}_m(m_1(\tau))\mathbb{L}_m(m_1(\tau)) = M_1(s)M_2(s), \tag{2.6}$$

or equivalently

$$\mathbb{L}_m^{-1}(M_1(s)M_2(s)) = m_1(\tau) * m_2(\tau). \tag{2.7}$$

Lemma 2.3 ([31])**.** *Let* $k - 1 < \beta < k \in \mathbb{Z}$. *Then,*

$$\mathbb{L}_m \left({}^{CH}D_{\alpha^+}^{\beta} g(t) \right) = s^\beta \mathbb{L}_m(h(t)) - \sum_{m=0}^{k-1} s^{\beta-m-1} \delta^m h(\alpha). \tag{2.8}$$

Definition 2.7 ([21])**.** The one-parameter Mittag-Leffler function is defined as

$$\mathbb{E}_\beta(x) = \sum_{m=0}^{+\infty} \frac{x^m}{\Gamma(\beta m + 1)}, \quad \beta > 0,\ x \in \mathbb{C}. \tag{2.9}$$

Definition 2.8 ([21])**.** The two-parameters Mittag-Leffler function is defined as

$$\mathbb{E}_{\beta,\alpha}(x) = \sum_{m=0}^{+\infty} \frac{x^m}{\Gamma(\beta m+\alpha)}, \quad \beta>0,\ \alpha>0. \tag{2.10}$$

The Mittag-Leffler function's modified Laplace transform is now provided. Using the formula that follows [35]

$$\int_0^{+\infty} e^{-st} t^{\beta k+\alpha-1} \mathbb{E}^{(k)}_{\beta,\alpha}\left(\pm\lambda_1 t^\beta\right) dt = \frac{k! s^{\beta-\alpha}}{(s^\beta \mp \lambda_1)^{k+1}}, \quad \Re(s) > |\lambda_1|^{\frac{1}{\beta}}.$$

When the variable $t=\log\frac{r}{c}$ is changed, it yields

$$\int_c^{+\infty} e^{-s\ln\frac{r}{c}} \left(\ln\frac{r}{c}\right)^{\beta k+\alpha-1} \mathbb{E}^{(k)}_{\beta,\alpha}\left(\pm\lambda_1\left(\ln\frac{r}{c}\right)^\beta\right)\frac{dr}{r} = \frac{k! s^{\beta-\alpha}}{(s^\beta \mp \lambda_1)^{k+1}}, \quad \Re(s) > |\lambda_1|^{\frac{1}{\beta}}, \tag{2.11}$$

or, equivalently,

$$L^{-1}\Bigg(\frac{k! s^{\beta-\alpha}}{(s^\beta \mp \lambda_1)^{k+1}}\Bigg) = \left(\ln\frac{r}{c}\right)^{\beta k+\alpha-1} \mathbb{E}^{(k)}_{\beta,\alpha}\left(\pm\lambda_1\left(\ln\frac{r}{c}\right)^\beta\right). \tag{2.12}$$

The subsequent lemma plays a crucial role in guaranteeing the MLHU of the main problem.

Lemma 2.4 ([48])**.** *If $x,y>0$ and $t>L_1>0$, then*

$$\int_{L_1}^t \left(\ln\frac{t}{s}\right)^{x-1}\left(\ln\frac{s}{L_1}\right)^{y-1}\frac{ds}{s} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\left(\ln\frac{t}{L_1}\right)^{x+y-1}. \tag{2.13}$$

Lemma 2.5. *Suppose $f\in C[L_1,+\infty)$. Then, the solution of the problem given by* (1.1)–(1.2) *is*

$$\begin{aligned} g(t) = {} & \left(\ln\frac{t}{L_1}\right)\mathbb{E}_{\lambda,2}\left(\beta\left(\ln\frac{t}{L_1}\right)^\lambda\right)\sigma_1 + \left(\ln\frac{t}{L_1}\right)^2\mathbb{E}_{\lambda,3}\left(\beta\left(\ln\frac{t}{L_1}\right)^\lambda\right)\sigma_2 \\ & + \int_{L_1}^t \left(\ln\frac{t}{s}\right)^{\lambda-1}\mathbb{E}_{\lambda,\lambda}\left(\beta\left(\ln\frac{t}{s}\right)^\lambda\right) f\left(s, g(s), I^\delta g(s)\frac{ds}{s}\right). \end{aligned} \tag{2.14}$$

Proof. Let $F(s)$ and $G(s)$ denote the modified LT of $f(t)$ and $g(t)$, respectively. Next, apply the modified Laplace transform and its related properties to the problem provided by (1.1)–(1.2), we have

$$\mathbb{L}_m\left({}^{CH}D^\lambda_{L_1^+}g(t) - \beta g(t) = f(t, g(t), I^\delta g(t))\right),$$

which results in

$$s^\lambda G(s) - s^{\lambda-2}\sigma_1 - s^{\lambda-3}\sigma_2 - \beta G(S) = F(s, G(s), I^\delta G(s)). \tag{2.15}$$

Solving for $G(s)$, we get

$$G(s) = \frac{s^{\lambda-2}\sigma_1 + s^{\lambda-3}\sigma_2 + F(s, G(s), I^\delta G(s))}{s^\lambda - \beta}. \tag{2.16}$$

Applying the inverse modified Laplace transform on the aforementioned relation (2.16) and using (2.7) and (2.12), we obtain the desired result (2.14). □

To establish the conditions required for the EU and stability of solutions to the problem (1.1)–(1.2), the following assumptions should be taken into account.

(Q1) $f : [L_1, +\infty] \times \mathbb{R}^2 \to \mathbb{R}$ is a given continuous function.

(Q2) For every $t \in K$ and $f_i, g_i \in \mathbb{R}$ $(i = 1, 2)$, there is a constant $k_1 > 0$ such that

$$|f(t, g_1, f_1) - f(t, g_2, f_2)| \leq k_1(\|g_1 - g_2\| + \|f_1 - f_2\|).$$

(Q3) Suppose that $\left\|\mathbb{E}_{\lambda,i}(\beta(\ln \frac{t}{s})^{\lambda}\right\| \leq M_i > 0$, for $i = 2, 3$ and $\left\|\mathbb{E}_{\lambda,\lambda}(\beta(\ln \frac{t}{s})^{\beta_i}\right\| \leq M_\lambda > 0$ for $s \in [L_1, t]$ and $t \in K$.

3. Main Results

We are now prepared to discuss our key findings.

We introduce the space $\mathbb{Q} = C(K, \mathbb{R})$ of all continuous functions $g : [L_1, L_2] \to \mathbb{R}$ with the norm $\|g\| = \sup\{|g(t)| : t \in K\}$. Observe that $\mathbb{Q}$ is a Banach space with this norm. By Lemma 2.5, we define an operator

$$J := \mathbb{Q} \to \mathbb{Q},$$

where

$$\begin{aligned}(Jg)(t) = &\left(\ln \frac{t}{L_1}\right) \mathbb{E}_{\lambda,2}\left(\beta\left(\ln \frac{t}{L_1}\right)^{\lambda}\right)\sigma_1 + \left(\ln \frac{t}{L_1}\right)^2 \mathbb{E}_{\lambda,3}\left(\beta\left(\ln \frac{t}{L_1}\right)^{\lambda}\right)\sigma_2 \\ &+ \int_{L_1}^{t} \left(\ln \frac{t}{s}\right)^{\lambda-1} \mathbb{E}_{\lambda,\lambda}\left(\beta\left(\ln \frac{t}{s}\right)^{\lambda}\right) f\left(s, g(s), I^{\delta}g(s)\right)\frac{ds}{s}.\end{aligned} \tag{3.1}$$

It is known that if operator J has a fixed point, then (3.1) has a solution.

For convenience, we set

$$\Delta = \frac{k_1 M_\lambda \left(\ln \frac{L_2}{L_1}\right)^{\lambda}}{\lambda}\left(1 + \frac{\left(\ln \frac{L_2}{L_1}\right)^{\delta}}{\Gamma(\delta + 1)}\right),$$

$$\Delta_1 = \frac{k_1 M_\lambda \left(\ln \frac{L_2}{L_1}\right)^{\lambda}}{\lambda},$$

$$\Delta_2 = M_2\left(\ln \frac{L_2}{L_1}\right)|\sigma_1| + M_3\left(\ln \frac{L_2}{L_1}\right)^2 |\sigma_2| + \frac{M_\lambda \alpha_1 \left(\ln \frac{L_2}{L_1}\right)^{\lambda}}{\lambda}.$$

We provide the following uniqueness result via the Banach contraction principle (Theorem 2.2).

Theorem 3.1. *Suppose* (Q1)-(Q3) *hold. Then, the IVP* (1.1)–(1.2) *has a unique solution on* $[T_1, T_2]$ *if*

$$\Delta < 1. \tag{3.2}$$

Proof. Set $\mathbb{D}_{\rho^*} = \{g \in \mathbb{Q} : \|g\| \leq \rho^*\}$. Define $\sup_{t\in[L_1,L_2]}\{|f(t,0,0)|\} = \alpha_i < +\infty$ such that $\rho^* \geq \frac{\Delta_2}{1-\Delta_1}$.

Now, we show that $J\mathbb{D}_{\rho^*} \subset \mathbb{D}_{\rho^*}$. For any $g \in \mathbb{D}_{\rho^*}$, we have

$$\begin{aligned}\|(Jg)(t)\| \leq & M_2\left(\ln\frac{L_2}{L_1}\right)|\sigma_1| + M_3\left(\ln\frac{L_2}{L_1}\right)^2|\sigma_2| + \frac{k_1M_\lambda\left(\ln\frac{L_2}{L_1}\right)^\lambda}{\lambda}\|g\| \\ & + \frac{\alpha_1M_\lambda\left(\ln\frac{L_2}{L_1}\right)^\lambda}{\lambda} \\ \leq & M_2\left(\ln\frac{L_2}{L_1}\right)|\sigma_1| + M_3\left(\ln\frac{L_2}{L_1}\right)^2|\sigma_2| + \frac{k_1M_\lambda\left(\ln\frac{L_2}{L_1}\right)^\lambda}{\lambda}\rho^* \\ & + \frac{\alpha_1M_\lambda\left(\ln\frac{L_2}{L_1}\right)^\lambda}{\lambda} \\ \leq & \rho^*,\end{aligned}$$

which shows that $J\mathbb{D}_{\rho^*} \subset \mathbb{D}_{\rho^*}$.

Next, for any $g_1(t), g_2(t) \in \mathbb{D}_{\rho^*}$ and for all t $\in [L_1, L_2]$, we obtain

$$\begin{aligned}\|J(g_1)(t) - J(g_2)(t)\| \leq & \int_{L_1}^t \left(\ln\frac{t}{s}\right)^{\lambda-1}\left\|\mathbb{E}_{\lambda,\lambda}\left(\beta\left(\ln\frac{t}{s}\right)^\lambda\right)\right\| \\ & \times\left|f\left(s, g_1(s), I^\delta g_1(s)\right) - f\left(s, g_2(s), I^\delta g_2(s)\right)\right|\frac{ds}{s} \\ \leq & \int_{L_1}^t \left(\ln\frac{t}{s}\right)^{\lambda-1} M_\lambda k_1\left(\|g_1-g_2\| + \frac{\left(\ln\frac{L_2}{L_1}\right)^\delta}{\Gamma(\delta+1)}\|g_1-g_2\|\right)\frac{ds}{s} \\ \leq & \frac{k_1M_\lambda\left(\ln\frac{L_2}{L_1}\right)^\lambda}{\lambda}\left(1 + \frac{\left(\ln\frac{L_2}{L_1}\right)^\delta}{\Gamma(\delta+1)}\right)\|g_1-g_2\|,\end{aligned} \tag{3.3}$$

which is a contraction from the assumption (3.2). As a result, by Banach contraction principle (Theorem 2.2), J has a unique fixed point that is the unique solution to the IVP (1.1)–(1.2) on $[L_1, L_2]$. □

Next, using nonlinear contractions, we provide a second existence result.

Theorem 3.2. *Let $f : [L_1, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ be a given continuous functions fulfilling the following assumptions.*

(Q4) $|f(t, v_1, w_1) - f(t, v_2, w_2)| \leq \bar{L}r_1(t)\left(1 - e^{-\|v_1-v_2\|}\right) + \bar{L}r_2(t)\left(1 - e^{-\|w_1-w_2\|}\right),$

for every $v_i, w_i \in \mathbb{R}$, for $i = 1, 2$ and for all $t \in [L_1, L_2]$, where $r_1, r_2 : [L_1, L_2] \to \mathbb{R}^+$ are continuous functions and $\bar{L} > 0$ is defined by

$$\bar{L} = \frac{\lambda}{M_\lambda\left(\ln\frac{L_2}{L_1}\right)^\lambda(\|r_1\| + \|r_2\|)}, \tag{3.4}$$

where $\|r_i\| = \sup_{t\in[L_1,L_2]}\{|r_i(t)| : for\ i = 1, 2\}$.

Moreover,

$$0 < \frac{\left(\ln \frac{L_2}{L_1}\right)^{\delta}}{\Gamma(1+\delta)} \leq 1.$$

Then, the IVP (1.1)–(1.2) *has a unique solution on* $[L_1, L_2]$.

Proof. Consider the continuous and increasing function $z : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$z(x) = 1 - e^{-x}, \quad \text{for all } x \geq 0.$$

Clearly, $z(0) = 0$ and $z(x) < x$ for all $x > 0$.

For any $v, w \in \mathbb{Q}$ and $t \in [L_1, L_2]$, it follows from (Q3), (Q4), (3.4)–(3.2) that

$$\begin{aligned}
&\|J(v)(t) - Jw(t)\| \\
\leq &\int_{L_1}^{t}\left(\ln\frac{t}{s}\right)^{\lambda-1}\left\|\mathbb{E}_{\lambda,\lambda}\left(\beta\left(\ln\frac{t}{s}\right)^{\lambda}\right)\right\|\frac{\bar{L}r_2(s)\left(1 - e^{-\|v-w\|}\right) + \bar{L}r_1(s)\left(1 - e^{-\|I^{\delta}v - I^{\delta}w\|}\right)}{s}ds \\
\leq &M_{\lambda}\bar{L}\int_{L_1}^{t}\left(\ln\frac{t}{s}\right)^{\lambda-1}\frac{r_1(s)\left(1 - e^{-\|v-w\|}\right) + r_2(s)\left(1 - e^{-\frac{\left(\ln\frac{L_2}{L_1}\right)^{\delta}}{\Gamma(\delta+1)}\|v-w\|}\right)}{s}ds \\
\leq &M_{\lambda}\bar{L}\int_{L_1}^{t}\left(\ln\frac{t}{s}\right)^{\lambda-1}\frac{r_1(s)\left(1 - e^{-\|v-w\|}\right) + r_2(s)\left(1 - e^{-\|v-w\|}\right)}{s}ds \\
\leq &M_{\lambda}\bar{L}z(\|v-w\|\int_{L_1}^{t}\left(\ln\frac{t}{s}\right)^{\lambda-1}\frac{r_1(s) + \ r_2(s)}{s}ds \\
\leq &\bar{L}z(\|v-w\|)M_{\lambda}\left(\|r_1\| + \|r_2\|\right)\frac{\left(\ln\frac{L_2}{L_1}\right)^{\lambda}}{\lambda} \\
= &z(\|v-w\|),
\end{aligned}$$

which shows that the contraction J is nonlinear. Therefore, J has a fixed point by Theorem 2.1, which is the unique solution of the IVP (1.1)–(1.2). ☐

We now provide an existence result using Schaefer's fixed point theorem. For the sake of simplicity, we use

$$\Delta_J = M_2\left(\ln\frac{L_2}{L_1}\right)|\sigma_1| + M_3\left(\ln\frac{L_2}{L_1}\right)^2|\sigma_2| + \frac{\psi_0 M_{\lambda}\left(\ln\frac{L_2}{L_1}\right)^{\lambda}}{\lambda} \tag{3.5}$$

and

$$\Delta_B = \psi_1 M_{\lambda}\frac{\left(\ln\frac{L_2}{L_1}\right)^{\delta}}{\Gamma(\lambda+1)}. \tag{3.6}$$

Theorem 3.3. *Assume that* (Q1) *and* (Q3) *hold true. Furthermore, constants* $\psi_0 > 0$ *and* $\psi_1 \geq 0$ *exist such that*

(Q5) $|f(t,g_1,g_2)| \leq \psi_0 + \psi_1(\|g_1\| + \|g_2\|$, *for all* $t \in [L_1, l_2]$, $g_1, g_2 \in \mathbb{R}$.

Then, at least one solution for the IVP (1.1)–(1.1) *exists on* $[L_1, L_2]$.

Proof. First we prove that the operator $J : \mathbb{Q} \to \mathbb{Q}$ is completely continous.

Step 1. Clearly, J is continuous as the function f is continuous.

Step 2. To prove that J is uniformly bounded, let $\Sigma \subset \mathbb{Q}$ be a bounded set. Thus, there exists a constant $e_1 > 0$ such that $|f(t, g(t), I^\delta g(t))| \leq e_1$ for all $g \in \Sigma$. Then, for all $g \in \Sigma$, we have

$$\|(Jg)(t)\| \leq M_2\left(\ln\frac{L_2}{L_1}\right)|\sigma_1| + M_3\left(\ln\frac{T_2}{T_1}\right)^2|\sigma_2| + \frac{e_1 M_\lambda\left(\ln\frac{T_2}{T_1}\right)^\lambda}{\lambda} = k > 0. \tag{3.7}$$

It follows from (3.7) that J is uniformly bounded.

Step 3. To show that J is equiccontinuous, we define

$$f^* = \sup\{|f(t,g_1,g_2)| : (t,g_1,g_2) \in ([L_1,L_2],\mathbb{Q},\mathbb{Q})\}. \tag{3.8}$$

For any $t_1, t_2 \in K$ with $t_1 < t_2$, we obtain

(3.9)

$$\begin{aligned}
&|(Jg)(t_2) - (Jg)(t_2)| \\
\leq &\left|\left(\ln\frac{t_2}{T_1}\right)\mathbb{E}_{\lambda,2}(\beta(\ln\frac{t_1}{L_1})^\lambda)\sigma_1 - \left(\ln\frac{t_1}{L_1}\right)\mathbb{E}_{\lambda,2}\left(\beta\left(\ln\frac{t_1}{L_1}\right)^\lambda\right)\sigma_2\right| \\
&+\left|\left(\ln\frac{t_2}{T_1}\right)\mathbb{E}_{\lambda,3}\left(\beta\left(\ln\frac{t_2}{L_1}\right)^\lambda\right)\sigma_2 - \left(\ln\frac{t_1}{L_1}\right)\mathbb{E}_{\lambda,3}\left(\beta\left(\ln\frac{t_1}{L_1}\right)^\lambda\right)\sigma_2\right| \\
&+ f^*\int_{L_1}^{t_1}\left|\left(\ln\frac{t_2}{s}\right)^{\lambda-1}\mathbb{E}_{\lambda,\lambda}\left(\beta\left(\ln\frac{t_2}{s}\right)^\lambda\right) - \left(\left(\ln\frac{t_1}{s}\right)^{\lambda-1}\mathbb{E}_{\lambda,\lambda}\right)\beta\left(\ln\frac{t_1}{s}\right)^\lambda\right)\right)\right|\frac{ds}{s} \\
&+ f^*\int_{t_1}^{t_2}\left|\left(\ln\frac{t_2}{s}\right)^{\lambda-1}\mathbb{E}_{\lambda,\lambda}\left(\beta\left(\ln\frac{t_2}{s}\right)^\lambda\right)\right|\frac{ds}{s}.
\end{aligned}$$

For (3.9), the R.H.S. is independent of g and approaches to zero as t_1 tends to t_2. As a result, J is equicontinous. Consequently, the operator J is a completely continuous operator, as follows from Steps 1. to 3. and the Arzelá-Ascoli theorem.

Step 4. Lastly, the boundlessness of the set $\Omega = \{g \in \mathbb{Q} : g = \zeta Jg, 0 \leq \zeta \leq 1\}$ is going to be verified. Let $g \in \Omega$. Then, for all t $\in [L_1, L_2]$, we have $g(t) = \zeta Jg(t)$. Using the hypotheses (Q1), (Q3) and (Q5), we obtain

$$\begin{aligned}
\|g\| \leq\ & M_2\left(\ln\frac{L_2}{L_1}\right)|\sigma_1| + M_3\left(\ln\frac{L_2}{L_1}\right)^2|\sigma_2| + \frac{\psi_0 M_\lambda\left(\ln\frac{L_2}{L_1}\right)^\lambda}{\lambda} \\
&+ \psi_1 M_\lambda\frac{\left(\ln\frac{L_2}{L_1}\right)^\delta}{\Gamma(\delta+1)}\int_{L_1}^{t}\left(\ln\frac{t}{s}\right)^{\lambda-1}\|g\|\frac{ds}{s}.
\end{aligned} \tag{3.10}$$

From (3.5), (3.6) and the Gronwall inequality theorem 2.4, it may be inferred that

$$\begin{aligned}\|g\| &\leq \Delta_J + \sum_{n=1}^{+\infty} \frac{\left(\Delta_B \Gamma(\lambda)\right)^n}{\Gamma(n\lambda)} \int_{L_1}^{t} \left(\ln \frac{t}{s}\right)^{n\lambda-1} \Delta_J \frac{ds}{s} \\ &\leq \Delta_J \left(1 + \sum_{n=1}^{+\infty} \frac{\left(\Delta_B \Gamma(\lambda)\right)^n}{\Gamma(n\lambda+1)} \left(\ln \frac{L_2}{L_1}\right)^{n\lambda}\right) \\ &\leq \Delta_J \mathbb{E}_q \left(\Delta_B \Gamma(\lambda) \left(\ln \frac{t}{L_1}\right)^{\lambda}\right),\end{aligned}$$

which shows that Ω is bounded. Thanks to Steps 1–4. and Lemma 2.2, we can deduce that the set Ω contains at least one fixed point, which is a solution of the system (1.1)–(1.2). □

4. Mittag-Leffler-Hyler Ulam Stability Results for the Problem

Motivated by the references [37, 46], we provide the MLHU stability of solutions for the problem (1.1)–(1.2). Consider the following equation and inequality

$$ {}^{CH}D^{\lambda}_{L_1^+} g(t) - \beta g(t) = f(t, g(t), I^{\delta} g(t)), \tag{4.1}$$

$$ \left\| {}^{CH}D^{\lambda}_{L_1^+} g(t) - \beta g(t) - f(t, g(t), I^{\delta} g(t)) \right\| \leq \epsilon \mathbb{E}_{\lambda} \left(\left(\ln \frac{t}{L_1}\right)^{\lambda}\right), \tag{4.2}$$

where $\mathbb{E}_{\lambda}$ is the Mitteg-Leffler function.

Definition 4.1 ([32])**.** Equation (4.1) is said to be MLHU stable with respect to $\mathbb{E}_{\lambda}((\ln \frac{t}{L_1})^{\lambda})$, if a positive real number $k_{\mathbb{E}_{\lambda}}$ exists such that for every $\epsilon > 0$ and for every solution $h \in C(K, Q)$ of (4.2), there exists a solution g of equation (4.1) with

$$|h(t) - g(t)| \leq k_{\mathbb{E}_{\lambda}} \epsilon \mathbb{E}_{\lambda} \left(\left(\ln \frac{t}{L_1}\right)^{\lambda}\right).$$

Remark 4.1. A function $g^* \in C(K, \mathbb{R})$ is a solution of (3.8) if and only if there exists a functiom $m(t) \in C(K, \mathbb{R})$ (which depends on g^*) such that

i. $|m(t)| \leq \epsilon \mathbb{E}_{\lambda}((\ln \frac{t}{L_1})^{\lambda})$;
ii. ${}^{CH}D^{\lambda}_{L_1^+} g^*(t) - \beta g^*(t) = f(t, v, I^{\delta} g^*(t)) + m(t)$.

Remark 4.2. Let $g^* \in C(K, \mathbb{R})$ be a solution of (3.8). Then, g^* is a solution of the integral inequality

$$\begin{aligned}&\Bigg| g^* - \left(\ln \frac{t}{L_1}\right) \mathbb{E}_{\lambda,2} \left(\beta \left(\ln \frac{t}{L_1}\right)^{\lambda}\right) \sigma_1 - \left(\ln \frac{t}{L_1}\right)^2 \mathbb{E}_{\lambda,3} \left(\beta \left(\ln \frac{t}{L_1}\right)^{\lambda}\right) \sigma_2 \\ &- \int_{L_1}^{t} \left(\ln \frac{t}{s}\right)^{\lambda-1} \mathbb{E}_{\lambda,\lambda} \left(\beta \left(\ln \frac{t}{s}\right)^{\lambda}\right) f(s, g(s), I^{\delta} g(s)) \frac{ds}{s} \Bigg|\end{aligned}$$

$$
\begin{aligned}
&\le M_\lambda \epsilon \int_{L_1}^{t} \left(\ln \frac{t}{s}\right)^{\lambda-1} \mathbb{E}_\lambda\left(\left(\ln \frac{s}{L_1}\right)^{\lambda}\right)\frac{ds}{s}\\
&\le M_\lambda \epsilon \sum_{i=1}^{+\infty} \frac{1}{\Gamma(i\lambda+1)} \int_{L_1}^{t} \left(\ln \frac{t}{s}\right)^{\lambda-1} \left(\ln \frac{s}{L_1}\right)^{\lambda i}\frac{ds}{s}\\
&\quad \text{(by Lemma 2.4, taking } x=\lambda,\ y=\lambda i+1)\\
&\le M_\lambda \epsilon \sum_{i=1}^{+\infty} \frac{1}{\Gamma(i\lambda+1)} \left(\ln \frac{t}{L_1}\right)^{(i+1)\lambda} \frac{\Gamma(\lambda)\Gamma(i\lambda+1)}{\Gamma((i+1)\lambda+1)}\\
&\le \Gamma(\lambda) M_\lambda \epsilon \sum_{k=1}^{+\infty} \frac{(\ln \frac{t}{L_1})^{k\lambda}}{\Gamma(k\lambda+1)} = \Gamma(\lambda) M_\lambda \mathbb{E}_\lambda\left(\left(\ln \frac{t}{L_1}\right)^{\lambda}\right)\epsilon.
\end{aligned}
$$

Theorem 4.1. *Suppose that* (Q1)-(Q3) *hold true. Then,* (4.1) *is MLHU stable with respect to* $\mathbb{E}_\lambda(\ln \frac{t}{L_1})^\lambda$.

Proof. Let $g^* \in C(K,\mathbb{R})$be a solution of (3.8) and g be a unique solution of (1.1)-(1.2). For $t \in [L_1, L_{12}]$, it follows from Remark 4.2, (Q2) and Gronwall inequality 2.4 that

$$
\begin{aligned}
|g^* - g| &\le \Gamma(\lambda) M_\lambda \mathbb{E}_\lambda\left(\left(\ln \frac{t}{L_1}\right)^{\lambda}\right)\epsilon\\
&\quad + \left| \int_{L_1}^{t} \left(\ln \frac{t}{s}\right)^{\lambda-1} \mathbb{E}_{\lambda,\lambda}\left(\lambda_1 \left(\ln \frac{t}{s}\right)^{\lambda}\right) f(s, g^*(s), I^\delta g^*(s))\frac{ds}{s}\right.\\
&\quad \left. - \int_{L_1}^{t} \left(\ln \frac{t}{s}\right)^{\lambda-1} \mathbb{E}_{\lambda,\lambda}\left(\lambda_1 \left(\ln \frac{t}{s}\right)^{\lambda}\right) f(s, g(s), I^\delta g(s))\frac{ds}{s}\right|\\
&\le \Gamma(\lambda) M_\lambda \mathbb{E}_\lambda\left(\left(\ln \frac{t}{L_1}\right)^{\lambda}\right)\epsilon\\
&\quad + k_1 \left(1 + \frac{\left(\ln \frac{L_2}{L_1}\right)^{\delta}}{\Gamma(\delta+1)}\right) M_\lambda \int_{L_1}^{t} \left(\ln \frac{t}{s}\right)^{\lambda-1} |g^* - g| \frac{ds}{s}\\
&\le \Gamma(\lambda) M_\lambda \mathbb{E}_\lambda\left(\left(\ln \frac{t}{L_1}\right)^{\lambda}\right)\epsilon\\
&\quad + \frac{k_1 M_\lambda \left(\ln \frac{L_2}{L_1}\right)^{\lambda}}{\lambda} \left(1 + \frac{\left(\ln \frac{L_2}{L_1}\right)^{\delta}}{\Gamma(\delta+1)}\right) |g^* - g|\\
&\le k_{\mathbb{E}_\lambda} \epsilon \mathbb{E}_\lambda\left(\left(\ln \frac{t}{L_1}\right)^{\lambda}\right),
\end{aligned}
$$

where

$$k_{\mathbb{E}_\lambda} = \frac{\Gamma(\lambda)M_\lambda}{1 - \frac{k_1 M_\lambda \left(\ln \frac{L_2}{L_1}\right)^\lambda}{\lambda}\left(1 + \frac{\left(\ln \frac{L_2}{L_1}\right)^\delta}{\Gamma(\delta+1)}\right)}, \quad \frac{k_1 M_\lambda \left(\ln \frac{L_2}{L_1}\right)^\lambda}{\lambda}\left(1 + \frac{\left(\ln \frac{L_2}{L_1}\right)^\delta}{\Gamma(\delta+1)}\right) < 1.$$

Thus, (4.1) is ML-UH stable by (3.2) and Definition 4.1. □

5. Examples

Here, we provide the following example to verify our main findings.

Example 5.1. Let us investigate the following nonlinear Caputo-Hadamard FDE

$$\begin{cases} {}^{CH}D^{2.7}g(\tau) = f(\tau, g(\tau), I^\delta g(\tau)), & \text{if } \tau \in K = [1, e], \\ g(1) = 0, \\ \xi g(1) = 1, \\ \xi^2 g(1) = 1, \end{cases} \tag{5.1}$$

where $\lambda = 2.7$, $\sigma_1 = \sigma_2 = 1$, $\delta = \frac{1}{4}$, $\beta = 0$, $L_1 = 1$, $L_2 = e$ and

$$f(\tau, g(\tau), I^\delta g(\tau)) = \frac{1}{\sqrt{\tau^2 + 2\tau + 6}}\left(\frac{\ln \tau}{2\sqrt{\tau^2 + 15}} \sin g(\tau) + \frac{1}{e^{\tau-1} + 2} g(\tau)\right).$$

Obviously,

$$\left|f(\tau, g_1(\tau), I^\delta g_1(\tau)) - f(\tau, g_2(\tau), I^\delta g_2(\tau))\right| \le \frac{11}{72}\left(\|g_1 - g_2\| + \left\|I^\delta g_1 - I^\delta g_2\right\|\right)$$

and hence $k_1 = \frac{11}{72}$. Moreover, $M_\lambda = \mathbb{E}_{2.7,2.7}(0) = \frac{1}{\Gamma(2.7)} = 0.64733$. Using the provided data, we obtain $\Delta = 0.07706 < 1$. Consequently, each condition of Theorem 3.1 is met. Therefore, there is only one solution to the fractional initial value problem (5.1) on $[1, e]$.

Example 5.2. Examine the problem (5.1) with $\lambda = 2.1,\ \delta = 2.3$,

$$f(\tau, g(\tau), I^\delta g(\tau)) = r_1(\tau)\left(1 - e^{-|g|}\right) + r_2(\tau) I^{2.3} e^{-|g|},$$

where $r_1(\tau) = \frac{\sin \tau}{4(\tau+7)}$ and $r_2(\tau) = \frac{\tau e^{-\tau}}{\sqrt{\tau^3+3}}$ with $\|r_1\| = \frac{1}{32}$ and $\|r_2\| = \frac{1}{2}$. Note that $\mathbb{E}_{2.1,2.1}(0) = \frac{1}{\Gamma(2.1)} = 1.04648$. Simple computations give that

$$\frac{\left(\ln \frac{L_2}{L_1}\right)^\delta}{\Gamma(\delta + 1)} = 0.3727 < 1 \quad \text{and} \quad \bar{L} = 3.5975.$$

Obviously,

$$|f(\tau, g_1, I^\delta g_1) - f(\tau, g_2, I^\delta g_2)| \le r_1(t)\left(e^{-|g_2|} - e^{-|g_1|}\right) + r_2(t)\left(I^\delta e^{-|g_1|} - I^\delta e^{-|g_2|}\right)$$

$$\le r_1(\tau)\left(1-e^{|g_2|-|g_1|}\right)+r_2(t)\frac{\left(\ln\frac{L_2}{L_1}\right)^{\delta}}{\Gamma(\delta+1)}\left(e^{-|g_1|}-e^{-|g_2|}\right)$$
$$\le r_1(\tau)\left(1-e^{-|g_1-g_2|}\right)+r_2(\tau)\left(e^{-|g_1|}-e^{-|g_2|}\right)$$
$$\le r_1(\tau)\left(1-e^{-|g_1-g_2|}\right)+r_2(\tau)\left(1-e^{-|g_1-g_2|}\right)$$
$$\le \bar{L}r_1(\tau)\left(1-e^{-|g_1-g_2|}\right)+\bar{L}r_2(\tau)\left(1-e^{-|g_1-g_2|}\right), \tag{5.2}$$

which satisfies the assumption (Q4). Therefore, assumptions of Theorem 3.2 are all met. Therefore, we can conclude that Problem (5.1) has a unique solution on $[1, e]$, by Theorem 3.2.

Example 5.3. Consider Problem (5.1), where

$$f(\tau, g(\tau), I^{\delta}g(\tau))=\frac{1}{\sqrt{\tau^2+\tau+7}}g(\tau)\sin\tau+\frac{\tau}{e^{\tau^2+2}}I^{\frac{3}{4}g(\tau)}+\frac{\tau e^{-\tau+\cos\tau}}{\tau^2+e^{-\tau}+e^{-e}}.$$

It is easy to compute that

$$\left|f(\tau, g(\tau), I^{\delta}g(\tau))\right|\le\frac{1}{3}(g(\tau)+I^{\delta}g(\tau))+e,$$

with $\psi_0 = e$ and $\psi_1 = \frac{1}{3}$. Therefore, conditions of Theorem 3.3 are all met. Consequently, we conclude from Theorem 3.3 that there is at least one solution for Problem (5.1).

6. Conclusions

In this article, we provided several conditions that establish the EU of a solution to the problem (1.1)–(1.2) by using the Banach contraction principle and BW and Schafer fixed point theorems. We then established the sufficient conditions for the MLHU stability results for the problem given by (1.1)–(1.2) using a Gronwall inequality theorem 2.4 and Lemma 2.4. Appropriate examples are also used to illustrate the existence and uniqueness of solutions.

References

[1] M. Ahmad, J. Jiang, A. Zada, Z. Ali, Z. Fu and J. Xu, *Hyers-Ulam-Mittag-Leffler Stability for a system of fractional neutral differential equations*, Discrete Dyn. Nat. Soc. **2020**(1) (2020), Article ID 2786041. https://doi.org/10.1155/2020/2786041

[2] A. Al-Khedhairi, A. E. Matouk and I. Khan, *Chaotic dynamics and chaos control for the fractional-order geomagnetic field model*, Chaos Solit. Fractals **128** (2019), 390–401. https://doi.org/10.1016/j.chaos.2019.07.019

[3] M. A. Almalahi, S. K. Panchal, F. Jarad and T. Abdeljawad, *Ulam-Hyers-Mittag-Leffler stability for tripled system of weighted fractional operators with TIME delay*, Adv. Differ. Equ. **2021**(1) (2021), Artcle ID 299. https://doi.org/10.1186/s13662-021-03455-0

[4] R. Almeida, A. B. Malinowska and T. Odzijewicz, *An extension of the fractional Gronwall inequality*, Bound. Value Probl. (2018), 20–28. https://doi.org/10.1007/978-3-030-17344-9_2

[5] A. Aphithana, S. K. Ntouyas, and J. Tariboon, *Existence and uniqueness of symmetric solutions for fractional differential equations with multi-order fractional integral conditions*, Bound. Value Probl. **2015**(1) (2015), Article ID 68. https://doi.org/10.1186/s13661-015-0329-1

[6] Y. Başci, S. Öğrekçi and A. Misir, *On Hyers-Ulam stability for fractional differential equations including the new Caputo-Fabrizio fractional derivative*, Mediterr. J. Math. **16**(5) (2019), Artcle ID 131. https://doi.org/10.1007/s00009-019-1407-x

[7] M. T. Beyene, M. D. Firdi and T. T. Dufera, *Existence and Ulam-Hyers stability results for Caputo-Hadamard fractional differential equations with non-instantaneous impulses*, Bound. Value Probl. **2025**(1) (2025), Article ID 6.

[8] D. W. Boyd and J. S. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20**(2) (1969), 458–464. https://doi.org/10.1186/s13661-024-01958-9

[9] M. Cai, G. Em Karniadakis and C. Li, *Fractional SEIR model and data-driven predictions of COVID-*19 *dynamics of Omicron variant*, Chaos **32**(7) (2022). https://doi.org/10.1063/5.0099450

[10] R. Cai, F. Ge, Y. Chen and C. Kou, *Regional gradient controllability of ultra-slow diffusions involving the Hadamard-Caputo time fractional derivative*, Mathematical Control & Related Fields **10**(1) (2020), 141–156. https://doi.org/10.3934/mcrf.2019033

[11] M. Caputo and C. Cametti, *Diffusion with memory in two cases of biological interest*, J. Theoret. Biol. **254**(3) (2008), 697–703. https://doi.org/10.1016/j.jtbi.2008.06.021

[12] F. Cesarone, M. Caputo and C. Cametti, *Memory formalism in the passive diffusion across a biological membrane*, J. Membr. Sci. **250** (2004), 79–84.

[13] A. Chen and Y. Chen, *Existence of solutions to anti-periodic boundary value problem for nonlinear fractional differential equations with impulses*, Adv. Differ. Equ. **2011**(1) (2011), Article ID 915689. https://doi.org/10.1155/2011/915689

[14] S. A. Ciplea, N. Lungu, D. Marian and T. M. Rassias, *On Hyers-Ulam-Rassias stability of a Volterra-Hammerstein functional integral equation*, in: *Approximation and Computation in Science and Engineering*, 2022, 147–156. https://link.springer.com/chapter/10.1007/978-3-030-84122-5_9

[15] K. Deimling, *Nonlinear Functional Analysis*, Courier Corporation Mineola, NY, 2010.

[16] C. Derbazi and Z. Baitiche, *Uniqueness and Ulam-Hyers-Mittag-Leffler stability results for the delayed fractional multiterm differential equation involving the* Φ*-Caputo fractional derivative*, Rocky Mountain J. Math. **52**(3) (2018), 887–897.

[17] N. Eghbali and V. Kalvandi, *A fixed point approach to the Mittag-Leffler-Hyers-Ulam stability of differential equations* $y(x) = F(x, y(x))$, Appl. Math. E-Notes **18**

(2018), 34–42. https://api.semanticscholar.org/CorpusID:13742495

[18] Y. Y. Gambo, F. Jarad, D. Baleanu and T. Abdeljawad, *On Caputo modification of the Hadamard fractional derivatives*, Adv. Differ. Equ. **2014**(1) (2014), Article ID 10. https://doi.org/10.1186/1687-1847-2014-10

[19] J. R. Graef, C. Tunç, M. Şengun and O. Tunć, *The stability of nonlinear delay integro-differential equations in the sense of Hyers-Ulam*, Nonauton. Dyn. Syst. **10**(1) (2023), Article ID 20220169. https://doi.org/10.1515/msds-2022-0169

[20] J. Hadamard, *Essai sur l'étude des fonctions données par leur développement de Taylor*, Journal de Mathématiques Pures et Appliquées **8** (1892), 101–186. http://www.numdam.org/item?id=JMPA_1892_4_8__101_0

[21] H. J. Haubold, A. M. Mathai and R. K. Saxena, *Mittag-Leffler functions and their applications*, J. Appl. Math. **2011**(1) (2011), Article ID 298628. https://doi.org/10.1155/2011/298628

[22] M. Houas and M. E. Samei, *Existence and Mittag-Leffler-Ulam-stability results for Duffing type problem involving sequential fractional derivatives*, Int. J. Appl. Comput. Math. **8**(4) (2022), Article ID 185. https://doi.org/10.1007/s40819-022-01398-y

[23] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. India Sect. A. **27**(4) (1941), 222–224. https://doi.org/10.1073/pnas.27.4.222

[24] G. Iaffaldano, M. Caputoand and S. Martino, *Experimental and theoretical memory diffusion of water in sand*, Hydrology and Earth System Sciences **10**(1) (2006), 93–100. https://doi.org/10.5194/hess-10-93-2006

[25] R. Ilyas Butt and M. ur Rehman, *Ulam-Hyers-Mittag-Leffler stability of fractional difference equations with delay*, Rocky Mountain J. Math. **51**(3) (2021), 891–901. https://doi.org/10.1216/rmj.2021.51.891

[26] F. Jarad, T. Abdeljawad and D. Baleanu, *Caputo-type modification of the hadamard fractional derivatives*, J. Appl. Anal. Comput. **2012**(1) (2012), Article ID 142. https://doi.org/10.1186/1687-1847-2012-142

[27] J. Jiang and H. Wang, *Existence and uniqueness of solutions for a fractional differential equation with multi-point boundary value problems*, J. Appl. Anal. Comput. **9**(6) (2019), 2156–2168. http://dx.doi.org/10.11948/20180286

[28] H. Khan, C. Tunc, W. Chen and A. Khan, *Existence theorems and Hyers-Ulam stability for a class of hybrid fractional differential equations with p-Laplacian operator*, J. Appl. Anal. Comput. **8**(4) (2018), 1211–1226. https://doi.org/10.11948/2018.1211

[29] K. Kilbas, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, Netherlands, 2006.

[30] S. Kundu and S. N. Bora, *On the Ulam-Hyers-Rassias-Mittag-Leffler stability of the solution to a ψ-Hilfer abstract fractional differential equation*, Internat. J. Theoret. Phys. **64**(7) (2025), Article ID 190. https://doi.org/10.1007/s10773-025-06055-w

[31] C. Li, Z. Li and Z. Wang, *Mathematical analysis and the local discontinuous Galerkin method for Caputo-Hadamard fractional partial differential equation*, J. Sci. Comput. **85**(2) (2020), Article ID 41. https://doi.org/10.1007/s10915-020-01353-3

[32] K. Liu, J. Wang and D. O'Regan, *Ulam-Hyers-Mittag-Leffler stability for ψ-Hilfer fractional-order delay differential equations*, Adv. Differ. Equ. **2019**(50) (2019), Article ID 50. https://doi.org/10.1186/s13662-019-1997-4

[33] A. U. K. Niazi, J. Wei, M. U. Rehman and P. Denghao, *Ulam-Hyers-Mittag-Leffler stability for nonlinear fractional neutral differential equations*, Sb. Math. **209**(9) (2018), 1337–1350. https://doi.org/10.1070/SM8958

[34] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution, and Some of their Applications*, **198**, Elsevier, Amsterdam, Netherlands, 1998.

[35] I. Podlubny, *Fractional differential equations*, Math. Sci. Eng. **198** (1999), 7–35.

[36] M. Rhaima, D. Boucenna, L. Mchiri, M. Benjemaa and A. B, *Ulam-Hyers-Rassias Mittag-Leffler stability of ϖ-fractional partial differential equations*, J. Inequal. Appl. **2024**(1) (2024), Article ID 109. https://doi.org/10.1186/s13660-024-03170-w

[37] I. A. Rus, *Ulam stability of ordinary differential equations*, Studia Universitatis Babes-Bolyai Mathematica **(54)** (2009), 125–133.

[38] A. Salim, C. Derbazi, J. Alzabut and A. Küçükaslan, *Existence and κ-Mittag-Leffler-Ulam-Hyers stability results for implicit coupled (κ, ϑ)-fractional differential systems*, Arab Journal of Basic and Applied Sciences **31**(1) (2024), 225–241. https://doi.org/10.1080/25765299.2024.2334130

[39] R. Shah and N. Irshad, *Ulam-Hyers-Mittag-Leffler stability for a class of nonlinear fractional reaction-diffusion equations with delay*, Internat. J. Theoret. Phys. **64**(1) (2025), Article ID 20. https://doi.org/10.1007/s10773-025-05884-z

[40] V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles*, Springer Science & Business Media, New York, 2011.

[41] M. T. Beyene, M. D. Firdi and T. T. Dufera, *Analysis of Caputo-Hadamard fractional neutral delay differential equations involving Hadamard integral and unbounded delays: Existence and uniqueness*, Res. Math. **11**(1) (2024), Article ID 2321669. https://doi.org/10.1080/27684830.2024.2321669

[42] M. T. Beyene, M. D. Firdi and T. T. Dufera, *Existence and stability of solution for a coupled system of Caputo-Hadamard fractional differential equations*, Fixed Point Theory Algorithms Sci. Eng. **2024**(1) (2024), Article ID 17. https://doi.org/10.1186/s13663-024-00773-2

[43] O. Tunç, *New results on the Ulam-Hyers-Mittag-Leffler stability of Caputo fractional-order delay differential equations*, Mathematics **12**(8) (2024), Article ID 1342. https://doi.org/10.3390/math12091342

[44] O. Tunç, C. Tunç, G. Petruşel and J.-C. Yao, *On the Ulam stabilities of nonlinear integral equations and integro-differential equations*, Math. Methods Appl. Sci.

47(6) (2014), 4014–4028. https://doi.org/10.1002/mma.9800

[45] S. M. Ulam, *Problems in Modern Mathematics*, Courier Corporation, New York, 2004.

[46] J. Wang and Y. Zhang, *Ulam-Hyers-Mittag-Leffler stability of fractional-order delay differential equations*, Optimization **63**(8) (2014), 1181–1190. https://doi.org/10.1080/02331934.2014.906597

[47] J. Wang and Y. Zhou, *Mittag-Leffler-Ulam stabilities of fractional evolution equations*, Appl. Math. Lett. **25**(4) (2012), 723–728. https://doi.org/10.1016/j.aml.2011.10.009

[48] Z. Yang, X. Zheng and H. Wang, *Well-posedness and regularity of Caputo-Hadamard fractional stochastic differential equations*, Zeitschrift für Angewandte Mathematik und Physik **72**(4) (2021), Article ID 141. https://doi.org/10.1007/s00033-021-01566-y

[49] Y. Zhou, L. Shangerganesh, J. Manimaran and A. Debbouche, *A class of time-fractional reaction-diffusion equation with nonlocal boundary condition*, Math. Methods Appl. Sci. **41**(8) (2018), 2987–2999. https://doi.org/10.1002/mma.4796

[1]Department of Mathematics,
Dilla University , Dilla, Ethiopia
Email address: saytigal@gmail.com

[2]Department of Applied Mathematics,
Adama Science & Technology University, Adama, Ethiopia.
Email address: mitbru2007@yahoo.com
Email address: tamirat.temesgen@astu.edu.et