Kragujevac Journal of Mathematics
Volume 50(10) (2026), Pages 1605–1620.

NONLINEAR LEFT BI-SKEW LIE TYPE DERIVATIONS ON $*$-ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a unital $*$-algebra over $\mathbb{C}$ (the field of complex number). For any $\alpha, \beta \in \mathcal{A}$, define $\alpha \circ \beta = \alpha^*\beta - \beta^*\alpha$. In this article, it is shown that a map $\vartheta : \mathcal{A} \to \mathcal{A}$ (need not be linear) satisfies $\vartheta(P_n(\lambda_1, \lambda_2, \ldots, \lambda_n)) = \sum_{i=1}^n P_n(\lambda_1, \ldots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \ldots, \lambda_n)$ for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive $*$-derivation. As applications, we apply our main result to various special classes of unital $*$-algebras, such as prime $*$-algebras, factor von Neumann algebras and von Neumann algebra with no central summands of type I_1.

1. Introduction

Let $\mathcal{A}$ be a $*$-algebra on the complex field $\mathbb{C}$. A map $\vartheta : \mathcal{A} \to \mathcal{A}$ is called an additive derivation if $\vartheta(\alpha + \beta) = \vartheta(\alpha) + \vartheta(\beta)$ and $\vartheta(\alpha\beta) = \vartheta(\alpha)\beta + \alpha\vartheta(\beta)$ hold for all $\alpha, \beta \in \mathcal{A}$. An algebra with an involution '$*$' is called $*$-algebra, then an additive derivation $\vartheta : \mathcal{A} \to \mathcal{A}$ is called an additive $*$-derivation, if $\vartheta(\alpha^*) = \vartheta(\alpha)^*$ hold for all $\alpha \in \mathcal{A}$. For $\alpha, \beta \in \mathcal{A}$, describe the skew Lie product and bi-skew Lie product of α and β by $[\alpha, \beta]_* = \alpha\beta - \beta\alpha^*$ and $[\alpha, \beta]_\triangledown = \alpha\beta^* - \beta\alpha^*$, respectively. A map $\vartheta : \mathcal{A} \to \mathcal{A}$ (need not be linear) is said to be nonlinear skew Lie derivation (resp. nonlinear skew Lie triple derivation) if $\vartheta([\alpha, \beta]_*) = [\vartheta(\alpha), \beta]_* + [\alpha, \vartheta(\beta)]_*$ (resp. $\vartheta([[\alpha, \beta]_*, \gamma]_*) = [[\vartheta(\alpha), \beta]_*, \gamma]_* + [\alpha, \vartheta(\beta)]_*, \gamma]_* + [[\alpha, \beta]_*, \vartheta(\gamma)]_*$), holds for all $\alpha, \beta, \gamma \in \mathcal{A}$. Analogously, a map $\vartheta : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) is called nonlinear bi-skew Lie derivation (resp. nonlinear bi-skew Lie triple derivation) if $\vartheta([\alpha, \beta]_\triangledown) = [\vartheta(\alpha), \beta]_\triangledown + [\alpha, \vartheta(\beta)]_\triangledown$ (resp. $\vartheta([[\alpha, \beta]_\triangledown, \gamma]_\triangledown) = [[\vartheta(\alpha), \beta]_\triangledown, \gamma]_\triangledown + [[\alpha, \vartheta(\beta)]_\triangledown, \gamma]_\triangledown + [[\alpha, \beta]_\triangledown, \vartheta(\gamma)]_\triangledown$) holds for

Key words and phrases. Additive $*$-derivation, $*$-algebras, left bi-skew Lie derivation, factor von Neumann algebra.
2020 *Mathematics Subject Classification.* Primary: 16W10. Secondary: 46L10, 47B47.
DOI
Received: June 06, 2024.
Accepted: April 01, 2025.

all $\alpha, \beta, \gamma \in \mathcal{A}$. Many mathematicians have spent the last decade to the studying mappings involving different products on various types of rings and algebras. Yu et al. in [15], proved that every $*$-Lie derivation on factor von Neumann algebras is an additive $*$-derivation. Li et al. [9] discussed the skew Lie triple derivation between factor von Neumann algebras and showed that it is an additive $*$-derivation. Kong et al. [7], extended the results of [15] to prime $*$-rings. Similarly, skew Jordan derivations, skew Jordan triple derivations and skew Jordan-type derivations have been studied under certain conditions on various types of algebras such as factor von Neumann algebras, von Neumann algebras and $*$-algebras (see [8, 10, 14, 16, 17]).

For any $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{A}$ and integer $n \geq 2$, define a sequence of polynomials as follows: $P_1(\lambda_1) = \lambda_1, P_2(\lambda_1, \lambda_2) = \lambda_1 \circ \lambda_2 = \lambda_1^* \lambda_2 - \lambda_2^* \lambda_1$ and $P_n(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n) = P_{n-1}(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \circ \lambda_n$. The polynomial $P_n(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n)$ is called the left bi-skew Lie n-product of elements $\lambda_1, \lambda_2, \ldots, \lambda_n$. Now, we define a map $\vartheta : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) is said to be nonlinear left bi-skew Lie n-derivation if

$$\vartheta(P_n(\lambda_1, \lambda_2, \ldots, \lambda_n)) = \sum_{i=1}^{n} P_n(\lambda_1, \ldots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \ldots, \lambda_n)$$

holds for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{A}$. Obviously, a nonlinear left bi-skew Lie 2-derivation is a nonlinear left bi-skew Lie derivation and a nonlinear left bi-skew Lie 3-derivation is a nonlinear left bi-skew Lie triple derivation. Nonlinear left bi-skew Lie 2-derivations, nonlinear left bi-skew Lie 3-derivations and nonlinear left bi-skew Lie n-derivations are collectively known as nonlinear left bi-skew Lie-type derivations. In 2018, Lin [11], studied nonlinear $*$-Lie type derivations on standard operator algebras and proved that every nonlinear $*$-Lie n-derivation on a standard operator algebra is an additive $*$-derivation. In the same year Lin [12], proved that every nonlinear $*$-Lie n-derivation on von Neumann algebras is an additive $*$-derivation. Recently Madni et al. [13], proved that if $\mathcal{A}$ is a unital $*$-algebra and $\vartheta : \mathcal{A} \to \mathcal{A}$ is a skew Lie n-derivation, then ϑ is additive. Moreover, if $\vartheta(\frac{iI}{2})$ is self-adjoint, then ϑ is a $*$-derivation. Kong et al. in [5], proved that if $\mathcal{A}$ is a factor von Neumann algebra and a map $\vartheta : \mathcal{A} \to \mathcal{A}$ (need not be linear) satisfy $\vartheta([\alpha, \beta]_\triangledown) = [\vartheta(\alpha), \beta]_\triangledown + [\alpha, \vartheta(\beta)]_\triangledown$ if and only if ϑ is an additive $*$-derivation. Khan [4], proved that every nonlinear bi-skew Lie triple derivation on factor von Neumann algebra is an additive $*$-derivation. Ashraf et al. [1], generalized the result of [4] and proved that every nonlinear bi-skew Lie n-derivation on factor von Neumann algebra is an additive $*$-derivation.

Inspired by the above-mentioned works, our focus will be on providing a description of nonlinear left bi-skew Lie n-derivations on $*$-algebras. More accurately, we show that, under mild assumptions, every nonlinear left bi-skew Lie n-derivation on a unital $*$-algebra is an additive $*$-derivation.

2. Main Results

The main result of this article is as follows.

Theorem 2.1. *Let $\mathcal{A}$ be a unital $*$-algebra with the unity I and containing a non-trivial projection P_1. Write $P_2 = I - P_1$ and suppose that $\mathcal{A}$ satisfies*

$$(\spadesuit) \qquad X\mathcal{A}P_k = 0 \Rightarrow X = 0 \quad (k = 1, 2),$$

where $X \in \mathcal{A}$. Then a map $\vartheta : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) satisfies

$$\vartheta(P_n(\lambda_1, \lambda_2, \dots, \lambda_n)) = \sum_{i=1}^{n} P_n(\lambda_1, \dots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \dots, \lambda_n),$$

for all $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive $*$-derivation.

Proof. Assume $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ for $i, j = 1, 2$. Then, by Peirce decomposition of $\mathcal{A}$, we have $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$. Clearly, any $\Pi \in \mathcal{A}$ can be written as $\Pi = \Pi_{11} + \Pi_{12} + \Pi_{21} + \Pi_{22}$, where $\Pi_{ij} \in \mathcal{A}_{ij}$ for $i, j = 1, 2$. Let $\mathcal{A}^+ = \{A \in \mathcal{A} : A^* = A\}$ and $\mathcal{A}^- = \{A \in \mathcal{A} : A^* = -A\}$, $\mathcal{A}^-_{12} = \{P_1AP_2 + P_2AP_1 : A \in \mathcal{A}^-\}$, $\mathcal{A}^-_{kk} = P_k\mathcal{A}^-P_k$, $k = 1, 2$. Thus, for every $A \in \mathcal{A}^-$, we have $A = A_{11} + A_{12} + A_{22}$, where $A_{11} \in \mathcal{A}^-_{11}, A_{12} \in \mathcal{A}^-_{12}$ and $A_{22} \in \mathcal{A}^-_{22}$. Obviously, we only need to prove the necessity part. To complete the proof of the theorem stated above, several lemmas are required. These lemmas are presented as follows. $\square$

Lemma 2.1. $\vartheta(0) = 0$.

Proof. It is obvious that

$$\begin{aligned}\vartheta(0) &= \vartheta(P_n(0, 0, \dots, 0)) \\ &= P_n(\vartheta(0), 0, \dots, 0) + P_n(0, \vartheta(0), \dots, 0) + \cdots + P_n(0, 0, \dots, \vartheta(0)) \\ &= 0.\end{aligned}$$

$\square$

Lemma 2.2. *$\vartheta(H)^* = -\vartheta(H)$ for any $H \in \mathcal{A}^-$.*

Proof. We have the relation $P_n\Big(H, -\frac{I}{2}, \dots, -\frac{I}{2}\Big) = H$ and

$$\begin{aligned}\vartheta(H) =&\vartheta\Big(P_n\Big(H, -\frac{I}{2}, \dots, -\frac{I}{2}\Big)\Big) \\ =&P_n\Big(\vartheta(H), -\frac{I}{2}, \dots, -\frac{I}{2}\Big) + P_n\Big(H, \vartheta\Big(-\frac{I}{2}\Big), \dots, -\frac{I}{2}\Big) \\ &+ \cdots + P_n\Big(H, -\frac{I}{2}, \dots, \vartheta\Big(-\frac{I}{2}\Big)\Big) \\ =&\frac{I}{2}\Big(\vartheta(H) - \vartheta(H)^*\Big) + \Big(-\vartheta\Big(-\frac{I}{2}\Big)^* H - H\vartheta\Big(-\frac{I}{2}\Big)\Big) \\ &+ \cdots + \Big(-\vartheta\Big(-\frac{I}{2}\Big)^* H - H\vartheta\Big(-\frac{I}{2}\Big)\Big) \\ =&\frac{I}{2}\Big(\vartheta(H) - \vartheta(H)^*\Big) + (n-1)\Big(-\vartheta\Big(-\frac{I}{2}\Big)^* H - H\vartheta\Big(-\frac{I}{2}\Big)\Big).\end{aligned}$$

On solving further we get

$$-\vartheta(H)^* = \frac{I}{2}\Big(\vartheta(H) - \vartheta(H)^*\Big) + (n-1)\Big(-\vartheta\Big(-\frac{I}{2}\Big)^* H - H\vartheta\Big(-\frac{I}{2}\Big)\Big) = \vartheta(H).$$ $\square$

Lemma 2.3. *For any $H_{11} \in \mathcal{A}_{11}^-, H_{12} \in \mathcal{A}_{12}^-$ and $H_{22} \in \mathcal{A}_{22}^-$, we have*

$$\vartheta(H_{11} + H_{12}) = \vartheta(H_{11}) + \vartheta(H_{12})$$

and

$$\vartheta(H_{12} + H_{22}) = \vartheta(H_{12}) + \vartheta(H_{22}).$$

Proof. Let $T = \vartheta(H_{11} + H_{12}) - \vartheta(H_{11}) - \vartheta(H_{12})$. We need to demonstrate this $T = 0$. Utilizing the reality $P_n(P_2, H_{11}, -I, -I, \ldots, -I) = 0$ and Lemma 2.1, we have

$$\begin{aligned}
&\vartheta(P_n(P_2, H_{11} + H_{12}, -I, -I, \ldots, -I)) \\
=&\vartheta(P_n(P_2, H_{11}, -I, -I, \ldots, -I)) + \vartheta(P_n(P_2, H_{12}, -I, -I, \ldots, -I)) \\
=&P_n(\vartheta(P_2), H_{11}, -I, -I, \ldots, -I) + P_n(P_2, \vartheta(H_{11}), -I, -I, \ldots, -I) \\
&+ P_n(P_2, H_{11}, \vartheta(-I), \ldots, -I) + \cdots + P_n(P_2, H_{11}, -I, -I, \ldots, \vartheta(-I)) \\
&+ P_n(\vartheta(P_2), H_{12}, -I, \ldots, -I) + P_n(P_2, \vartheta(H_{12}), -I, -I, \ldots, -I) \\
&+ P_n(P_2, H_{12}, \vartheta(-I), \ldots, -I) + \cdots + P_n(P_2, H_{12}, -I, -I, \ldots, \vartheta(-I)) \\
=&P_n(\vartheta(P_2), (H_{11} + H_{12}), -I, -I, \ldots, -I) \\
&+ P_n(P_2, (\vartheta(H_{11}) + \vartheta(H_{12})), -I, -I, \ldots, -I) \\
&+ P_n(P_2, (H_{11} + H_{12}), \vartheta(-I), -I, \ldots, -I) \\
&+ \cdots + P_n(P_2, (H_{11} + H_{12}), -I, -I, \ldots, \vartheta(-I)).
\end{aligned}$$

Further, we have

$$\begin{aligned}
&\vartheta(P_n(P_2, H_{11} + H_{12}, -I, -I, \ldots, -I)) \\
=&P_n(\vartheta(P_2), (H_{11} + H_{12}), -I, -I, \ldots, -I) + P_n(P_2, (\vartheta(H_{11} + H_{12})), -I, -I, \ldots, -I) \\
&+ P_n(P_2, (H_{11} + H_{12}), \vartheta(-I), -I, \ldots, -I) \\
&+ \cdots + P_n(P_2, (H_{11} + H_{12}), -I, -I, \ldots, \vartheta(-I)).
\end{aligned}$$

On equating the above two expressions, we obtain that $P_n(P_2, T, -I, \ldots, -I) = 0$. This gives us $T_{12} = T_{21} = T_{22} = 0$.

Invoking the fact $P_n(P_2 - P_1, H_{12}, -I, \ldots, -I) = 0$ and using Lemma

$$\begin{aligned}
&\vartheta(P_n((P_2 - P_1), H_{11} + H_{12}, -I, \ldots, -I)) \\
=&\vartheta((P_n(P_2 - P_1), H_{11}, -I, \ldots, -I)) + \vartheta(P_n((P_2 - P_1), H_{12}, -I, \ldots, -I)) \\
=&P_n(\vartheta(P_2 - P_1), H_{11}, -I, \ldots, -I) + P_n((P_2 - P_1), \vartheta(H_{11}), -I, \ldots, -I) \\
&+ P_n((P_2 - P_1), H_{11}, \vartheta(-I), \ldots, -I) + \cdots + P_n((P_2 - P_1), H_{11}, -I, \ldots, \vartheta(-I)) \\
&+ P_n(\vartheta(P_2 - P_1), H_{12}, -I, \ldots, -I) + P_n((P_2 - P_1), \vartheta(H_{12}), -I, \ldots, -I) \\
&+ P_n((P_2 - P_1), H_{12}, \vartheta(-I), \ldots, -I) + \cdots + P_n((P_2 - P_1), H_{12}, -I, \ldots, \vartheta(-I)) \\
=&P_n(\vartheta(P_2 - P_1), H_{11} + H_{12}, -I, \ldots, -I) + P_n((P_2 - P_1), \vartheta(H_{11}) + \vartheta(H_{12}), \\
&- I, \ldots, -I) + P_n((P_2 - P_1), H_{11} + H_{12}, \vartheta(-I), \ldots, -I) + \cdots + P_n((P_2 - P_1), \\
&H_{11} + H_{12}, -I, \ldots, \vartheta(-I)).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\vartheta(P_n((P_2-P_1),H_{11}+H_{12},-I,\dots,-I))\\=&P_n(\vartheta(P_2-P_1),H_{11}+H_{12},-I,\dots,-I)+P_n((P_2-P_1),\vartheta(H_{11}+H_{12}),-I,\dots,-I)\\&+P_n((P_2-P_1),H_{11}+H_{12},\vartheta(-I),\dots,-I)+\cdots+P_n((P_2-P_1),H_{11}+H_{12},\\&-I,\dots,\vartheta(-I)).\end{aligned}$$

When comparing the two expressions above, we obtain that

$$P_n((P_2-P_1),T,-I,\dots,-I)=0.$$

This further implies that $T_{11}=0$. Hence $T=0$, that is,

$$\vartheta(H_{11}+H_{12})=\vartheta(H_{11})+\vartheta(H_{12}).$$

Similarly, we can show that $\vartheta(H_{12}+H_{22})=\vartheta(H_{12})+\vartheta(H_{22})$. □

Lemma 2.4. *For any $H_{11}\in\mathcal{A}_{11}^-, H_{12}\in\mathcal{A}_{12}^-$ and $H_{22}\in\mathcal{A}_{22}^-$, we have*

$$\vartheta(H_{11}+H_{12}+H_{22})=\vartheta(H_{11})+\vartheta(H_{12})+\vartheta(H_{22}).$$

Proof. Let $T=\vartheta(H_{11}+H_{12}+H_{22})-\vartheta(H_{11})-\vartheta(H_{12})-\vartheta(H_{22})$. We show that $T=0$. Using the fact that $P_n(P_1,H_{22},-I,\dots,-I)=0$ and Lemmas 2.1 and 2.3, we find that

$$\begin{aligned}&\vartheta(P_n(P_1,H_{11}+H_{12}+H_{22},-I,\dots,-I))\\=&\vartheta(P_n(P_1,H_{11}+H_{12},-I,\dots,-I))+\vartheta(P_n(P_1,H_{22},-I,\dots,-I))\\=&P_n((\vartheta(P_1),H_{11}+H_{12},-I,\dots,-I))+P_n(P_1,\vartheta(H_{11}+H_{12}),-I,\dots,-I)\\&+P_n(P_1,H_{11}+H_{12},\vartheta(-I),\dots,-I)+\cdots+P_n(P_1,H_{11}+H_{12},-I,\dots,\vartheta(-I))\\&+P_n(\vartheta(P_1),H_{22},-I,\dots,-I)+P_n(P_1,\vartheta(H_{22}),-I,\dots,-I)\\&+P_n(P_1,H_{22},\vartheta(-I),\dots,-I)+\cdots+P_n(P_1,H_{22},-I,\dots,\vartheta(-I))\\=&P_n((\vartheta(P_1),H_{11}+H_{12},-I,\dots,-I))+P_n(P_1,\vartheta(H_{11})+\vartheta(H_{12}),-I,\dots,-I)\\&+P_n(P_1,H_{11}+H_{12},\vartheta(-I),\dots,-I)+\cdots+P_n(P_1,H_{11}+H_{12},-I,\dots,\vartheta(-I))\\&+P_n(\vartheta(P_1),H_{22},-I,\dots,-I)+P_n(P_1,\vartheta(H_{22}),-I,\dots,-I)\\&+P_n(P_1,H_{22},\vartheta(-I),\dots,-I)+\cdots+P_n(P_1,H_{22},-I,\dots,\vartheta(-I))\\=&P_n((\vartheta(P_1),H_{11}+H_{12}+H_{22},-I,\dots,-I))\\&+P_n(P_1,\vartheta(H_{11})+\vartheta(H_{12})+\vartheta(H_{22}),-I,\dots,-I)\\&+P_n(P_1,H_{11}+H_{12}+H_{22},\vartheta(-I),\dots,-I)\\&+\cdots+P_n(P_1,H_{11}+H_{12}+H_{22},-I,\dots,\vartheta(-I)).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\vartheta(P_n(P_1,H_{11}+H_{12}+H_{22},-I,\dots,-I))\\=&P_n((\vartheta(P_1),H_{11}+H_{12}+H_{22},-I,\dots,-I))\\&+P_n(P_1,\vartheta(H_{11}+H_{12}+H_{22}),-I,\dots,-I)\end{aligned}$$

$$+ P_n(P_1, H_{11} + H_{12} + H_{22}, \vartheta(-I), \dots, -I)$$
$$+ \cdots + P_n(P_1, H_{11} + H_{12} + H_{22}, -I, \dots, \vartheta(-I)).$$

On solving the above two expressions for $\vartheta(P_n(P_1, H_{11} + H_{12} + H_{22}, -I, \dots, -I))$, we obtain that $P_n(P_1, T, -I, \dots, -I) = 0$ which futher implies that $T_{11} = T_{12} = T_{21} = 0$. Using the fact that $P_n(P_2, H_{11}, -I, \dots, -I) = 0$ and Lemmas 2.1 and 2.3, we find that

$$\begin{aligned}
&\vartheta(P_n(P_2, H_{11} + H_{12} + H_{22}, -I, \dots, -I)) \\
=&\vartheta(P_n(P_2, H_{11}, -I, \dots, -I)) + \vartheta(P_n(P_2, H_{12} + H_{22}, -I, \dots, -I)) \\
=&P_n((\vartheta(P_2), H_{11}, -I, \dots, -I)) + P_n(P_2, \vartheta(H_{11}), -I, \dots, -I) \\
&+ P_n(P_2, H_{11}, \vartheta(-I), \dots, -I) + \cdots + P_n(P_2, H_{11}, -I, \dots, \vartheta(-I)) \\
&+ P_n(\vartheta(P_2), H_{12} + H_{22}, -I, \dots, -I) + P_n(P_2, \vartheta(H_{12}) + \vartheta(H_{22}), -I, \dots, -I) \\
&+ P_n(P_2, H_{12} + H_{21}, \vartheta(-I), \dots, -I) + \cdots + P_n(P_2, H_{12} + H_{21}, -I, \dots, \vartheta(-I)) \\
=&P_n((\vartheta(P_2), H_{11} + H_{12} + H_{22}, -I, \dots, -I)) \\
&+ P_n(P_2, \vartheta(H_{11}) + \vartheta(H_{12}) + \vartheta(H_{22}), -I, \dots, -I) \\
&+ P_n(P_2, H_{11} + H_{12} + H_{22}, \vartheta(-I), \dots, -I) \\
&+ \cdots + P_n(P_2, H_{11} + H_{12} + H_{22}, -I, \dots, \vartheta(-I)).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\vartheta(P_n(P_2, H_{11} + H_{12} + H_{22}, -I, \dots, -I)) \\
=&P_n((\vartheta(P_2), H_{11} + H_{12} + H_{22}, -I, \dots, -I)) \\
&+ P_n((P_2, \vartheta(H_{11}) + H_{12} + H_{22}), -I, \dots, -I) \\
&+ P_n(P_2, H_{11} + H_{12} + H_{22}, \vartheta(-I), \dots, -I) \\
&+ \cdots + P_n(P_2, H_{11} + H_{12} + H_{22}, -I, \dots, \vartheta(-I)).
\end{aligned}$$

On comparing the above two expressions, we obtain that $P_n(P_2, T, -I, \dots, -I) = 0$ which further implies that $T_{22} = 0$. Thus, $T = 0$, that is,

$$\vartheta(H_{11} + H_{12} + H_{21} + H_{22}) = \vartheta(H_{11}) + \vartheta(H_{12}) + \vartheta(H_{21}) + \vartheta(H_{22}). \qquad \square$$

Lemma 2.5. *For any $H_{12}, H'_{12} \in \mathcal{A}^-_{12}$ we have*

$$\vartheta(H_{12} + H'_{12}) = \vartheta(H_{12}) + \vartheta(H'_{12}).$$

Proof. Take $H_{12} = A_{12} - A^*_{12}$ and $H'_{12} = B_{12} - B^*_{12}$ for $A_{12}, B_{12} \in \mathcal{A}_{12}$. Using the facts $P_n(iP_1 + iA_{12} + iA^*_{12}, iP_2 + iB_{12} + iB^*_{12}, -\frac{I}{2}, \dots, -\frac{I}{2}) = H_{12} + H'_{12} + H_{12}(H'_{12})^* - H'_{12}H^*_{12}$, where $H_{12}(H'_{12})^* - H'_{12}H^*_{12} = (A^*_{12}B_{12} - B^*_{12}A_{12}) + (A_{12}B^*_{12} - B_{12}A^*_{12})$ and making use of Lemma 2.4, we have

$$\begin{aligned}
&\vartheta(H_{12} + H'_{12}) + \vartheta(H_{12}(H'_{12})^* - H'_{12}H^*_{12}) \\
=&\vartheta\Big(H_{12} + H'_{12} + H_{12}(H'_{12})^* - H'_{12}H^*_{12}\Big)
\end{aligned}$$

$$
\begin{aligned}
=&\vartheta\Big(P_n\Big(iP_1+iA_{12}+iA_{12}^*,iP_2+iB_{12}+iB_{12}^*,-\frac{I}{2},\dots,-\frac{I}{2}\Big)\Big)\\
=&P_n\Big(\vartheta(iP_1)+\vartheta(iA_{12}+iA_{12}^*),iP_2+iB_{12}+iB_{12}^*,-\frac{I}{2},\dots,-\frac{I}{2}\Big)\\
&+P_n\Big(iP_1+iA_{12}+iA_{12}^*,\vartheta(iP_2)+\vartheta(iB_{12}+iB_{12}^*),-\frac{I}{2},\dots,-\frac{I}{2}\Big)\\
&+P_n\Big(iP_1+iA_{12}+iA_{12}^*,iP_2+iB_{12}+iB_{12}^*,\vartheta\Big(-\frac{I}{2}\Big),\dots,-\frac{I}{2}\Big)\\
&+\cdots+P_n\Big(iP_1+iA_{12}+iA_{12}^*,iP_2+iB_{12}+iB_{12}^*,-\frac{I}{2},\dots,\vartheta\Big(-\frac{I}{2}\Big)\Big)\\
=&\vartheta\Big(P_n\Big(iP_1,iP_2,-\frac{I}{2},\dots,-\frac{I}{2}\Big)\Big)+\vartheta\Big(P_n(iP_1,iB_{12}+iB_{12}^*,-\frac{I}{2},\dots,-\frac{I}{2}\Big)\Big)\\
&+\vartheta\Big(P_n\Big(iA_{12}+iA_{12}^*,iP_2,-\frac{I}{2},\dots,-\frac{I}{2}\Big)\Big)\\
&+\vartheta\Big(P_n\Big(iA_{12}+iA_{12}^*,iB_{12}+iB_{12}^*,-\frac{I}{2},\dots,-\frac{I}{2}\Big)\Big)\\
=&\vartheta(H_{12})+\vartheta(H_{12}^{'})+\vartheta(H_{12}(H_{12}^{'})^*-H_{12}^{'}H_{12}^*).
\end{aligned}
$$

Hence, $\vartheta(H_{12}+H_{12}^{'})=\vartheta(H_{12})+\vartheta(H_{12}^{'})$ for any $H_{12},H_{12}^{'}\in\mathcal{A}_{12}^-$. □

Lemma 2.6. *For any $H_{ii},H_{ii}^{'}\in\mathcal{A}_{ii}$ for $i=1,2$, we have*

$$\vartheta(H_{11}+H_{11}^{'})=\vartheta(H_{11})+\vartheta(H_{11}^{'})\quad and\quad \vartheta(H_{22}+H_{22}^{'})=\vartheta(H_{22})+\vartheta(H_{22}^{'}).$$

Proof. Let $T=\vartheta(H_{11}+H_{11}^{'})-\vartheta(H_{11})-\vartheta(H_{11}^{'})$, we show that $T=0$. Using the fact that $P_n(P_2,H_{11},-I\dots,-I)=P_n(P_2,H_{11}^{'},-I,\dots,-I)=0$ and making use of Lemma 2.1, we obtain

$$
\begin{aligned}
&\vartheta(P_n(P_2,H_{11}+H_{11}^{'},-I\dots,-I))\\
=&\vartheta(P_n(P_2,H_{11},-I\dots,-I))+\vartheta(P_n(P_2,H_{11}^{'},-I\dots,-I))\\
=&P_n(\vartheta(P_2),H_{11},-I,\dots,-I)+P_n(P_2,\vartheta(H_{11}),-I,\dots,-I)\\
&+P_n(P_2,H_{11},\vartheta(-I),\dots,-I)+\cdots+P_n(P_2,H_{11},-I,\dots,\vartheta(-I))\\
&+P_n(\vartheta(P_2),H_{11}^{'},-I,\dots,-I)+P_n(P_2,\vartheta(H_{11}^{'}),-I,\dots,-I)\\
&+P_n(P_2,H_{11}^{'},\vartheta(-I),\dots,-I)+\cdots+P_n(P_2,H_{11}^{'},-I,\dots,\vartheta(-I))\\
=&P_n(\vartheta(P_2),H_{11}+H_{11}^{'},-I,\dots,I)+P_n(P_2,\vartheta(H_{11})+\vartheta(H_{11}^{'}),-I,\dots,-I)\\
&+P_n(P_2,H_{11}+H_{11}^{'},\vartheta(-I),\dots,-I)+\cdots+P_n(P_2,H_{11}+H_{11}^{'},-I,\dots,\vartheta(-I)).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\vartheta(P_n(P_2,H_{11}+H_{11}^{'},-I\dots,-I))\\
=&P_n(\vartheta(P_2),H_{11}+H_{11}^{'},-I,\dots,I)+P_n(P_2,\vartheta(H_{11}+H_{11}^{'}),-I,\dots,-I)\\
&+P_n(P_2,H_{11}+H_{11}^{'},\vartheta(-I),\dots,-I)+\cdots+P_n(P_2,H_{11}+H_{11}^{'},-I,\dots,\vartheta(-I)).
\end{aligned}
$$

Comparing the above two expressions, we find that $P_n(P_2, T, -I, \ldots, -I) = 0$, which gives us $T_{12} = T_{21} = T_{22} = 0$.

Next, we show that $T_{11} = 0$. Let $X = A_{12} - A_{12}^*$ for any $A_{12} \in \mathcal{A}_{12}$ and it is easy to observe that $P_n(X, H_{11}, -\frac{I}{2}, \ldots, -\frac{I}{2}), P_n(X, H_{11}^{'}, -\frac{I}{2}, \ldots, -\frac{I}{2}) \in \mathcal{A}_{12}$. Thus, using Lemma 2.5, we find that

$$\begin{aligned}
&\vartheta\Big(P_n\Big(X, H_{11} + H_{11}^{'}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\Big)\\
=&\vartheta\Big(P_n\Big(X, H_{11}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\Big) + \vartheta\Big(P_n\Big(X, H_{11}^{'}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\Big)\\
=&P_n\Big(\vartheta(X), H_{11}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big) + P_n\Big(X, \vartheta(H_{11}), -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\\
&+ P_n\Big(X, H_{11}, \vartheta\Big(-\frac{I}{2}\Big), \ldots, -\frac{I}{2}\Big) + \cdots + P_n\Big(X, H_{11}, -\frac{I}{2}, \ldots, \vartheta\Big(-\frac{I}{2}\Big)\Big)\\
&+ P_n\Big(\vartheta(X), H_{11}^{'}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big) + P_n\Big(X, \vartheta(H_{11}^{'}), -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\\
&+ P_n\Big(X, H_{11}^{'}, \vartheta\Big(-\frac{I}{2}\Big), \ldots, -\frac{I}{2}\Big) + \cdots + P_n\Big(X, H_{11}^{'}, -\frac{I}{2}, \ldots, \vartheta\Big(-\frac{I}{2}\Big)\Big)\\
=&P_n\Big(\vartheta(X), H_{11} + H_{11}^{'}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big) + P_n\Big(X, \vartheta(H_{11}) + \vartheta(H_{11}^{'}), -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\\
&+ P_n\Big(X, H_{11} + H_{11}^{'}, \vartheta\Big(-\frac{I}{2}\Big), \ldots, -\frac{I}{2}\Big)\\
&+ \cdots + P_n\Big(X, H_{11} + H_{11}^{'}, -\frac{I}{2}, \ldots, \vartheta\Big(-\frac{I}{2}\Big)\Big).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\vartheta\Big(P_n\Big(X, H_{11} + H_{11}^{'}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\Big)\\
=&P_n\Big(\vartheta(X), H_{11} + H_{11}^{'}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big) + P_n\Big(X, \vartheta(H_{11} + H_{11}^{'}), -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\\
&+ P_n\Big(X, H_{11} + H_{11}^{'}, \vartheta\Big(-\frac{I}{2}\Big), \ldots, -\frac{I}{2}\Big)\\
&+ \cdots + P_n\Big(X, H_{11} + H_{11}^{'}, -\frac{I}{2}, \ldots, \vartheta\Big(-\frac{I}{2}\Big)\Big).
\end{aligned}$$

From the last two expressions, we get $P_n(X, T, -\frac{I}{2}, \ldots, -\frac{I}{2}) = 0$, on solving further, we obtain $-XT + TX = 0$. Multiplying by P_1 and P_2 from left and right, we obtain $P_1TP_1AP_2 = 0$ for all $A \in \mathcal{A}$. Application of the condition (♠) yields $T_{11} = 0$. Hence, $T = 0$, that is, $\vartheta(H_{11} + H_{11}^{'}) = \vartheta(H_{11}) + \vartheta(H_{11}^{'})$. Symmetrically, we can prove that $\vartheta(H_{22} + H_{22}^{'}) = \vartheta(H_{22}) + \vartheta(H_{22}^{'})$. □

Lemma 2.7. *ϑ is additive on $\mathcal{A}^-$.*

Proof. For any $H, K \in \mathcal{A}^-$, we have $H = H_{11} + H_{12} + H_{22}$ and $K = K_{11} + K_{12} + K_{22}$. With the help of Lemmas 2.4, 2.5 and 2.6, we obtain

$$\begin{aligned}\vartheta(H+K) =& \vartheta(H_{11} + H_{12} + H_{22} + K_{11} + K_{12} + K_{22})\\ =& \vartheta(H_{11} + K_{11}) + \vartheta(H_{12} + K_{12}) + \vartheta(H_{22} + K_{22})\\ =& \vartheta(H_{11}) + \vartheta(K_{11}) + \vartheta(H_{12}) + \vartheta(K_{12}) + \vartheta(H_{22}) + \vartheta(K_{22})\\ =& \vartheta(H_{11} + H_{12} + H_{22}) + \vartheta(K_{11} + K_{12} + K_{22})\\ =& \vartheta(H) + \vartheta(K).\end{aligned}$$

□

Lemma 2.8. $\vartheta(-I) = 0$.

Proof. Using Lemmas 2.2 and 2.7, it follows that $2^{n-1}V = P_n(V, -I, -I, \ldots, -I)$ for any $V \in \mathcal{A}^-$, we have

$$\begin{aligned}2^{n-1}\vartheta(V) =& \vartheta(P_n(V, -I, -I, \ldots, -I)).\\ =& P_n(\vartheta(V), -I, -I, \ldots, -I) + P_n(V, \vartheta(-I), -I, \ldots, I)\\ &+ P_n(V, -I, \vartheta(-I), \ldots, -I) + \cdots + P_n(V, -I, -I, \ldots, \vartheta(-I))\\ =& 2^{n-1}\vartheta(V) + 2^{n-2}(n-1)\{-A\vartheta(-I) - \vartheta(-I)^*V\}.\end{aligned}$$

On simplifying further, we obtain

$$0 = V\vartheta(-I) + \vartheta(-I)^*V.$$

Taking $V = iI$ in the above expression, we obtain $\vartheta(-I) = -\vartheta(-I)^*$, which implies $\vartheta(-I) \in \mathcal{A}^-$. We have now $V\vartheta(-I) = \vartheta(-I)V$ for all $V \in \mathcal{A}^-$. For every $F \in \mathcal{A}$, we have $F = F_1 + iF_2$, where $F_1 = \frac{F-F^*}{2} \in \mathcal{A}^-$ and $F_2 = \frac{F+F^*}{2i} \in \mathcal{A}^-$. Thus, we get $F\vartheta(-I) = \vartheta(-I)F$ for all $F \in \mathcal{A}$, this further implies that

$$\vartheta(-I) \in Z(\mathcal{A}) \cap \mathcal{A}^-. \tag{2.1}$$

Using Lemmas 2.1 and 2.2, it follows that $0 = P_n(V, -iI, -I, \ldots, -I)$ for any $V \in \mathcal{A}^-$, we have

$$\begin{aligned}0 =& \vartheta(P_n(V, -iI, -I, \ldots, -I))\\ =& P_n(\vartheta(V), -iI, -I, \ldots, -I) + P_n(V, \vartheta(-I), -I, \ldots, I)\\ &+ P_n(V, -I, \vartheta(-iI), \ldots, -I) + \cdots + P_n(V, -iI, -I, \ldots, \vartheta(-I))\\ =& 2^{n-2}(-V\vartheta(-iI) - \vartheta(-iI)^*V).\end{aligned}$$

By following similar steps as done earlier, we get

$$\vartheta(-iI) \in Z(\mathcal{A}) \cap \mathcal{A}^-. \tag{2.2}$$

Assume $\vartheta(P_1) = L_1 + iL_2$, where $L_1, L_2 \in \mathcal{A}^+$, with the help of equation (2.1) and fact $P_n(-I, P_1, -\frac{I}{2}, \ldots, -\frac{I}{2}) = 0$, we get

$$0 = \vartheta\Big(P_n\Big(-I, P_1, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\Big)$$

$$
\begin{aligned}
=&P_n\Big(\vartheta(-I), P_1, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big) + P_n\Big(-I, \vartheta(P_1), -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\\
&+ P_n\Big(-I, P_1, \vartheta\Big(-\frac{I}{2}\Big), \ldots, -\frac{I}{2}\Big) + \cdots + P_n\Big(-I, P_1, -\frac{I}{2}, \ldots, \vartheta\Big(-\frac{I}{2}\Big)\Big)\\
=&-2\vartheta(-I)P_1 - 2iL_2.
\end{aligned}
$$

Thus, we obtain

$$\vartheta(P_1) = L_1 - \vartheta(-I)P_1. \tag{2.3}$$

With the help of equations (2.2) and (2.3) and fact $P_n(-iI, P_1, -I, \ldots, -I) = 2^{n-1}iP_1$, we obtain

$$
\begin{aligned}
2^{n-1}\vartheta(iP_1) =&\vartheta(P_n(-iI, P_1, -I, \ldots, -I))\\
=&P_n(\vartheta(-iI), P_1, -I, \ldots, -I) + P_n(-iI, \vartheta(P_1), -I, \ldots, -I)\\
&+ P_n(-iI, P_1, \vartheta(-I), \ldots, -I) + \cdots + P_n(-iI, P_1, -I, \ldots, \vartheta(-I))\\
=&-2^{n-1}\vartheta(-iI)P_1 + 2^{n-1}iL_1.
\end{aligned}
$$

On solving further, we obtain

$$\vartheta(iP_1) = -\vartheta(-iI)P_1 + iL_1. \tag{2.4}$$

Applying (2.3) and (2.4), we get

$$
\begin{aligned}
2^{n-1}\vartheta(iP_1) =&\vartheta(P_n(P_1, iP_1, -I, \ldots, -I))\\
=&P_n(\vartheta(P_1), iP_1, -I, \ldots, -I) + P_n(P_1, \vartheta(iP_1), -I, \ldots, -I)\\
&+ P_n(P_1, iP_1, \vartheta(-I), \ldots, -I) + \cdots + P_n(P_1, iP_1, -I, \ldots, \vartheta(-I))\\
=&-2^{n-1}\vartheta(-iI)P_1 + 2^{n-1}i(P_1L_1 + L_1P_1).
\end{aligned}
$$

Therefore, we get

$$\vartheta(iP_1) = -\vartheta(-iI)P_1 + i(P_1L_1 + L_1P_1). \tag{2.5}$$

Comparing (2.4) and (2.5), we obtain $L_1 = P_1L_1+L_1P_1$, implies $P_1L_1P_1 = P_2L_1P_2 = 0$. Invoking the last relation in (2.3), we obtain

$$\vartheta(P_1) = P_1L_1P_2 + P_2L_1P_1 - \vartheta(-I)P_1. \tag{2.6}$$

Taking $X = A_{12} - A_{12}^*$ for any $A_{12} \in \mathcal{A}_{12}$, then $X \in \mathcal{A}^-$. Invoking Lemmas 2.2 and 2.7, we possess

$$
\begin{aligned}
2^{n-2}\vartheta(X) =&\vartheta(P_n(P_1, X, -I, \ldots, -I))\\
=&P_n(\vartheta(P_1), X, -I, \ldots, -I) + P_n(P_1, \vartheta(X), -I, \ldots, -I)\\
&+ P_n(P_1, X, \vartheta(-I), \ldots, -I) + \cdots + P_n(P_1, X, -I, \ldots, \vartheta(-I))\\
=&2^{n-2}(\vartheta(P_1)^*X + X\vartheta(P_1)) + 2^{n-2}(P_1\vartheta(X) + \vartheta(X)P_1).
\end{aligned}
$$

Accordingly, we get

$$\vartheta(X) = \vartheta(P_1)^*X + X\vartheta(P_1) + P_1\vartheta(X) + \vartheta(X)P_1. \tag{2.7}$$

Multiplying (2.7) by P_1 and P_2 from left and right, respectively, and applying (2.6), we obtain $\vartheta(-I)A_{12}$. Implementation of the condition (♠) gives us $\vartheta(-I)P_1 = 0$, employing (2.1) and condition (♠) in the last relation, we get $\vartheta(-I)P_2 = 0$, finally we obtain $\vartheta(-I) = 0$. □

Lemma 2.9. *For any $S, S_1, S_2 \in \mathcal{A}^+$*
(1) $\vartheta(S)^* = \vartheta(S)$;
(2) $\vartheta(iS) = i\vartheta(S) - \vartheta(-iI)S$;
(3) ϑ *is additive on* $\mathcal{A}^+$;
(4) $\vartheta(S_1 + iS_2) = \vartheta(S_1) + i\vartheta(S_2) - \vartheta(-iI)S_2$.

Proof. (1) Using the fact that $P_n(-I, S, -I, \dots, -I) = 0$ and Lemma 2.8, where $S \in \mathcal{A}^+$, we obtain

$$0 = \vartheta(P_n(-I, S, -I, \dots, -I)) = P_n(-I, \vartheta(S), -I, \dots, -I) = 2^{n-2}(\vartheta(S)^* - \vartheta(S)),$$

which allows us $\vartheta(S)^* = \vartheta(S)$.

(2) Using the fact that $P_n(-iI, S, -I \dots, -I) = 2^{n-1}iS$, together with Lemmas 2.7, 2.8 and equation (2.2), where $S \in \mathcal{A}^+$, we obtain

$$\begin{aligned} 2^{n-1}\vartheta(iS) =& \vartheta(P_n(-iI, S, -I \dots, -I)) \\ =& P_n(\vartheta(-iI), S, -I, \dots, -I) + P_n(-iI, \vartheta(S), -I, \dots, -I) \\ =& 2^{n-1}i(\vartheta(S) - \vartheta(-iI)S), \end{aligned}$$

which gives us

$$\vartheta(iS) = i\vartheta(S) - \vartheta(-iI)S. \tag{2.8}$$

(3) For any $S_1, S_2 \in \mathcal{A}^+$, invoking Lemma 2.7 and (2.8), we have

$$\vartheta(iS_1 + iS_2) = \vartheta(iS_1) + \vartheta(iS_2) = i\vartheta(S_1) - \vartheta(-iI)S_1 + i\vartheta(S_2) - \vartheta(-iI)S_2.$$

On the other hand, we have $\vartheta(i(S_1+S_2)) = i\vartheta(S_1+S_2) - \vartheta(-iI)(S_1+S_2)$. Comparing the above two expressions for $\vartheta(i(S_1 + S_2))$, we obtain $\vartheta(S_1 + S_2) = \vartheta(S_1) + \vartheta(S_2)$.

(4) Using the fact that $P_n(S_1 + iS_2, -I, -I, \dots, -I) = 2^{n-1}iS_2$, together with Lemmas 2.7, 2.8 and (2.8), where $S_1, S_2 \in \mathcal{A}^+$, we obtain

$$\begin{aligned} 2^{n-1}(i\vartheta(S_2) - \vartheta(-iI)S_2) =& 2^{n-1}\vartheta(iS_2) = \vartheta(P_n(S_1 + iS_2, -I, -I, \dots, -I)) \\ =& P_n(\vartheta(S_1 + iS_2), -I, -I, \dots, -I) \\ =& 2^{n-2}(\vartheta(S_1 + iS_2) - \vartheta(S_1 + iS_2)^*). \end{aligned}$$

Thus, we have

$$2^{n-1}(i\vartheta(S_2) - \vartheta(-iI)S_2) = 2^{n-2}(\vartheta(S_1 + iS_2) - \vartheta(S_1 + iS_2)^*). \tag{2.9}$$

Invoking the fact that $P_n(-iI, S_1 + iS_2, -I, \dots, -I) = 2^{n-1}iS_1$ and using Lemmas 2.7, 2.8 and (2.2), (2.8), where $S_1, S_2 \in \mathcal{A}^+$, we obtain

$$\begin{aligned} 2^{n-1}(i\vartheta(S_1) - \vartheta(-iI)S_1) =& 2^{n-1}\vartheta(iS_1) \\ =& \vartheta(P_n(-iI, S_1 + iS_2, -I, \dots, -I)) \end{aligned}$$

$$\begin{aligned}=&P_n(\vartheta(-iI), S_1 + iS_2, -I, \ldots, -I)\\ &+ P_n(-iI, \vartheta(S_1 + iS_2), -I, \ldots, -I)\\ =&-2^{n-1}\vartheta(-iI)S_1 + 2^{n-2}i(\vartheta(S_1 + iS_2) + \vartheta(S_1 + iS_2)^*).\end{aligned}$$

On solving further, we obtain

$$2^{n-1}\vartheta(S_1) = 2^{n-2}(\vartheta(S_1 + iS_2) + \vartheta(S_1 + iS_2)^*). \tag{2.10}$$

By adding (2.9) and (2.10), we get $\vartheta(S_1 + iS_2) = \vartheta(S_1) + i\vartheta(S_2) - \vartheta(-iI)S_2$. □

Lemma 2.10. *For all* $A \in \mathcal{A}$

(1) $\vartheta(A^*) = \vartheta(A)^*$*;*

(2) ϑ *is additive on* $\mathcal{A}$.

Proof. (1) For $A \in \mathcal{A}$, we have $A = A_1 + iA_2$, where $A_1, A_2 \in \mathcal{A}^+$ and using Lemma 2.9 and (2.2), we obtain

$$\begin{aligned}\vartheta(A)^* =&\vartheta(A_1 + iA_2)^* = (\vartheta(A_1) + i\vartheta(A_2) - \vartheta(-iI)A_2)^*\\ =&\vartheta(A_1) - i\vartheta(A_2) + \vartheta(-iI)A_2 = \vartheta(A_1 + i(-A_2)) = \vartheta(A_1 - iA_2) = \vartheta(A^*).\end{aligned}$$

(2) For $A, B \in \mathcal{A}$, then $A = A_1 + iA_2$ and $B = B_1 + iB_2$, where $A_1, A_2, B_1, B_2 \in \mathcal{A}^+$. Using Lemma 2.9, we obtain

$$\begin{aligned}\vartheta(A + B) =&\vartheta(A_1 + iA_2 + B_1 + iB_2)\\ =&\vartheta((A_1 + B_1) + i(A_2 + B_2))\\ =&\vartheta(A_1 + B_1) + i\vartheta(A_2 + B_2) - \vartheta(-iI)(A_2 + B_2)\\ =&\vartheta(A_1) + \vartheta(B_1) + i\vartheta(A_2) + i\vartheta(B_2) - \vartheta(-iI)A_2 - \vartheta(-iI)(B_2)\\ =&(\vartheta(A_1) + i\vartheta(A_2) - \vartheta(-iI)A_2) + (\vartheta(B_1) + i\vartheta(B_2) - \vartheta(-iI)B_2)\\ =&\vartheta(A) + \vartheta(B).\end{aligned}$$

□

Lemma 2.11. *For all* $A \in \mathcal{A}$

(1) $\vartheta(-iI) = 0$*;*

(2) $\vartheta(iA) = i\vartheta(A)$.

Proof. (1) Equation (2.4) implies

$$\vartheta(iP_1) = -\vartheta(-iI)P_1 + i(P_1L_1P_2 + P_2L_1P_1). \tag{2.11}$$

Observe that $P_n(iP_1, (A_{12} - A_{12}^*), -I, \ldots, -I) = -2^{n-2}i(A_{12} + A_{12}^*)$, for any $A_{12} \in \mathcal{A}_{12}$. Using Lemmas 2.8, 2.9 and 2.10, we find that

$$\begin{aligned}-2^{n-2}\vartheta(i(A_{12} + A_{12}^*)) =&\vartheta(P_n(iP_1, (A_{12} - A_{12}^*), -I, \ldots, -I))\\ =&P_n(\vartheta(iP_1), (A_{12} - A_{12}^*), -I, \ldots, -I)\\ &+ P_n(iP_1, \vartheta(A_{12} - A_{12}^*), -I, \ldots, -I).\end{aligned}$$

Using (2.11), in the above expression, we obtain

$$-2^{n-2}\vartheta(i(A_{12} + A_{12}^*)) = 2^{n-2}\Big(\vartheta(-iI)A_{12} - iP_2L_1A_{12} + iP_1L_1A_{12}^* + iA_{12}L_1P_1$$

$$+ A_{12}^*\vartheta(-iI) - iA_{12}^*L_1P_2 - iP_1\vartheta(A_{12}) + iP_1\vartheta(A_{12})^* - i\vartheta(A_{12})^*P_1 + i\vartheta(A_{12})P_1\Big).$$

On the other hand, we have

$$-2^{n-2}\vartheta(i(A_{12} + A_{12}^*)) = -\, 2^{n-2}i\vartheta(A_{12} + A_{12}^*) + \vartheta(-iI)(A_{12} + A_{12}^*).$$

Comparing the above two expressions, we obtain

$$\begin{aligned}&-i\vartheta(A_{12}) - i\vartheta(A_{12})^* + \vartheta(-iI)A_{12} + \vartheta(-iI)A_{12}^*\\ =&\vartheta(-iI)A_{12} - iP_2L_1A_{12} + iP_1L_1A_{12}^* + iA_{12}L_1P_1 + A_{12}^*\vartheta(-iI)\\ &- iA_{12}^*L_1P_2 - iP_1\vartheta(A_{12}) + iP_1\vartheta(A_{12})^* - i\vartheta(A_{12})^*P_1 + i\vartheta(A_{12})P_1.\end{aligned}$$

Multiplying the above expression by P_1 and P_2 from left and right, respectively, we obtain

$$P_1\vartheta(A_{12})^*P_2 = 0. \tag{2.12}$$

Alternatively observed that $P_n(iP_1, i(A_{12} + A_{12}^*), -I, \ldots, -I) = 2^{n-2}(A_{12} - A_{12}^*)$, for any $A_{12} \in \mathcal{A}_{12}$. Using Lemmas 2.8, 2.9 and 2.10, we find that

$$\begin{aligned}2^{n-2}\vartheta(A_{12} - A_{12}^*) =&\vartheta(P_n(iP_1, i(A_{12} + A_{12}^*), -I, \ldots, -I))\\ =&P_n(\vartheta(iP_1), i(A_{12} + A_{12}^*), -I, \ldots, -I)\\ &+ P_n(iP_1, \vartheta(i(A_{12} + A_{12}^*)), -I, \ldots, -I).\end{aligned}$$

Using (2.11) in the above expression, we obtain

$$\begin{aligned}\vartheta(A_{12} - A_{12}^*) =&i\vartheta(-iI)A_{12} + P_2L_1A_{12} + P_1L_1A_{12}^* - iA_{12}L_1P_1\\ &- iA_{12}^*\vartheta(-iI) - iA_{12}^*L_1P_2 + P_1\vartheta(A_{12}) + P_1\vartheta(A_{12})^*\\ &+ iP_1\vartheta(-iI)A_{12} - \vartheta(A_{12})P_1 - \vartheta(A_{12})^*P_1 - i\vartheta(-iI)A_{12}^*.\end{aligned}$$

Multiplying the above expression by P_1 and P_2 from left and right, respectively, and applying (2.12), we obtain $\vartheta(-iI)A_{12} = 0$. Implementation of the condition (♠) gives us $\vartheta(-iI)P_1 = 0$, employing (2.2) and condition (♠) in $\vartheta(-iI)A_{12} = 0$, we get $\vartheta(-I)P_2 = 0$, hence we obtain $\vartheta(-iI) = 0$.

(2) Applying the fact $\vartheta(-iI) = 0$ in the Lemma 2.9 (2), we obtain $\vartheta(iS) = i\vartheta(S)$ for all $S \in \mathcal{A}^+$. Let any $A \in \mathcal{A}$ then $A = A_1 + iA_2$, where $A_1, A_2 \in \mathcal{A}^+$

$$\begin{aligned}\vartheta(iA) =&\vartheta(iA_1 - A_2) = \vartheta(iA_1) + \vartheta(-A_2) = i\vartheta(A_1) - \vartheta(A_2)\\ =&i(\vartheta(A_1) + i\vartheta(A_2)) = i\vartheta(A_1 + iA_2) = i\vartheta(A).\end{aligned}$$

□

Lemma 2.12. *$\vartheta(AB) = \vartheta(A)B + A\vartheta(B)$, for all $A, B \in \mathcal{A}$.*

Proof. For any $A, B \in \mathcal{A}$, applying the fact

$$P_n(A, iB, -I, \ldots, -I) = 2^{n-2}i(A^*B + B^*A),$$

and using Lemmas 2.8, 2.10 (2) and 2.11 (2), we obtain

$$2^{n-2}i\vartheta(A^*B + B^*A) =\vartheta(P_n(A, iB, -I, \ldots, -I))$$

$$=P_n(\vartheta(A), iB, -I, \ldots, -I) + P_n(A, \vartheta(iB), -I, \ldots, -I)$$
$$=2^{n-2}i(\vartheta(A)^*B + B^*\vartheta(A)) + 2^{n-2}i(A^*\vartheta(B) + \vartheta(B)^*A).$$

Hence, we obtain

$$\vartheta(A^*B + B^*A) =\vartheta(A)^*B + B^*\vartheta(A) + A^*\vartheta(B) + \vartheta(B)^*A. \tag{2.13}$$

Implementing the fact $P_n(A, B, -I, \ldots, -I) = 2^{n-2}(A^*B - B^*A)$ and using Lemmas 2.8 and 2.10 (2), we obtain

$$\begin{aligned}2^{n-2}\vartheta(A^*B - B^*A) =&\vartheta(P_n(A, B, -I, \ldots, -I))\\ =&P_n(\vartheta(A), B, -I, \ldots, -I) + P_n(A, \vartheta(B), -I, \ldots, -I)\\ =&2^{n-2}(\vartheta(A)^*B - B^*\vartheta(A)) + 2^{n-2}(A^*\vartheta(B) - \vartheta(B)^*A).\end{aligned}$$

Therefore, we obtain

$$\vartheta(A^*B - B^*A) =\vartheta(A)^*B - B^*\vartheta(A) + A^*\vartheta(B) - \vartheta(B)^*A. \tag{2.14}$$

Adding equations (2.13) and (2.14) and adopting Lemma 2.10 (1), we obtain

$$\vartheta(A^*B) =\vartheta(A^*)B + A^*\vartheta(B).$$ □

Accordingly, we obtain $\vartheta(AB) = \vartheta(A)B + A\vartheta(B)$.

Invoking Lemmas 2.10 and 2.12, we obtain that ϑ is an additive $*$-derivation on $\mathcal{A}$. Hence, the proof of Theorem 2.1, is completed.

3. Corollaries

Recall that an algebra $\mathcal{A}$ is prime if for any $\alpha, \beta \in \mathcal{A}$, $\alpha\mathcal{A}\beta = \{0\}$ implies that either $\alpha = 0$ or $\beta = 0$. It is simple to check that every prime $*$-algebra satisfies (♠). Hence, as a direct significance of Theorem 2.1, we have the following result.

Corollary 3.1. *Let $\mathcal{A}$ be a unital prime $*$-algebra containing a nontrivial projection. Then, a map $\vartheta : \mathcal{A} \to \mathcal{A}$ (need not be linear) satisfies*

$$\vartheta(P_n(\lambda_1, \lambda_2, \ldots, \lambda_n)) = \sum_{i=1}^{n} P_n(\lambda_1, \ldots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \ldots, \lambda_n),$$

for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive $*$-derivation.

It is commonly known that a factor von Neumann algebra is prime, thus, it always satisfies (♠). Thus, due to the quick aftereffects of Corollary 3.1, we obtain the subsequent result.

Corollary 3.2. *Let $\mathcal{A}$ be a factor von Neumann algebra with* $\dim(\mathcal{A}) \geq 2$. *Then, a map $\vartheta : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\vartheta(P_n(\lambda_1, \lambda_2, \ldots, \lambda_n)) = \sum_{i=1}^{n} P_n(\lambda_1, \ldots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \ldots, \lambda_n),$$

for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive $*$-derivation.

Further, it is well-known that every von Neumann algebra with no central summands of type I_1 satisfies (♠) (see [3, 8] for details). Therefore, applying Theorem 2.1, we have the following result.

Corollary 3.3. *Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1. Then, a map $\vartheta : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\vartheta(P_n(\lambda_1, \lambda_2, \dots, \lambda_n)) = \sum_{i=1}^{n} P_n(\lambda_1, \dots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \dots, \lambda_n),$$

for all $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive $*$-derivation.

4. Conclusion

In this paper, we have studied nonlinear left bi-skew Lie n-derivations on arbitrary $*$-algebras. In fact, it is shown that, under certain assumptions, every nonlinear left bi-skew Lie n-derivation on a unital $*$-algebra is an additive $*$-derivation.

Acknowledgements. The authors are greatly indebted to the referee for his/her valuable suggestions, which have immensely improved the paper.

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