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SOME GRÜSS TYPE INEQUALITIES FOR FRÉCHET DIFFERENTIABLE MAPPINGS

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Abstract. Let X be a Hilbert C^*-module on C^*-algebra A and $p \in A$. We denote by $D_p(A, X)$ the set of all continuous functions $f : A \to X$, which are Fréchet differentiable on a open neighborhood U of p. Then, we introduce some generalized semi-inner products on $D_p(A, X)$, and using them some Grüss type inequalities in semi-inner product C^*-module $D_p(A, X)$ and $D_p(A, X^n)$ are established.

1. Introduction

Let A, X be two normed vector spaces over $\mathbb{K}(\mathbb{K} = \mathbb{C}, \mathbb{R})$, we recall that a function $f : A \to X$ is Fréchet differentiable in $p \in A$, if there exists a bounded linear mapping $u : A \to X$ such that

$$\lim_{h \to 0} \frac{\|f(p+h) - f(p) - u(h)\|_X}{\|h\|_A} = 0,$$

and in this case, we denote u by $Df(p)$. Let $D_p(A, X)$ denotes the set of all continuous functions $f : A \to X$, which are Fréchet differentiable on a open neighborhood (say U) of p. The main purpose of differential calculus consists in getting some information using an affine approximation to a given nonlinear map around a given point. In many applications it is important to have Fréchet derivatives of f, since they provide genuine local linear approximation to f. For instance, let U be an open subset of A containing the segment $[x, y] = \{(1-\theta)x + \theta y : 0 \leq \theta \leq 1\}$, and let $f : A \to X$ be Fréchet differentiable on U, then the following mean value formula holds

$$\|f(x) - f(y) \leq \|x - y\| \sup_{0<\theta<1} \|Df((1-\theta)x + \theta y)\| .$$

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For two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{R}$, consider the Čebyšev functional:

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \frac{1}{b-a}\int_a^b g(t)dt.$$

In 1934, G. Grüss [4] showed that

$$|T(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property $-\infty < m \leq f \leq M < \infty$ and $-\infty < n \leq g \leq N < \infty$ a.e. on $[a, b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity and is achieved for

$$f(x) = g(x) = \operatorname{sgn}\left(x - \frac{a+b}{2}\right).$$

The discrete version of (1.1) states that: if $a \leq a_i \leq A,\ b \leq b_i \leq B,\ i = 1, ..., n$, where a, A, b, B, a_i, b_i are real numbers, then

$$\left|\frac{1}{n}\sum_{i=1}^n a_i b_i - \frac{1}{n}\sum_{i=1}^n a_i . \frac{1}{n}\sum_{i=1}^n b_i\right| \leq \frac{1}{4}(A-a)(B-b), \tag{1.2}$$

where the constant $\frac{1}{4}$ is the best possible for an arbitrary $n \geq 1$. Some refinements of the discrete version of Grüss inequality (1.2) for inner product spaces are available in [1, 6].

Theorem 1.1. ([2, Theorem 2]). *Let $(H; \langle\cdot,\cdot\rangle)$ and $\mathbb{K}$ be as above and $\overline{x} = (x_1, \ldots, x_n) \in H^n$, $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{K}^n$ and $\overline{p} = (p_1, \ldots, p_n)$ a probability vector. If $x, X \in H$ are such that*

$$Re\,\langle X - x_i, x_i - x\rangle \geq 0, \quad \textit{for all } i \in \{1, \ldots, n\},$$

or, equivalently,

$$\left\|x_i - \frac{x+X}{2}\right\| \leq \frac{1}{2}\|X - x\|, \quad \textit{for all } i \in \{1, \ldots, n\},$$

holds, then the following inequality holds

$$\begin{aligned}\left\|\sum_{i=1}^n p_i\alpha_i x_i - \sum_{i=1}^n p_i\alpha_i \sum_{i=1}^n p_i x_i\right\| &\leq \frac{1}{2}\|X - x\| \sum_{i=1}^n p_i \left|\alpha_i - \sum_{j=1}^n p_j\alpha_j\right| \\ &\leq \frac{1}{2}\|X - x\| \left[\sum_{i=1}^n p_i|\alpha_i|^2 - \left|\sum_{i=1}^n p_i\alpha_i\right|^2\right]^{\frac{1}{2}}.\end{aligned}$$

The constant $\frac{1}{2}$ in the first and second inequalities is the best possible.

In recent years several refinements and generalizations have been considered for the Grüss inequality. We would like to refer the reader to [2–6, 8, 9] and references therein for more information.

In this paper, for every Hilbert C^*-module X over a C^*-algebra A, some Grüss type inequalities in semi-inner product C^*-module $D_p(A, X^n)$ are established. We also for two arbitrary Banach $*$-algebras, define a norm and an involution map on $D_p(A, B)$ and prove that $D_p(A, B)$ is a Banach $*$-algebra.

2. Grüss Type Inequalities for Differentiable Mappings

Let A be a C^*-algebra. A semi-inner product module over A is a right module X over A together with a generalized semi-inner product, that is with a mapping $\langle ., . \rangle$ on $X \times X$, which is A-valued and has the following properties:

(i) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in X$;
(ii) $\langle x, ya \rangle = \langle x, y \rangle\, a$ for $x, y \in X, a \in A$;
(iii) $\langle x, y \rangle^* = \langle y, x \rangle$ for all $x, y \in X$;
(iv) $\langle x, x \rangle \geq 0$ for $x \in X$.

We will say that X is a semi-inner product C^*-module. If, in addition,

(v) $\langle x, x \rangle = 0$ implies $x = 0$,

then $\langle \cdot, \cdot \rangle$ is called a generalized inner product and X is called an inner product module over A or an inner product C^*-module. An inner product C^*-module which is complete with respect to its norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$, is called a Hilbert C^*-module.

As we can see, an inner product module obeys the same axioms as an ordinary inner product space, except that the inner product takes values in a more general structure rather than in the field of complex numbers. If A is a C^*-algebra and X is a semi-inner product A-module, then the following Schwarz inequality holds:

$$\langle x, y \rangle \langle y, x \rangle \leq \|\langle x, x \rangle\| \langle y, y \rangle, \quad x, y \in X$$

(e.g., [7, Proposition 1.1]).

Theorem 2.1 ([3]). *Let A be a C^*- Algebra, X a Hilbert C^*- module. If $x, y, e \in X$, $\langle e, e \rangle$ is an idempotent in A and $\alpha, \beta, \lambda, \mu$ are complex numbers such that*

$$\left\| x - \frac{\alpha+\beta}{2} e \right\| \leq \frac{1}{2} |\alpha - \beta|, \quad \left\| y - \frac{\lambda+\mu}{2} e \right\| \leq \frac{1}{2} |\lambda - \mu|,$$

hold, then one has the following inequality:

$$\|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle\| \leq \frac{1}{4} |\alpha - \beta|\, |\lambda - \mu|.$$

Example 2.1. Let A be a C^*-algebra and X be a semi-inner product C^*-module on a C^*-algebra B. If functions $f, g \in D_p(A, X)$, then function $k : A \to B$ as $k(a) = \langle f(a), g(a) \rangle$ is differentiable in $p \in A$ and derivative of that is a linear mapping $Dk(p) : A \to B$ defined by

$$Dk(p)(a) = \langle Df(p)(a^*), g(p) \rangle + \langle f(p), Dg(p)(a) \rangle.$$

Because

$$\begin{aligned}&\langle f(p+h), g(p+h)\rangle - \langle f(p), g(p)\rangle - \langle Df(p)(h^*), g(p)\rangle - \langle f(p), Dg(p)(h)\rangle\\ =&\langle f(p+h), g(p+h) - g(p) - Dg(p)(h)\rangle\rangle + \langle f(p+h) - f(p), Dg(p)(h)\rangle\\ &+ \langle f(p+h^*) - f(p) - Df(p)(h^*), g(p)\rangle + \langle f(p+h) - f(p+h^*), g(p)\rangle.\end{aligned}$$

Let A be a C^*-algebra and X a semi-inner product A-module. If $f \in D_p(A,X)$ and $a \in A$, we define the function $f_a : A \to X$ by $f_a(t) = f(t)a$.

Theorem 2.2. *Let X be a semi-inner product C^*-module on C^*-algebra A, and $p \in A, e \in X$. If $\langle e,e\rangle$ is an idempotent element in A, and $f,g \in D_p(A,X)$, then for every $a \in A$, the map $[\cdot,\cdot]_a : D_p(A,X) \times D_p(A,X) \to A$ with*

$$[f,g]_a := \Big\langle Df(p)(a), Dg(p)(a)\Big\rangle_1 + \Big\langle f(p), g(p)\Big\rangle_1 - D\Big\langle f(\cdot), g(\cdot)\Big\rangle_1(p)(a),$$

is a generalized semi-inner product on $D_p(A,X)$, where

$$\langle f(a), g(a)\rangle_1 = \langle f(a), g(a)\rangle - \langle f(a), e\rangle\langle e, g(a)\rangle.$$

Proof. First, we show that $f_a \in D_p(A,X)$ and $Df_a(p) = (Df(p))a$. There exists a bounded convex set $V(= B(p,r))$ containing p such that $V \subseteq U$. Let $p,h \in V$, $a \in A$, then

$$\begin{aligned}\|f_a(p+h) - f_a(p) - (Df(p)(h))a\| &= \|[f(p+h) - f(p) - Df(p)(h)]a\|\\ &\le \|f(p+h) - f(p) - Df(p)(h)\|\|a\|.\end{aligned}$$

This implies that $f_a \in D_p(A,X)$.

A simple calculation shows

$$\begin{aligned}[f,g]_a =&\Big\langle Df(p)(a) - f(p), Dg(p)(a) - g(p)\Big\rangle\\ &- \Big\langle Df(p)(a) - f(p), e\rangle\langle e, Dg(p)(a) - g(p)\Big\rangle\\ =&\Big\langle (Df(p)(a) - f(p)) - e\Big\langle e, (Df(p)(a) - f(p))\Big\rangle,\\ &(Dg(p)(a) - g(p)) - e\Big\langle e, (Dg(p)(a) - g(p))\Big\rangle\Big\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}[f,f]_a =&\Big\langle (Df(p)(a) - f(p)) - e\Big\langle e, (Df(p)(a) - f(p))\Big\rangle,\\ &(Df(p)(a) - f(p)) - e\Big\langle e, (Df(p)(a) - f(p))\Big\rangle\Big\rangle \ge 0.\end{aligned}$$

It is easy to show that $[\cdot,\cdot]_a$ is a generalized semi-inner product on $D_p(A,X)$. □

Lemma 2.1. *Let X be a semi-inner product C^*-module on C^*-algebra A, and $p,a \in A, e \in X$. If $\langle e,e\rangle$ is an idempotent element in A, $f,g \in D_p(A,X)$ and $\alpha, \beta, \alpha', \beta', \mu$,*

λ, μ', λ' *are complex numbers such that*

$$\begin{aligned}\left\|f(p) - \frac{\alpha+\beta}{2}e\right\| \leq & \frac{1}{2}|\alpha-\beta|,\\ \left\|Df(p)(a) - \frac{\alpha'+\beta'}{2}e\right\| \leq & \frac{1}{2}|\alpha'-\beta'|,\\ \left\|g(p) - \frac{\lambda+\mu}{2}e\right\| \leq & \frac{1}{2}|\lambda-\mu|,\\ \left\|Dg(p)(a) - \frac{\mu'+\lambda'}{2}e\right\| \leq & \frac{1}{2}|\mu'-\lambda'|,\end{aligned}$$

then the following inequality holds

$$\begin{aligned}&\|\langle Df(p)(a), Dg(p)(a)\rangle_1 + \langle f(p), g(p)\rangle_1 - D\Big\langle f(\cdot), g(\cdot)\Big\rangle_1(p)(a)\|\\ \leq & \frac{1}{2}(|\alpha-\beta| + |\alpha'-\beta'|)(|\lambda-\mu| + |\lambda'-\mu'|).\end{aligned}$$

Proof. Since $[\cdot,\cdot]_a$ is a generalized semi-inner product on $D_p(A,X)$, the Schwartz inequality holds, i.e,

$$\|[f,g]_a\|^2 \leq \|[f,f]_a\| \; \|[g,g]_a\|.$$

We know that

$$\begin{aligned}\|[f,f]_a\| \leq & \Big\|\langle Df(p)(a), Df(p)(a)\rangle - \langle Df(p)(a), e\rangle\langle e, Df(p)(a)\rangle\Big\|\\ & + \Big\|\langle f(p), f(p)\rangle - \langle f(p), e\rangle\langle e, f(p)\rangle\Big\|\\ & + \Big\|\langle Df(p)(a), f(p)\rangle - \langle Df(p)(a), e\rangle\langle e, f(p)\rangle\Big\|\\ & + \Big\|\langle f(p), Df(p)(a)\rangle - \langle f(p), e\rangle\langle e, Df(p)(a)\rangle\Big\|.\end{aligned}$$

This inequality and Theorem 2.1 imply that

$$\begin{aligned}\|[f,f]_a\| \leq & \frac{1}{4}|\alpha'-\beta'|^2 + \frac{1}{4}|\alpha-\beta|^2 + \frac{1}{2}|\alpha'-\beta'||\alpha-\beta|\\ = & \frac{1}{4}(|\alpha-\beta| + |\alpha'-\beta'|)^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\|[g,g]_a\| \leq & \frac{1}{4}|\lambda'-\mu'|^2 + \frac{1}{4}|\lambda-\mu|^2 + \frac{1}{2}|\lambda'-\mu'||\lambda-\mu|\\ = & \frac{1}{4}(|\lambda-\mu| + |\lambda'-\mu'|)^2.\end{aligned}$$

□

Let X be a semi-inner product C^*-module over C^*-algebra A. For every $x \in X$, we define the map $\hat{x} : A \to X^n$ by $\hat{x}(a) = (xa, \ldots, xa)$, $a \in A$.

Lemma 2.2. *Let X be a semi-inner product C^*-module, $x_0, y_0, x_1, y_1 \in X$ and $(r_1, r_2, \dots, r_n) \in \mathbb{R}^n$ a probability vector. If $p \in A$ and $f = (f_1, \dots, f_n)$, $g = (g_1, \dots, g_n) \in D_p(A, X^n)$ such that*

$$\left\| Df(p) - \widehat{\frac{x_0 + y_0}{2}} \right\| \leq \left\| \frac{x_0 - y_0}{2} \right\|$$

and

$$\left\| Dg(p) - \widehat{\frac{x_1 + y_1}{2}} \right\| \leq \left\| \frac{x_1 - y_1}{2} \right\|,$$

then for all $a \in A$, we have

$$\begin{aligned}&\left\| \sum_{i=1}^n r_i \left\langle Df_i(p)(a), Dg_i(p)(a) \right\rangle - \left\langle \sum_{i=1}^n r_i Df_i(p)(a), \sum_{i=1}^n r_i Dg_i(p)(a) \right\rangle \right\| \\ \leq& \frac{1}{4} \|x_0 - y_0\| \|x_1 - y_1\| \|a\|^2.\end{aligned}$$

Proof. For every $a \in A$, we define the map $\Big(\cdot, \cdot\Big)_a : D_p(A, X^n) \times D_p(A, X^n) \to A$ with

$$\Big(f, g\Big)_a = \sum_{i=1}^n r_i \Big\langle Df_i(p)(a), Dg_i(p)(a) \Big\rangle - \Big\langle \sum_{i=1}^n r_i Df_i(p)(a), \sum_{i=1}^n r_i Dg_i(p)(a) \Big\rangle.$$

The following Korkine type inequality for differentiable mappings holds:

$$\Big(f, g\Big)_a = \frac{1}{2} \sum_{i=1, j=1}^n r_i r_j \Big\langle Df_i(p)(a) - Df_j(p)(a), Dg_i(p)(a) - Dg_j(p)(a) \Big\rangle,$$

Therefore, $\Big(f, f\Big)_a \geq 0$. It is easy to show that $\Big(\cdot, \cdot\Big)_a$ is a generalized semi-inner product on $D_p(A, X^n)$.

A simple calculation shows that

$$\begin{aligned}\Big(f, g\Big)_a =& \sum_{i=1}^n r_i \Big\langle Df_i(p)(a) - \frac{x_0 + y_0}{2} a,\ Dg_i(p)(a) - \frac{x_1 + y_1}{2} a \Big\rangle \\ &- \Big\langle \sum_{i=1}^n r_i Df_i(p)(a) - \frac{x_0 + y_0}{2} a,\ \sum_{i=1}^n r_i Dg_i(p)(a) - \frac{x_1 + y_1}{2} a \Big\rangle.\end{aligned}$$

From Schwartz inequality, we have

$$\begin{aligned}\left\| \Big(f, g\Big)_a \right\|^2 &\leq \sum_{i=1}^n r_i \left\| Df_i(p)(a) - \frac{x_0 + y_0}{2} a \right\|^2 \sum_{i=1}^n r_i \left\| Dg_i(p)(a) - \frac{x_1 + y_1}{2} a \right\|^2 \\ &\leq \left\| Df(p) - \widehat{\frac{x_0 + y_0}{2}} \right\|^2 \left\| Dg(p) - \widehat{\frac{x_1 + y_1}{2}} \right\|^2 \|a\|^4 \\ &\leq \frac{1}{16} \|x_0 - y_0\|^2 \|x_1 - y_1\|^2 \|a\|^4.\end{aligned}$$

□

Corollary 2.1. *Let X be a semi-inner product C^*-module, $x_0, y_0 \in X$, $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ and $(r_1, r_2, \dots, r_n) \in \mathbb{R}^n$ a probability vector. If $p \in A$ and $f = (f_1, \dots, f_n) \in D_p(A, X^n)$ such that*

$$\left\| Df(p) - \widehat{\frac{x_0 + y_0}{2}} \right\| \le \left\| \frac{x_0 - y_0}{2} \right\|,$$

then for all $a \in A$, we have

$$\left\| \sum_{i=1}^{n} r_i \alpha_i Df_i(p)(a) - \sum_{i=1}^{n} r_i \alpha_i \sum_{i=1}^{n} r_i Df_i(p)(a) \right\|$$

$$\le \|a\| \left\| \frac{x_0 - y_0}{2} \right\| \left[\sum_{i=1}^{n} r_i |\alpha_i|^2 - \left| \sum_{i=1}^{n} r_i \alpha_i \right|^2 \right]^{\frac{1}{2}}. \tag{2.1}$$

Proof. We have

$$\begin{aligned} &\left\| \sum_{i=1}^{n} r_i \alpha_i Df_i(p)(a) - \sum_{i=1}^{n} r_i \alpha_i \sum_{i=1}^{n} r_i Df_i(p)(a) \right\| \\ =&\left| \sum_{i=1}^{n} r_i \Big(\alpha_i - \sum_{j=1}^{n} r_j \alpha_j \Big) \right| \left\| Df_i(p)(a) - \frac{x_0 + y_0}{2} a \right\| \\ \le& \sum_{i=1}^{n} r_i \left| \alpha_i - \sum_{j=1}^{n} r_j \alpha_j \right| \left\| Df(p) - \widehat{\frac{x_0 + y_0}{2}} \right\| \|a\| \\ \le& \|a\| \left\| \frac{x_0 - y_0}{2} \right\| \left[\sum_{i=1}^{n} r_i |\alpha_i|^2 - \left| \sum_{i=1}^{n} r_i \alpha_i \right|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

□

Corollary 2.2. *Let X be a semi-inner product C^*-module, $x_0, y_0 \in X$. If $p \in A$ and $f = (f_1, \dots, f_n) \in D_p(A, X^n)$ such that*

$$\left\| Df(p) - \widehat{\frac{x_0 + y_0}{2}} \right\| \le \left\| \frac{x_0 - y_0}{2} \right\|,$$

then for all $a \in A$, we have

$$\left\| \sum_{k=1}^{n} k Df_k(p)(a) - \frac{n+1}{2} . \sum_{k=1}^{n} Df_k(p)(a) \right\| \le \frac{\|a\| \|x_0 - y_0\| n}{4} \left[\frac{(n-1)(n+1)}{3} \right]^{\frac{1}{2}}, \tag{2.2}$$

and

$$\left\| \sum_{k=1}^{n} k^2 Df_k(p)(a) - \frac{(n+1)(2n+1)}{6} . \sum_{k=1}^{n} Df_k(p)(a) \right\| \tag{2.3}$$

$$\le \frac{\|a\| \|x_0 - y_0\| n}{12\sqrt{5}} \sqrt{(n-1)(n+1)(2n+1)(8n+11)}.$$

Proof. If we put $ri = \frac{1}{n}$, $\alpha_i = k$ in inequality (2.1), then we get (2.2), and if $ri = \frac{1}{n}, \alpha_i = k^2$ in inequality (2.1), then we get (2.3). □

3. Differentiable Mappings on Banach *-Algebras

Theorem 3.1. *Let A, B be two Banach $*$-algebras and $p \in A$, then $D_p(A,B)$ is a Banach $*$-algebra with the point-wise operations and the involution $f^*(a) = (f(a))^*$, $a \in A$, and the norm*

$$\|f\| := \max\left\{\sup_{x\in U}\|Df(x)\|,\ \sup_{a\in A}\|f(a)\|\right\} < \infty.$$

Proof. First we show that the involution $f \mapsto f^*$ is differentiable and $Df^*(x)(h) = (Df(x)(h^*))^*$, $x, h \in U$. It is trivial that $Df^*(x)$ is a bounded linear map with $\|Df^*(x)\| = \|Df(x)\|$ and

$$\begin{aligned}
&\|f^*(x+h) - f^*(x) - Df^*(x)(h)\| \\
=&\|\Big(f(x+h) - f(x) - Df(x)(h^*)\Big)^*\| \\
=&\|f(x+h) - f(x) - Df(x)(h^*)\| \\
=&\|f(x+h) - f(x) - Df(x)(h) + Df(x)(h) - Df(x)(h^*)\| \\
\leq&\epsilon\|h\| + \|Df(x)(h-h^*)\| \leq \epsilon\|h\| + 2\|Df(x)\|\|h\|.
\end{aligned}$$

From $\|Df^*(x)\| = \|Df(x)\|$ and $\|f^*(a)\| = \|f(a)\|$, we obtain

$$\begin{aligned}
\|f^*\| &= \max\left\{\sup_{x\in U}\|Df^*(x)\|,\ \sup_{a\in A}\|f^*(a)\|\right\} \\
&= \max\left\{\sup_{x\in U}\|Df(x)\|,\ \sup_{a\in A}\|f(a)\|\right\} = \|f\|.
\end{aligned}$$

Now, we show that $D_p(A,B)$ is complete. There exists a bounded convex set $V(= B(p,r))$ containing p such that $V \subseteq U$. Suppose that (f_n) is a Cauchy sequence in $D_p(A,B)$, i.e.,

$$\|f_n(a) - f_m(a)\| \to 0, \quad a \in A, \qquad \text{and} \qquad \|Df_n(x) - Df_m(x)\| \to 0, \quad x \in V.$$

Since B is complete, therefore $L(A,B)$ the space of all bounded linear maps from A into B, is complete. So, there are functions f, g such that $\sup_{a\in A}\|f_n(a) - f(a)\| \to 0$ and $\sup_{x\in V}\|Df_n(x) - g(x)\| \to 0$. Given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that for $m > n \geq N$ one has

$$\|Df_m - Df_n\|_\infty = \sup_{x\in V}\|Df_m(x) - Df_n(x)\| < \frac{\varepsilon}{3},$$

$$\|g - Df_n\|_\infty = \sup_{x\in V}\|g(x) - Df_n(x)\| < \frac{\varepsilon}{3}. \tag{3.1}$$

We may suppose that there exist $a \in A$ such that $p + a \in V$. Using Lipschitzian functions $f_m - f_n$, we obtain that

$$\begin{aligned}
&\|f_m(p+a) - f_m(p) - (f_n(p+a) - f_n(p))\| \\
&\leq \sup_{0<\theta<1}\|Df_m(p+\theta a) - Df_n(p+\theta a)\|\|a\| \leq \frac{\varepsilon}{3}\|a\|.
\end{aligned}$$

Passing to the limit on m, we get

$$\|f(p+a)-f(p)-(f_n(p+a)-f_n(p))\| \leq \frac{\varepsilon}{3}\|a\|. \tag{3.2}$$

Utilizing differentiability f_N and (3.1), we have

$$\begin{aligned}\|f_N(p+a)-f_N(p)-g(p)(a)\| \leq &\|f_N(p+a)-f_N(p)-Df_N(p)(a)\|\\ &+\|Df_N(p)(a)-g(p)(a)\| \leq \frac{\varepsilon}{3}\|a\|+\frac{\varepsilon}{3}\|a\|.\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$\|f(p+a)-f(p)-g(p)(a)\| \leq \varepsilon\|a\|.$$

Therefore, $D_p(A,B)$ is a Banach $*$-algebra. □

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