Kragujevac Journal of Mathematics
Volume 43(2) (2019), Pages 267–273.

INNER HIGHER DERIVATIONS ON ALGEBRAS

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Abstract. Let $\mathcal{A}$ be an algebra. A sequence $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ of linear operators on $\mathcal{A}$ is called a *higher derivation* if d_0 is the identity mapping on $\mathcal{A}$ and $d_n(xy) = \sum_{k=0}^{n} d_k(x)d_{n-k}(y)$, for each $n = 0, 1, 2, \ldots$ and $x, y \in \mathcal{A}$. We say that a higher derivation $\mathbf{d}$ is *inner* if there is a sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ such that $(n+1)d_{n+1}(x) = \sum_{k=0}^{n} a_{k+1}d_{n-k}(x) - d_{n-k}(x)a_{k+1}$, for each $n = 0, 1, 2, \ldots$ and $x \in \mathcal{A}$. Giving a characterization for inner higher derivations on a torsion free algebra $\mathcal{A}$, we show that each higher derivation on $\mathcal{A}$ is inner provided that each derivation on $\mathcal{A}$ is inner.

1. Introduction

Let $\mathcal{A}$ be an algebra. A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a *derivation* if it satisfies the *Leibniz rule* $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. If a is a fixed element of $\mathcal{A}$ then the linear mapping $\delta : \mathcal{A} \to \mathcal{A}$, defined by $\delta(x) = [a, x] = ax - xa$ for $x \in \mathcal{A}$, is a derivation. Such a derivation is called *inner* and is denoted by δ_a. Thus a derivation δ is inner if $\delta = \delta_a$ for some $a \in \mathcal{A}$. In this case we say that δ is an inner derivation *implemented by a*. For a discussion about derivations, inner derivations, automatic continuity of derivations and the related topics the reader can see [5] and [9].

There are known algebras on which each derivation is inner by an element of the algebra. Furthermore, there are algebras $\mathcal{A}$ for which each derivation is inner implemented by an element of an algebra $\mathcal{B}$ containing $\mathcal{A}$. For example, each derivation on a C^*-algebra $\mathfrak{A}$ is inner implemented by an element of its weak closure (see [11] and [6]).

When $\delta : \mathcal{A} \to \mathcal{A}$ is a derivation, the sequence $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$, defined by $d_n = \frac{\delta^n}{n!}$, satisfies the *generalized Leibniz rule* $d_n(xy) = \sum_{k=0}^{n} d_k(x)d_{n-k}(y)$, for each

Key words and phrases. Derivation, inner derivation, higher derivation, inner higher derivation.
2010 *Mathematics Subject Classification.* Primary: 16W25. Secondary: 47L57, 47B47, 13N15.
Received: August 20, 2017.
Accepted: November 20, 2017.

$n = 0, 1, 2, \dots$ and $x, y \in \mathcal{A}$. Since the sequence $\mathbf{d}$ deals with higher powers of δ, such a sequence is called a *higher derivation*. Though this is not the only example of a higher derivation, such a sequence is a typical example and is called an *ordinary higher derivation*. A theorem proved by the second named author [8] gives a one to one correspondence between higher derivations and the family of sequences of derivations in the sense that for each higher derivation $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ on a torsion free algebra $\mathcal{A}$, there is a sequence $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ of derivations on $\mathcal{A}$ such that

$$d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) \delta_{r_1} \cdots \delta_{r_i} \right), \tag{1.1}$$

where the inner summation is taken over all positive integers r_j, with $\sum_{j=1}^{i} r_j = n$. Conversely, if for a sequence $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ of derivations on $\mathcal{A}$ we define the sequence $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ by (1.1), then $\mathbf{d}$ is a higher derivation. If we denote this higher derivation by $\mathbf{d}_{\boldsymbol{\delta}}$ then we can say that each higher derivation on a torsion free algebra $\mathcal{A}$ is of the form $\mathbf{d}_{\boldsymbol{\delta}}$ for some sequence $\boldsymbol{\delta}$ of derivations on $\mathcal{A}$. For a study about higher derivations, their generalizations, automatic continuity of higher derivations the reader is referred to [1–4, 7, 10] and [12].

In the present paper, to a sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ of elements of $\mathcal{A}$, we correspond a sequence $\mathbf{d_a} = \{d_n\}_{n=0}^{\infty}$ of linear mappings on $\mathcal{A}$ in such a way that $\mathbf{d_a}$ is a higher derivation. We say that $\mathbf{d_a}$ is the inner higher derivation implemented by $\mathbf{a}$. A higher derivation is then inner if it is implemented by a sequence of elements of $\mathcal{A}$. Giving a characterization for a higher derivation to be inner, we show that each higher derivation on a torsion free algebra $\mathcal{A}$ is inner if and only if each derivation on $\mathcal{A}$ is inner.

In the following $\mathcal{A}$ is an algebra and I denotes the identity mapping on it. When we talk about a higher derivation $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ we will assume that d_0 is I. This implies that d_1 is a derivation on $\mathcal{A}$. When d_0 is not the identity mapping, we deals with the σ-derivation d_1, where σ is d_0.

2. The Results

Inner higher derivations will be defined as a special case of higher derivations. In the next definition we define an inner property for a sequence of linear operators on an algebra. As we will see in Lemma 2.1, a sequence of linear mapping with the inner property is automatically a higher derivation.

Definition 2.1. Let $\mathcal{A}$ be an algebra and $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ be a sequence of linear operators on $\mathcal{A}$ with $d_0 = I$. We say that $\mathbf{d}$ *has the inner property* if there is a sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ such that $(n+1)d_{n+1}(x) = \sum_{k=0}^{n} a_{k+1}d_{n-k}(x) - d_{n-k}(x)a_{k+1}$, for each $n = 0, 1, 2, \dots$ and $x \in \mathcal{A}$.

Prior to anything, we show that a sequence possessing the inner property is indeed a higher derivation.

Lemma 2.1. *Let $\mathcal{A}$ be an algebra and $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ have the inner property. Then* $\mathbf{d}$ *is a higher derivation.*

Proof. We use induction on n to show that $d_n(xy) = \sum_{k=0}^{n} d_k(x)d_{n-k}(y)$, for each $n = 0, 1, 2, \ldots$ and $x, y \in \mathcal{A}$. This is clear for $n = 0$, since $d_0 = I$. Suppose that this is true for n. For $n + 1$ we have

$$\begin{aligned}(n+1)d_{n+1}(xy) &= \sum_{k=0}^{n} \delta_{a_{k+1}}(d_{n-k}(xy)) \\ &= \sum_{k=0}^{n} \delta_{a_{k+1}}(\sum_{\ell=0}^{n-k} d_\ell(x)d_{n-k-\ell}(y)) \\ &= \sum_{k=0}^{n}\sum_{\ell=0}^{n-k} \delta_{a_{k+1}}(d_\ell(x))d_{n-k-\ell}(y) + \sum_{k=0}^{n}\sum_{\ell=0}^{n-k} d_\ell(x)\delta_{a_{k+1}}(d_{n-k-\ell}(y)).\end{aligned}$$

Changing variables implies

$$\begin{aligned}(n+1)d_{n+1}(xy) =&\sum_{r=0}^{n}\left[\sum_{k=0}^{r} \delta_{a_{k+1}}(d_{r-k}(x))\right] d_{n-r}(y) + \sum_{\ell=0}^{n} d_\ell(x)\left[\sum_{k=0}^{n-\ell}\delta_{a_{k+1}}(d_{n-k-\ell}(y))\right] \\ =&\sum_{r=0}^{n}\Big[(r+1)d_{r+1}(x)\Big]d_{n-r}(y) + \sum_{\ell=0}^{n} d_\ell(x)\left[(n-\ell+1)d_{n-\ell+1}(y)\right] \\ =&\sum_{\ell=1}^{n+1}\left[\ell d_\ell(x)\right] d_{n-\ell+1}(y) + \sum_{\ell=0}^{n} d_\ell(x)\Big[(n-\ell+1)d_{n-\ell+1}(y)\Big] \\ =&\sum_{\ell=1}^{n}(n+1)d_\ell(x)d_{n-\ell+1}(y) + (n+1)d_0(x)d_{n+1}(y) \\ &+ (n+1)d_{n+1}(x)d_0(y) \\ =&(n+1)\sum_{\ell=0}^{n+1} d_\ell(x)d_{n+1-\ell}(y).\end{aligned}$$

□

Definition 2.2. A higher derivation $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ is called *inner* if it has the inner property.

Lemma 2.1 does indeed show that any inner higher derivation is *a fortiori* a higher derivation. A natural question is: When does a higher derivation inner? Prior to provide an answer for the question, we give a characterization for inner higher derivations.

Lemma 2.2. *Let n, k and $r_1, \ldots, r_k$ be positive integers, $r_1 + \cdots + r_k = n + 1$ and $\alpha_{r_1,\ldots,r_i} = \prod_{j=1}^{i} \frac{1}{r_j+\cdots+r_i}$. Then $\alpha_{r_1,\ldots,r_i} = \frac{1}{n+1}\alpha_{r_2,\ldots,r_i}$.*

Proposition 2.1. *Let $\mathcal{A}$ be an algebra, $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{A}$ and $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ be defined by*

$$d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) \delta_{a_{r_1}} \cdots \delta_{a_{r_i}} \right), \tag{2.1}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$. Then $\mathbf{d}$ is an inner higher derivation.

Proof. Let $n+1 = r_1 + \cdots + r_i$, where r_j's are positive integers. If $r_1 = k+1$, where $0 \le k \le n$, then $n-k = r_2 + \cdots + r_i$. Putting $\alpha_{r_1,\ldots,r_i} = \prod_{j=1}^{i} \frac{1}{r_j+\cdots+r_i}$ and using Lemma 2.2 we have

$$\begin{aligned} d_{n+1} =& \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \alpha_{r_1,\ldots,r_i} \, \delta_{a_{r_1}} \cdots \delta_{a_{r_i}} \right) \\ =& \frac{1}{n+1} \sum_{k=0}^{n} \delta_{a_{k+1}} \sum_{i=1}^{n-k} \left(\sum_{\sum_{j=2}^{i} r_j = n-k} \alpha_{r_2,\ldots,r_i} \, \delta_{a_{r_2}} \cdots \delta_{a_{r_i}} \right) \\ =& \frac{1}{n+1} \sum_{k=0}^{n} \delta_{a_{k+1}} d_{n-k}. \end{aligned}$$

This shows that $\mathbf{d}$ has the inner property and hence is an inner higher derivation. □

The inner higher derivation $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$, defined as in Proposition 2.1, is denoted by $\mathbf{d_a}$ and is called *the inner higher derivation implemented by the sequence* $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$. When $\mathbf{a}$ is the sequence defined by $a_1 = a$ and $a_n = 0$, for $n \ge 2$, then $\mathbf{d_a}$ is called *the ordinary inner higher derivation implemented by* the element $a \in \mathcal{A}$.

Theorem 2.1. *Let $\mathcal{A}$ be a torsion free algebra and $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ be a higher derivation. Then $\mathbf{d}$ is inner if and only if $\mathbf{d} = \mathbf{d_a}$ for some sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ in $\mathcal{A}$. Furthermore, $\mathbf{d_a}$ is an ordinary inner higher derivation if and only if $d_n = \frac{\delta_a^n}{n!}$ for a fixed element $a \in \mathcal{A}$.*

Proof. Let $\mathbf{d}$ be an inner higher derivation. Then d_n satisfies the recursive relation $(n+1)d_n = \sum_{k=0}^{n} \delta_{a_{k+1}} d_{n-k}$, with the initial value $d_0 = I$. We know that the answer of this recursive relation with a fixed initial value is unique. Thus if we show that the sequence defined as in (2.1) of Proposition 2.1 satisfies the same recursive relation, then we can deduce that $\mathbf{d} = \mathbf{d_a}$. To show this we have

$$(n+1)d_{n+1} = \sum_{i=2}^{n+1} \left(\sum_{\sum_{j=1}^{i} r_j = n+1} (n+1)\alpha_{r_1,\ldots,r_i} \delta_{a_{r_1}} \cdots \delta_{a_{r_i}} \right) + \delta_{a_{n+1}}$$

$$
\begin{aligned}
&= \sum_{i=2}^{n+1} \left(\sum_{r_1=1}^{n+2-i} \delta_{r_1} \sum_{\sum_{j=2}^{i} r_j = n+1-r_1} \alpha_{r_2,\dots,r_i} \delta_{a_{r_2}} \cdots \delta_{a_{r_i}} \right) + \delta_{a_{n+1}} \\
&= \sum_{r_1=1}^{n} \delta_{r_1} \sum_{i=2}^{n-(r_1-1)} \left(\sum_{\sum_{j=2}^{i} r_j = n-(r_1-1)} \alpha_{r_2,\dots,r_i} \delta_{a_{r_2}} \cdots \delta_{a_{r_i}} \right) + \delta_{a_{n+1}} \\
&= \sum_{r_1=1}^{n} \delta_{a_{r_1}} d_{n-(r_1-1)} + \delta_{a_{n+1}} \\
&= \sum_{k=0}^{n} \delta_{a_{k+1}} d_{n-k}.
\end{aligned}
$$

Furthermore, if $a_1 = a$ and $a_n = 0$, for $n \geq 2$ then $\delta_{a_{r_j}} = 0$ whenever $r_j \geqslant 2$. Thus the inner summation of (2.1) is non-zero only if $r_1 = \cdots = r_i = 1$. In this case we should have $i = n$ and thus $\alpha_{r_1,\dots,r_i} = \frac{1}{n!}$. We therefore have $d_n = \frac{1}{n!}\delta_a^n$, since $\delta_{a_{r_1}} = \cdots = \delta_{a_{r_n}} = \delta_{a_1} = \delta_a$. □

We can construct various examples of inner higher derivations using different sequences of elements of $\mathcal{A}$. Let us see what can occur if we consider the constant sequence $\mathbf{a} = \{a\}_{n=1}^{\infty}$. We need a lemma and a notation.

Lemma 2.3. *Let n, i be positive integers with $i \leq n$. If $\beta_{n,i} = \sum_{\sum_{j=1}^{i} r_j = n} \alpha_{r_1,\dots,r_i}$ then $\beta_{n,i}$ satisfies the recursive relation $\beta_{n,i} = \frac{\beta_{n-1,i-1}+(n-1)\beta_{n-1,i}}{n}$ with the initial value $\beta_{1,1} = 1$. Moreover, $\beta_{n,1} = \frac{1}{n}, \beta_{n,n} = \frac{1}{n!}$ and $\sum_{i=1}^{n} \beta_{n,i} = 1$.*

Proof. At first we note that

$$
\beta_{n,i} = \sum_{\sum_{j=1}^{i} r_j = n} \alpha_{r_1,\dots,r_i} = \sum_{r_1=1}^{n-(i-1)} \frac{1}{n} \sum_{\sum_{j=2}^{i} r_j = n} \alpha_{r_2,\dots,r_i} = \frac{1}{n} \sum_{k=1}^{n+1-i} \beta_{n-k,i-1}.
$$

Now we have

$$
\begin{aligned}
\beta_{n,i} =& \frac{1}{n} \left(\beta_{n-1,i-1} + \sum_{k=2}^{n+1-i} \beta_{n-k,i-1} \right) \\
=& \frac{1}{n} \left(\beta_{n-1,i-1} + \sum_{\ell=1}^{n-i} \beta_{n-(\ell+1),i-1} \right) \\
=& \frac{1}{n} (\beta_{n-1,i-1} + (n-1)\beta_{n-1,i}).
\end{aligned}
$$

Note that $\beta_{n,0} = \beta_{n,n+1} = 0$. An inductive argument thus shows

$$
\beta_{n,1} = \frac{1}{n}(0 + (n-1)\beta_{n-1,1}) = \frac{1}{n}
$$

and

$$\beta_{n,n} = \frac{1}{n}(\beta_{n-1,n-1} + (n-1) \times 0) = \frac{1}{n} \cdot \frac{1}{(n-1)!} = \frac{1}{n!}.$$

Moreover,

$$\begin{aligned}\sum_{i=1}^{n} \beta_{n,i} =& \sum_{i=1}^{n} \frac{\beta_{n-1,i-1} + (n-1)\beta_{n-1,i}}{n} \\ =& \frac{1}{n}\left(\sum_{j=1}^{n-1} \beta_{n-1,j} + (n-1)\sum_{i=1}^{n-1} \beta_{n-1,i}\right) \\ =& \frac{1}{n}(1 + (n-1)) = 1.\end{aligned}$$

□

Values of $\beta_{n,i}$ for $1 \le i \le n \le 7$ are evaluated in the Table 1.

TABLE 1. Values of $\beta_{n,i}$ for $1 \le i \le n \le 7$

$n \diagdown i$	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1	0	0	0	0	0	0
2	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0
3	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$	0	0	0	0
4	$\frac{1}{4}$	$\frac{11}{24}$	$\frac{1}{4}$	$\frac{1}{24}$	0	0	0
5	$\frac{1}{5}$	$\frac{5}{12}$	$\frac{7}{24}$	$\frac{1}{12}$	$\frac{1}{120}$	0	0
6	$\frac{1}{6}$	$\frac{137}{360}$	$\frac{5}{16}$	$\frac{17}{144}$	$\frac{1}{48}$	$\frac{1}{720}$	0
7	$\frac{1}{7}$	$\frac{7}{20}$	$\frac{29}{90}$	$\frac{7}{48}$	$\frac{5}{144}$	$\frac{1}{240}$	$\frac{1}{5040}$

Example 2.1. Let $p_n(x) = \beta_{n,1}x + \beta_{n,2}x^2 + \cdots + \beta_{n,n}x^n$ and a be a fixed element of an algebra $\mathcal{A}$. Then $d_0 = I$ and $d_n = p_n(\delta_a)$ for positive integers n defines an inner higher derivation $\mathbf{d} = \{d_n\}_{n=0}^{\infty}$ on $\mathcal{A}$. To see this note that $\mathbf{d} = \mathbf{d_a}$, where $\mathbf{a}$ is the constant sequence $\{a\}_{n=1}^{\infty}$.

Our ultimate goal is to characterize those torsion free algebras on which all higher derivations are inner.

Theorem 2.2. *Let $\mathcal{A}$ be a torsion free algebra. Then each higher derivation on $\mathcal{A}$ is inner if and only if each derivation on $\mathcal{A}$ is inner.*

Proof. Theorem 2.3 of [8] shows that each higher derivation $\mathbf{d}$ on a torsion free algebra is of the form $\mathbf{d}_{\boldsymbol{\delta}}$ for some sequence $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ of derivations on $\mathcal{A}$. Theorem 2.1 says that $\mathbf{d}$ is inner if and only if it is defined by (2.1) of Proposition 2.1. This is equivalent to the fact that δ_n is of the form δ_{a_n} for an $a_n \in \mathcal{A}$.

For the converse note that if each higher derivation on $\mathcal{A}$ is inner and δ is an arbitrary derivation on $\mathcal{A}$ then innerness of the ordinary higher derivation $\mathbf{d} = \{d_n\}_{n=0}^{\infty} = \{\frac{\delta^n}{n!}\}_{n=0}^{\infty}$ implies that d_1 is of the form δ_{a_1} for some $a_1 \in \mathcal{A}$. This shows that $\delta = d_1$ is inner. □

Acknowledgements. This article is resulted from research project, named "Inner Higher Derivations on Algebras" supported by "Islamic Azad University-Bojnourd Branch".

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