

JORDAN IDEALS AND THE COMMUTATIVITY OF PRIME RINGS WITH HOMODERIVATIONS

MAHMOUD M. EL-SOUFI^{1,2}

ABSTRACT. This work examines the influence of homoderivations on Jordan ideals within prime rings, particularly regarding their effect on commutativity. We identify several algebraic conditions on a Jordan ideal $\mathfrak{J}$ of a prime ring $\mathfrak{B}$ (with $\text{char}(\mathfrak{B}) \neq 2$) under which the ring must be commutative. Among the various conditions studied are the following: (1) $\mathcal{H}$ is nilpotent on $\mathfrak{B}$ and centralizing on $\mathfrak{J}$; (2) $\mathcal{H}(v \circ \ell) = 0$; (3) $\mathcal{H}(v \circ \ell) = v \circ \ell$; (4) $\mathcal{H}([v, \ell]) = 0$; and (5) $\mathcal{H}$ is nilpotent on $\mathfrak{B}$, $\mathcal{H}(\mathcal{Z}(\mathfrak{B})) \neq (0)$, and $\mathcal{H}([v, \ell]) \in \mathcal{Z}(\mathfrak{B})$, for all $v, \ell \in \mathfrak{J}$. Our findings show that, under any of these conditions, $\mathfrak{B}$ is necessarily commutative.

1. INTRODUCTION

The present study focuses on the behavior of homoderivations acting on Jordan ideals in prime rings. While extensive research has been conducted on homoderivations acting on ideals, left ideals, and Lie ideals, demonstrating how certain algebraic conditions ensure these subsets lie within the center of the ring, relatively fewer investigations have addressed similar conditions applied specifically to Jordan ideals. This research aims to fill this gap by exploring algebraic constraints involving homoderivations on Jordan ideals, thereby advancing the understanding of commutativity and structural properties in prime rings.

Prior results concerning ideals, left ideals, and Lie ideals form an essential background for our approach. For instance, Alharfie and Muthana [2] investigated the commutativity of a prime ring $\mathfrak{B}$ with $\text{char}(\mathfrak{B}) \neq 2$ under the action of a non-zero homoderivation $\mathcal{H}$ defined on a non-zero left ideal $\mathfrak{U}$. They established that if, for all

Key words and phrases. Prime and semiprime ring, Jordan ideal, homoderivation.
2020 Mathematics Subject Classification. Primary: 16W25, 16N60, 16U80, 16R50.
DOI

Received: October 08, 2025.

Accepted: July 23, 2026.

$x, y \in \mathfrak{U}$, one of the following expressions lies in the center $\mathcal{Z}(\mathfrak{B})$: (1) $x\mathcal{H}(y) \pm xy$; (2) $x\mathcal{H}(y) \pm yx$; (3) $x\mathcal{H}(y) \pm [x, y]$; (4) $\mathcal{H}(xy) \pm xy$ or (5) $[\mathcal{H}(x), y] \pm yx$, then $\mathfrak{B}$ is commutative. Furthermore, they showed that if $\mathcal{H}$ satisfies $\mathcal{H}([x, y]) = \pm[x, y]$ for all x, y in a non-zero left ideal $\mathfrak{U} \subseteq \mathfrak{B}$, then the ring is necessarily commutative.

Abdelkarim et al. [4] proved that if a 2-torsion free semiprime ring $\mathfrak{B}$ admits a generalized homoderivation $\mathcal{H}$ (associated with a non-zero homoderivation h) such that $\mathcal{H}h(\mathfrak{J}) \neq (0)$ for a non-zero ideal $\mathfrak{J}$ of $\mathfrak{B}$, and one of the following holds for all $v, \ell \in \mathfrak{J}$: (i) $\mathcal{H}(v \circ \ell) = 0$; (ii) $\mathcal{H}(v \circ \ell) = v \circ \ell$ or (iii) $\mathcal{H}(v \circ \ell) = [v, \ell]$, then $\mathfrak{B}$ contains a non-zero central ideal. Additionally, Taoufiq and Belkadi [15] obtained several results regarding the nilpotency of homoderivations in semiprime rings.

Ali [11] demonstrated that, in a 2-torsion free semiprime ring $\mathfrak{B}$, a homoderivation $\mathcal{H}$ exhibits commuting behavior on any non-central square-closed Lie ideal $\mathfrak{L}$, provided that for each pair $\omega, \tau \in \mathfrak{L}$, at least one among several specified criteria is met. Illustrative examples of these criteria include: $\mathcal{H}(\omega \circ \tau) = 0$, $\mathcal{H}([\omega, \tau]) = 0$ or $\mathcal{H}([\omega, \tau]) = \omega \circ \tau$. In a related development, Güven [9] investigated the action of homoderivations on Lie ideals within prime rings and established conditions under which such Lie ideals are compelled to reside in the center. Specifically, for a prime ring $\mathfrak{B}$ with $\text{char}(\mathfrak{B}) \neq 2$ and a nontrivial homoderivation $\mathcal{H}$, any non-zero Lie ideal $\mathfrak{L}$ satisfying $[v, \mathcal{H}(\ell)] \in \mathcal{Z}(\mathfrak{B})$ for all $v, \ell \in \mathfrak{L}$ must be entirely contained in the center, i.e., $\mathfrak{L} \subseteq \mathcal{Z}(\mathfrak{B})$. Furthermore, Engin et al. [8] showed that in a prime ring $\mathfrak{B}$ of characteristic not equal to 2, equipped with a homoderivation $\mathcal{H}$, any nontrivial square-closed Lie ideal $\mathfrak{L}$ is necessarily central if at least one of the following scenarios holds: the image of $\mathfrak{L}$ under $\mathcal{H}$ is zero; $\mathcal{H}(\mathfrak{L})$ is contained in the center, for all $x, y \in \mathfrak{L}$, $\mathcal{H}(xy)$ equals either xy or yx or $\mathcal{H}$ annihilates all commutators $[\ell, v]$ with $\ell, v \in \mathfrak{L}$. Under these circumstances, the centrality of $\mathfrak{L}$ within $\mathfrak{B}$ is guaranteed.

Nilpotent homoderivations have become a subject of significant interest in recent mathematical research. In a notable result, Belkadi et al. [3] demonstrated that if $\mathfrak{B}$ is a prime ring, $m \geq 1$ is an integer, and $\mathcal{H}_1, \mathcal{H}_2$ are homoderivations on $\mathfrak{B}$ such that $[\mathcal{H}_1^m(t), \mathcal{H}_2^m(s)] = 0$ for all $t, s \in \mathfrak{B}$, then one of the following alternatives must occur: the ring $\mathfrak{B}$ is commutative, $\mathcal{H}_1^{3m-1} = 0$ or $\mathcal{H}_2^{3m-1} = 0$.

A Jordan ideal plays a significant role in understanding the symmetric structure of noncommutative rings. In this work, we investigate how certain algebraic identities imposed by homoderivations can force Jordan ideals to lie in the center of $\mathfrak{B}$, thereby influencing the global commutativity of the ring.

Motivated by previous studies, we shift our focus from left ideals, ideals, and Lie ideals to Jordan ideals, and investigate analogous identities involving homoderivations. We show that such identities yield new insights when applied to Jordan ideals, particularly when the homoderivation $\mathcal{H}$ is nilpotent.

Our aim is to establish conditions on a Jordan ideal $\mathfrak{J}$ of an associative prime ring $\mathfrak{B}$ (with $\text{char}(\mathfrak{B}) \neq 2$) under which $\mathfrak{B}$ is commutative, provided that any of the following identities holds for all $v, \ell \in \mathfrak{J}$.

If $\mathfrak{J}$ is a Jordan ideal that is also a subring, then $\mathfrak{B}$ is commutative whenever any of the following conditions holds: $\mathcal{H}$ is nilpotent on $\mathfrak{B}$ and centralizing on $\mathfrak{J}$; $\mathcal{H}(v \circ \ell) = 0$; $\mathcal{H}(v \circ \ell) = v \circ \ell$; $\mathcal{H}([v, \ell]) = 0$; $\mathcal{H}$ is surjective on $\mathfrak{J}$, nilpotent on $\mathfrak{B}$, and satisfies $\mathcal{H}(v) \circ \mathcal{H}(\ell) = 0$; $[\mathcal{H}(v), \ell] = [v, \mathcal{G}(\ell)]$, where $\mathcal{H}$ and $\mathcal{G}$ are nonzero homoderivations on $\mathfrak{B}$ that are zero-power valued on $\mathfrak{B}$; or $[\mathcal{H}(v), \mathcal{G}(\ell)] + [v, \ell] = 0$ where both $\mathcal{H}$ and $\mathcal{G}$ are nonzero homoderivations on $\mathfrak{B}$, and one of them is nilpotent on $\mathfrak{B}$ and surjective on $\mathfrak{J}$. On the other hand, if $\mathfrak{J}$ is a Jordan ideal not necessarily a subring, then the same conclusion holds if one of the following further identities is satisfied: $\mathcal{H}$ is zero-power valued on $\mathfrak{J}$ and $[\mathcal{H}(v), \mathcal{H}(\ell)] = 0$; or $\mathcal{H}$ is zero-power valued on $\mathfrak{J}$ and $[\mathcal{H}(v), \mathcal{H}(\ell)] = [v, \ell]$; or $\mathcal{H}$ is nilpotent on $\mathfrak{B}$, $\mathcal{H}(\mathcal{Z}(\mathfrak{B})) \neq (0)$, and $\mathcal{H}([v, \ell]) \in \mathcal{Z}(\mathfrak{B})$.

These findings highlight how Jordan ideals respond differently to homoderivations than Lie ideals do. Moreover, the presence of such identities often provides sufficient conditions for the full commutativity of the ring when applied to suitable Jordan ideals.

Let $\mathfrak{B}$ be an associative ring, and let $\mathcal{Z}(\mathfrak{B})$ denote its center. For any $v, \omega, \ell \in \mathfrak{B}$, the standard commutator is defined as $[v, \omega] = v\omega - \omega v$, while the corresponding anticommutator is given by $v \circ \omega = v\omega + \omega v$. It follows immediately that the following identity can be established:

$$(1.1) \quad [v, \omega\ell] = \omega[v, \ell] + [v, \omega]\ell,$$

$$(1.2) \quad v \circ (\omega\ell) = (v \circ \omega)\ell - \omega[v, \ell] = \omega(v \circ \ell) + [v, \omega]\ell.$$

We start with basic algebraic concepts. A ring $\mathfrak{B}$ earns the distinction of being prime if, whenever the product $a\mathfrak{B}b$ evaporates into nothingness, one must concede that either a or b was already zero. The algebraic fabric of $\mathfrak{B}$ may also feature additive subgroups $\mathfrak{J}$ with a penchant for commutators: if $v \circ r$ always returns to $\mathfrak{J}$ for any $v \in \mathfrak{J}$ and $r \in \mathfrak{B}$, then $\mathfrak{J}$ is a Jordan ideal.

In the exposition of [7], one encounters the notion of a homoderivation: an additive mapping $\mathcal{H} : \mathfrak{B} \rightarrow \mathfrak{B}$ that, for any $v, \ell \in \mathfrak{B}$, satisfies the relation $\mathcal{H}(v\ell) = \mathcal{H}(v)\ell + v\mathcal{H}(\ell) + \mathcal{H}(v)\mathcal{H}(\ell)$. Suppose S is a nonempty subset of $\mathfrak{B}$. We say that $\mathcal{H}$ is zero-power valued on S if it preserves S and, for each $\ell \in S$, there exists some integer $n(\ell) > 1$ such that $\mathcal{H}^{n(\ell)}(\ell) = 0$. When, for some fixed m , the m -fold iteration of $\mathcal{H}$ annihilates the entirety of $\mathfrak{B}$, while no smaller exponent achieves this feat, we declare $\mathcal{H}$ nilpotent of index m . $\mathcal{H}$ is commuting (centralizing) on S if $[\mathcal{H}(\ell), \ell] = 0$ ($[\mathcal{H}(\ell), \ell] \in \mathcal{Z}(\mathfrak{B})$) for all $\ell \in S$.

2. PRELIMINARIES

We summarize here several known results that will be frequently employed in the proofs.

Lemma 2.1. ([16, Lemma 2.6]). *Let $\mathfrak{B}$ be a prime ring with $\text{char}(\mathfrak{B}) \neq 2$, and let $\mathfrak{J}$ be a non-zero Jordan ideal of $\mathfrak{B}$. If $v, \ell \in \mathfrak{B}$ and $v\mathfrak{J}\ell = (0)$, then $v = 0$ or $\ell = 0$.*

Lemma 2.2. ([10, Lemma 2. 6]). *No group can coincide with the union of two of its proper subgroups.*

Lemma 2.3. ([1, Remark 2.1]). *In a prime ring $\mathfrak{B}$, the centralizer of any non-zero one-sided ideal is equal to the center of $\mathfrak{B}$; in particular, if $\mathfrak{B}$ has non-zero central ideal, $\mathfrak{B}$ must be commutative.*

Lemma 2.4 ([12]). *In a prime ring $\mathfrak{B}$ with $\text{char}(\mathfrak{B}) \neq 2$, if there exists an automorphism Ψ that acts as a centralizing and a nontrivial map on a Lie ideal $\mathfrak{L}$, then this condition necessarily implies that the Lie ideal $\mathfrak{L}$ lies entirely within the center of $\mathfrak{B}$, that is, $\mathfrak{L} \subseteq \mathcal{Z}(\mathfrak{B})$.*

Lemma 2.5. *Let $\mathfrak{B}$ be a 2-torsion free prime ring and $\mathfrak{J}$ a non-zero Jordan ideal of $\mathfrak{B}$.*

- (I) $4\mathfrak{B}v^2 \subseteq \mathfrak{J}$ and $4v^2\mathfrak{B} \subseteq \mathfrak{J}$ for all $v \in \mathfrak{J}$ (see [13]).
- (II) *If $\mathfrak{B}$ is noncommutative such that $\omega[r, v\ell]\tau = 0$ for all $v, \ell \in \mathfrak{J}, r \in \mathfrak{B}$, then $\omega = 0$ or $\tau = 0$ (see [13, Fact 3]).*
- (III) *If $[r, \ell^2] = 0$ for each $\ell \in \mathfrak{J}$, then $r \in \mathcal{Z}(\mathfrak{B})$ (see [13, Fact 5]).*

Lemma 2.6. ([6, Theorem 3.1]). *Let $\mathfrak{B}$ be a 2-torsion free semiprime ring, $\mathfrak{J}$ a non-zero Jordan ideal, and $\mathcal{H}$ an additive mapping on $\mathfrak{B}$. If $\mathcal{H}$ is centralizing on $\mathfrak{J}$, then $\mathcal{H}$ is commuting on $\mathfrak{J}$.*

Lemma 2.7. ([16, Lemma 2.4]). *Let $\mathfrak{B}$ be a ring and $\mathfrak{J}$ a non-zero Jordan ideal of $\mathfrak{B}$. Then, $2[\mathfrak{B}, \mathfrak{B}]\mathfrak{J} \subseteq \mathfrak{J}$ and $2\mathfrak{J}[\mathfrak{B}, \mathfrak{B}] \subseteq \mathfrak{J}$.*

Lemma 2.8. ([16, Lemma 2.5]). *Let $\mathfrak{B}$ be a prime ring and $\mathfrak{J}$ a non-zero Jordan ideal of $\mathfrak{B}$. If $a \in \mathfrak{B}$ and $\mathfrak{J}a = (0)$ (or $a\mathfrak{J} = (0)$), then $a = 0$.*

Lemma 2.9. ([16, Lemma 2.7]). *Let $\mathfrak{B}$ be a 2-torsion free prime ring and $\mathfrak{J}$ a non-zero Jordan ideal of $\mathfrak{B}$. If $\mathfrak{J}$ is a commutative Jordan ideal, then $\mathfrak{J} \subseteq \mathcal{Z}(\mathfrak{B})$.*

Lemma 2.10. ([5, Lemma 1]). *Let $\mathfrak{B}$ be a semiprime ring and let $\mathcal{L}$ be a non-zero ideal of $\mathfrak{B}$. Suppose that $b \in \mathfrak{B}$ is such that $[b, [v, \ell]] = 0$ for all $v, \ell \in \mathcal{L}$. Then, $[b, v] = 0$ for all $v \in \mathcal{L}$.*

Lemma 2.11. ([14, Lemma 2.2]). *Let $\mathfrak{B}$ be a 2-torsion free semiprime ring and $\mathfrak{J} \not\subseteq \mathcal{Z}(\mathfrak{B})$ be a Jordan ideal of $\mathfrak{B}$. Then, $\mathfrak{J}$ contains a non-zero ideal of $\mathfrak{B}$.*

Lemma 2.12. *Let $\mathfrak{B}$ be a 2-torsion free prime ring with characteristic not equal to 2 and $\mathfrak{J}$ a Jordan ideal of $\mathfrak{B}$. If $v^2 = 0$ for all $v \in \mathfrak{J}$, then $\mathfrak{J} = (0)$.*

Proof. We begin by supposing that $\mathfrak{J} \neq (0)$ and

$$(2.1) \quad v^2 = 0, \quad \text{for all } v \in \mathfrak{J}.$$

Then, there exists a non-zero element $\ell \in \mathfrak{J}$ and $\ell^2 = 0$. In (2.1), replacing v by $v + \ell$, we get $v\ell + \ell v = 0$ for all $v \in \mathfrak{J}$. That is, $v\ell = -\ell v$ for all $v \in \mathfrak{J}$. Therefore,

$$\ell v \ell = -\ell^2 v = 0, \quad \text{for all } v \in \mathfrak{J}.$$

So, $\ell\mathfrak{J}\ell = (0)$. By Lemma 2.1, $\ell = 0$, a contradiction. Therefore, $\mathfrak{J} = (0)$. □

Lemma 2.13. *Let $\mathfrak{B}$ be a 2-torsion free prime ring and $\mathfrak{J}$ a non-zero Jordan ideal of $\mathfrak{B}$. If $\mathfrak{J} \subseteq \mathcal{Z}(\mathfrak{B})$, then $\mathfrak{B}$ is commutative.*

Proof. Suppose that $\mathfrak{J} \subseteq \mathcal{Z}(\mathfrak{B})$. Since $\mathfrak{J}$ is a non-zero Jordan ideal of $\mathfrak{B}$, then, by Lemma 2.7, $2[a, b]\ell \in \mathcal{Z}(\mathfrak{B})$ for all $a, b \in \mathfrak{B}, \ell \in \mathfrak{J}$. Therefore,

$$[2[a, b]\ell, t] = 0, \quad \text{for all } a, b, t \in \mathfrak{B}, \ell \in \mathfrak{J}.$$

This implies that

$$[[a, b], t]\ell = 0, \quad \text{for all } a, b, t \in \mathfrak{B}, \ell \in \mathfrak{J}.$$

By Lemma 2.8, we get

$$(2.2) \quad [[a, b], t] = 0, \quad \text{for all } a, b, t \in \mathfrak{B}.$$

Replacing a by ab in (2.2) and using (2.2), we obtain $[a, b][b, t] = 0$ for all $a, b, t \in \mathfrak{B}$. Replacing t by rt , we get $[a, b]r[b, t] = 0$ for all $a, b, t, r \in \mathfrak{B}$. In particular, $[a, t]\mathfrak{B}[a, t] = (0)$ for all $a, t \in \mathfrak{B}$. By primeness of $\mathfrak{B}$, $\mathfrak{B}$ is commutative. $\square$

Lemma 2.14. *Let $\mathfrak{B}$ be a prime ring with $\text{char}(\mathfrak{B}) \neq 2$ and let $\mathfrak{J} \neq (0)$ be a Jordan ideal of $\mathfrak{B}$ that is also a subring. If $\mathcal{H}$ is a homoderivation of $\mathfrak{B}$ that vanishes identically on $\mathfrak{J}$, then $\mathcal{H}$ vanishes identically on $\mathfrak{B}$.*

Proof. Suppose that $\mathcal{H}$ is a homoderivation of $\mathfrak{B}$ that vanishes identically on $\mathfrak{J}$. Then,

$$(2.3) \quad \mathcal{H}(\ell) = 0, \quad \text{for any } \ell \in \mathfrak{J}.$$

We substitute ℓ with $\ell \circ r$, where $r \in \mathfrak{B}$, and using (2.3), we obtain

$$(2.4) \quad \ell\mathcal{H}(r) + \mathcal{H}(r)\ell = 0, \quad \text{for any } \ell \in \mathfrak{J}, r \in \mathfrak{B}.$$

We substitute ℓ with $v\ell$, where $v \in \mathfrak{J}$, we obtain

$$(2.5) \quad v\ell\mathcal{H}(r) + \mathcal{H}(r)v\ell = 0, \quad \text{for any } \ell \in \mathfrak{J}, r \in \mathfrak{B}.$$

By (2.4), (2.5), we get

$$v[\ell, \mathcal{H}(r)] = 0, \quad \text{for any } \ell, v \in \mathfrak{J}, r \in \mathfrak{B}.$$

By Lemma 2.8,

$$(2.6) \quad [\ell, \mathcal{H}(r)] = 0, \quad \text{for any } \ell \in \mathfrak{J}, r \in \mathfrak{B}.$$

By (2.4), (2.6), and $\text{char}(\mathfrak{B}) \neq 2$, we obtain

$$\ell\mathcal{H}(r) = 0, \quad \text{for any } \ell \in \mathfrak{J}, r \in \mathfrak{B}.$$

Again, by Lemma 2.8, $\mathcal{H}(r) = 0$ for any $r \in \mathfrak{B}$. $\square$

3. MAIN RESULTS ABOUT JORDAN IDEAL SUBRING

Here, we delve into the interplay between homoderivations and Jordan ideals, seeking to unravel how specific algebraic stipulations imposed on these maps reverberate through the architecture of prime rings. We anchor our discussion in the context where $\mathfrak{B}$ stands as a prime ring of characteristic not equal to 2, $\mathfrak{J}$ emerges as a non-zero Jordan ideal that simultaneously forms a subring, and $\mathcal{H}$ and $\mathcal{G}$ represent non-zero homoderivations on $\mathfrak{B}$. This constellation of assumptions serves as the launchpad for the central insights presented in this work.

Theorem 3.1. *If $\mathcal{H}$ is nilpotent on $\mathfrak{B}$ and centralizing on $\mathfrak{J}$, then $\mathfrak{J} \subseteq \mathcal{Z}(\mathfrak{B})$ and hence, $\mathfrak{B}$ is commutative.*

Proof. Since $\mathcal{H}$ is homoderivation of $\mathfrak{B}$ and centralizing on $\mathfrak{J}$, it follows from Lemma 2.6 that $\mathcal{H}$ is commuting on $\mathfrak{J}$.

Given that $\mathcal{H}$ is a nilpotent additive map on $\mathfrak{B}$, it follows that $\mathcal{H} + I_{id}$, where I_{id} denotes the identity map, is invertible. Consequently, $\mathcal{H} + I_{id}$ is a bijection on $\mathfrak{B}$. Now, we define $\Psi : \mathfrak{B} \rightarrow \mathfrak{B}$ as follows: $\Psi = \mathcal{H} + I_{id}$. Hence, Ψ is an automorphism of $\mathfrak{B}$.

Suppose $\mathfrak{J} \not\subseteq \mathcal{Z}(\mathfrak{B})$, then, by Lemma 2.11, there exists a non-zero ideal $\mathfrak{L}$ of $\mathfrak{B}$ such that $\mathfrak{L} \subseteq \mathfrak{J}$.

Assume that Ψ vanishes on $\mathfrak{L}$. Then,

$$(3.1) \quad \mathcal{H} = -I_{id}, \quad \text{on } \mathfrak{L}.$$

Using the nilpotency of $\mathcal{H}$ together with (3.1), it follows that

$$(3.2) \quad \mathcal{H} = 0, \quad \text{on } \mathfrak{L}.$$

By (3.1) and (3.2), then $\mathfrak{L} = (0)$, a contradiction.

Suppose that $\Psi(v) = v$ for any $v \in \mathfrak{L}$. Then $\mathcal{H} = 0$ on $\mathfrak{L}$. Therefore, $\mathcal{H}(vr) = 0$ for all $v \in \mathfrak{L}, r \in \mathfrak{B}$. So,

$$v\mathcal{H}(r) = 0, \quad \text{for all } v \in \mathfrak{L}, r \in \mathfrak{B}.$$

By Lemma 2.8 and fact that every ideal is Jordan ideal, $\mathcal{H}(r) = 0$ for all $r \in \mathfrak{B}$, a contradiction.

Hence, Ψ is nontrivial on $\mathfrak{L}$. Since Ψ is commuting on $\mathfrak{J}$, then Ψ is commuting on $\mathfrak{L}$. Therefore, by Lemma 2.4 and every ideal is Lie ideal, $\mathfrak{L} \subseteq \mathcal{Z}(\mathfrak{B})$. Hence, by Lemma 2.3, $\mathfrak{B}$ is commutative ring. Therefore, $\mathfrak{J} \subseteq \mathcal{Z}(\mathfrak{B})$, which contradicts our assumptions. Thus, $\mathfrak{J}$ must be contained in $\mathcal{Z}(\mathfrak{B})$. With the aid of Lemma 2.13, $\mathfrak{B}$ is commutative. $\square$

Theorem 3.2. *If $\mathcal{H}(v \circ \ell) = 0$ for each $v, \ell \in \mathfrak{J}$, then $\mathfrak{B}$ is commutative.*

Proof. By the theorem's assumption, we are given that

$$(3.3) \quad \mathcal{H}(v \circ \ell) = 0, \quad \text{for each } \ell, v \in \mathfrak{J}.$$

Substituting ℓ with $v\ell$ in (3.3), and using the identity (1.2) and (3.3), it follows that

$$(3.4) \quad \mathcal{H}(v)(v \circ \ell) = 0, \quad \text{for any } v, \ell \in \mathfrak{J}.$$

Substituting ℓ with $\ell\omega$ in (3.4), together with the identity (3.4), we get

$$(3.5) \quad \mathcal{H}(v)\ell[v, \omega] = 0, \quad \text{for any } \omega, v, \ell \in \mathfrak{J}.$$

Therefore, by Lemma 2.1, for each $v \in \mathfrak{J}$, either $\mathcal{H}(v) = 0$ or $[v, \omega] = 0$ for any $\omega \in \mathfrak{J}$. If $\mathfrak{J}_1 = \{v \in \mathfrak{J} : \mathcal{H}(v) = 0\}$ and $\mathfrak{J}_2 = \{v \in \mathfrak{J} : [v, \omega] = 0 \text{ for any } \omega \in \mathfrak{J}\}$, then by Lemma 2.2, $\mathfrak{J}_1 = \mathfrak{J}$ or $\mathfrak{J}_2 = \mathfrak{J}$. Therefore, either $\mathcal{H} = 0$ on $\mathfrak{J}$ or $[\mathfrak{J}, \mathfrak{J}] = (0)$. If $\mathcal{H} = 0$ on $\mathfrak{J}$, then by Lemma 2.14, $\mathcal{H} = 0$ on $\mathfrak{B}$, a contradiction. Hence, $[\mathfrak{J}, \mathfrak{J}] = (0)$. By Lemma 2.9, $\mathfrak{J} \subseteq \mathcal{Z}(\mathfrak{B})$. Thus, by Lemma 2.13, $\mathfrak{B}$ is commutative. $\square$

Theorem 3.3. *If $\mathcal{H}(v \circ \ell) = v \circ \ell$ for any $v, \ell \in \mathfrak{J}$, then $\mathfrak{B}$ is commutative.*

Proof. By the theorem's assumption, we have

$$(3.6) \quad \mathcal{H}(v \circ \ell) = v \circ \ell, \quad \text{for any } v, \ell \in \mathfrak{J}.$$

Substituting ℓ with ℓv in (3.6) and using the fact that $\text{char}(\mathfrak{B}) \neq 2$, (1.2) and (3.6), we obtain

$$(3.7) \quad (v \circ \ell)\mathcal{H}(v) = 0, \quad \text{for any } v, \ell \in \mathfrak{J}.$$

Substituting ℓ with ℓw in (3.7), and using (1.2), (3.7), we obtain

$$(3.8) \quad [v, \ell]w\mathcal{H}(v) = 0, \quad \text{for any } v, \ell, w \in \mathfrak{J}.$$

As in (3.5), (3.8) implies that $\mathfrak{B}$ is commutative. $\square$

Corollary 3.1. *If $\mathcal{H}(v\ell) = v\ell$ for any $v, \ell \in \mathfrak{J}$, then $\mathfrak{B}$ is commutative.*

Theorem 3.4. *If $\mathcal{H}([v, \ell]) = 0$ for each $v, \ell \in \mathfrak{J}$, then $\mathfrak{B}$ is commutative.*

Proof. By assumption of the theorem, we are given that

$$(3.9) \quad \mathcal{H}([v, \ell]) = 0, \quad \text{for each } \ell, v \in \mathfrak{J}.$$

Substituting ℓ with ℓv in (3.9), along with identity (3.9), we get

$$(3.10) \quad [v, \ell]\mathcal{H}(v) = 0, \quad \text{for any } v, \ell \in \mathfrak{J}.$$

Substituting ℓ with $\ell\omega$ in (3.10), along with identity (3.10), we get

$$(3.11) \quad [v, \ell]\omega\mathcal{H}(v) = 0, \quad \text{for any } \omega, v, \ell \in \mathfrak{J}.$$

As in (3.5), (3.11) implies to $\mathfrak{B}$ is commutative. $\square$

Theorem 3.5. *If $\mathcal{H}$ is surjective on $\mathfrak{J}$, nilpotent on $\mathfrak{B}$ and satisfies $\mathcal{H}(v) \circ \mathcal{H}(\ell) = 0$ for any $v, \ell \in \mathfrak{J}$, then it necessarily follows that $\mathfrak{B}$ is commutative.*

Proof. To explore the structural implications of our assumptions, we begin by supposing that $\mathfrak{J} \not\subseteq \mathcal{Z}(\mathfrak{B})$. Suppose that the following identity holds:

$$(3.12) \quad \mathcal{H}(v) \circ \mathcal{H}(\ell) = 0, \quad \text{for any } v, \ell \in \mathfrak{J}.$$

By substituting ℓ with ℓz , and using (3.12), we obtain

$$(\mathcal{H}(v) \circ \ell)\mathcal{H}(z) - \mathcal{H}(\ell)[\mathcal{H}(v), \mathcal{H}(z)] - \ell[\mathcal{H}(v), \mathcal{H}(z)] - \mathcal{H}(\ell)[\mathcal{H}(v), z] = 0,$$

for any $v, \ell, z \in \mathfrak{J}$. Setting $z = \mathcal{H}(v)$, we obtain

$$(\mathcal{H}(v) \circ \ell)\mathcal{H}^2(v) - (\ell + \mathcal{H}(\ell))[\mathcal{H}(v), \mathcal{H}^2(v)] = 0,$$

for any $v, \ell \in \mathfrak{J}$. Putting $\mathcal{H}(\ell)$ instead of ℓ and using (3.12), we arrive at $\mathcal{H}(\ell + \mathcal{H}(\ell))[\mathcal{H}(v), \mathcal{H}^2(v)] = 0$ for any $v, \ell \in \mathfrak{J}$. By assumption, $\mathcal{H}$ is surjective on $\mathfrak{J}$ and nilpotent on $\mathfrak{B}$. Hence, $\mathcal{H}$ is zero-power valued on $\mathfrak{J}$. Consequently,

$$(3.13) \quad \mathcal{H}(\ell)[\mathcal{H}(v), \mathcal{H}^2(v)] = 0, \quad \text{for any } v, \ell \in \mathfrak{J}.$$

Setting $\ell = \ell z$ in (3.13) and using (3.13), we get $\mathcal{H}(\ell)z[\mathcal{H}(v), \mathcal{H}^2(v)] = 0$ for any $v, \ell, z \in \mathfrak{J}$. Therefore, by Lemma 2.1, either $\mathcal{H}(\ell) = 0$ for any $\ell \in \mathfrak{J}$ or $[\mathcal{H}(v), \mathcal{H}^2(v)] = 0$ for any $v \in \mathfrak{J}$. Since $\mathcal{H}$ is surjective on $\mathfrak{J}$, it follows that either $\mathfrak{J} = (0)$, which is a contradiction, or, again using the surjectivity of $\mathcal{H}$ on $\mathfrak{J}$, $[v, \mathcal{H}(v)] = 0$ for every $v \in \mathfrak{J}$. Hence, by Theorem 3.1, $\mathfrak{B}$ is commutative. $\square$

Theorem 3.6. *If $\mathcal{H}, \mathcal{G}$ are non-zero homoderivations of $\mathfrak{B}$ such that $[\mathcal{H}(v), \ell] = [v, \mathcal{G}(\ell)]$ for any $v, \ell \in \mathfrak{J}$ and both $\mathcal{H}$ and $\mathcal{G}$ are zero-power valued on $\mathfrak{J}$, then $\mathfrak{B}$ is commutative.*

Proof. By assumption, we have the following identity holds:

$$(3.14) \quad [\mathcal{H}(v), \ell] = [v, \mathcal{G}(\ell)], \quad \text{for any } v, \ell \in \mathfrak{J}.$$

Substituting v with τv in (3.14) and using (3.14), this yields

$$(3.15) \quad \mathcal{H}(\tau)[v + \mathcal{H}(v), \ell] + [\tau + \mathcal{H}(\tau), \ell]\mathcal{H}(v) = 0, \quad \text{for any } v, \tau, \ell \in \mathfrak{J}.$$

Next, replacing ℓ by $\tau + \mathcal{H}(\tau)$ in (3.15), we gain $\mathcal{H}(\tau)[v + \mathcal{H}(v), \tau + \mathcal{H}(\tau)] = 0$ for any $v, \tau \in \mathfrak{J}$. The assumption that $\mathcal{H}$ is zero-power valued on $\mathfrak{J}$ leads to the identity

$$(3.16) \quad \mathcal{H}(\tau)[v, \tau + \mathcal{H}(\tau)] = 0, \quad \text{for any } v, \tau \in \mathfrak{J}.$$

By (3.14) and (3.16), we have $\mathcal{H}(\tau)[v + \mathcal{G}(v), \tau] = 0$ for any $\tau, v \in \mathfrak{J}$. Using the fact that $\mathcal{G}$ is also zero-power valued on $\mathfrak{J}$, we further conclude that

$$(3.17) \quad \mathcal{H}(\tau)[v, \tau] = 0, \quad \text{for any } \tau, v \in \mathfrak{J}.$$

Now, replacing v with $v\ell$ in (3.17), we obtain

$$(3.18) \quad \mathcal{H}(\tau)v[\ell, \tau] = 0, \quad \text{for any } \tau, \ell, v \in \mathfrak{J}.$$

But (3.18) is exactly the same identity as (3.5). Therefore, $\mathfrak{B}$ is commutative. $\square$

Theorem 3.7. *Suppose that one of $\mathcal{H}, \mathcal{G}$ is nilpotent on $\mathfrak{B}$ and surjective on $\mathfrak{J}$. If $[\mathcal{H}(v), \mathcal{G}(\ell)] + [v, \ell] = 0$ for any $\ell, v \in \mathfrak{J}$, then $\mathfrak{B}$ is commutative.*

Proof. Assume that $\mathcal{G}$ is nilpotent on $\mathfrak{B}$ and surjective on $\mathfrak{J}$. By hypotheses, we have

$$(3.19) \quad [\mathcal{H}(v), \mathcal{G}(\ell)] + [v, \ell] = 0, \quad \text{for any } v, \ell \in \mathfrak{J}.$$

By substituting $v \mapsto v\tau$, where $\tau \in \mathfrak{J}$, into (3.19), it follows that

$$[v, \mathcal{G}(\ell) - \ell]\mathcal{H}(\tau) + \mathcal{H}(v)[\tau, \mathcal{G}(\ell) - \ell] = 0, \quad \text{for any } v, \tau, \ell \in \mathfrak{J}.$$

Since $\mathcal{G}$ is surjective on $\mathfrak{J}$, we replace τ by $\mathcal{G}(\ell) - \ell$ to get

$$[v, \mathcal{G}(\ell) - \ell]\mathcal{H}(\mathcal{G}(\ell) - \ell) = 0, \quad \text{for any } v, \ell \in \mathfrak{J}.$$

By substituting v with τv , we gain $[\tau, \mathcal{G}(\ell) - \ell]v\mathcal{H}(\mathcal{G}(\ell) - \ell) = 0$ for any $v, \tau, \ell \in \mathfrak{J}$. In light of Lemma 2.1, it follows that for every $\ell \in \mathfrak{J}$, either $\mathcal{H}(\mathcal{G}(\ell) - \ell) = 0$ or $\mathcal{G}(\ell) - \ell$ belongs to the center $\mathcal{Z}(\mathfrak{J})$. Define $\mathfrak{J}_3 = \{\ell \in \mathfrak{J} : \mathcal{H}(\mathcal{G}(\ell) - \ell) = 0\}$ and $\mathfrak{J}_4 = \{\ell \in \mathfrak{J} : \mathcal{G}(\ell) - \ell \in \mathcal{Z}(\mathfrak{J})\}$. It is clear that both $\mathfrak{J}_3$ and $\mathfrak{J}_4$ are additive subgroups of $\mathfrak{J}$. Consequently, either $\mathfrak{J}_3 = \mathfrak{J}$ or $\mathfrak{J}_4 = \mathfrak{J}$.

If $\mathfrak{J}_3 = \mathfrak{J}$, then substituting v by $\mathcal{G}(\ell) - \ell$ in (3.19) shows that $\mathcal{G}$ commuting on $\mathfrak{J}$. Similarly, if $\mathfrak{J}_4 = \mathfrak{J}$, then $\mathcal{G}$ is also commuting on $\mathfrak{J}$. Therefore, by Theorem 3.1, it follows that $\mathfrak{B}$ is commutative, which completes the proof. $\square$

The following example illustrates that the primeness of the ring is an essential hypothesis in all Theorems from 3.1 to 3.7. Without this assumption, the ring need not be commutative.

Example 3.1. Let $\mathfrak{B} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{Z}_3 \right\}$ be the ring of 2×2 matrices over $\mathbb{Z}_3$ equipped with standard addition and multiplication. It is apparent that, $\mathcal{Z}(\mathfrak{B}) = (0)$, $\text{char}(\mathfrak{B}) = 3$ and $\mathfrak{B}$ is non-prime because

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathfrak{B} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = (0),$$

but neither $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ nor $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is zero. Let $\mathfrak{J} = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \in \mathfrak{B} \mid b \in \mathbb{Z}_3 \right\}$. Then, $\mathfrak{J}$ is a non-zero Jordan ideal and subring of $\mathfrak{B}$.

To proceed, we define the following mapping $\mathcal{H} : \mathfrak{B} \rightarrow \mathfrak{B}$ by $\mathcal{H} \left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$.

One can directly verify that $\mathcal{H}$ is a non-zero homoderivation of $\mathfrak{B}$ with $\mathcal{H}^2 = 0$, and that it is zero-power valued on the Jordan ideal $\mathfrak{J}$. Moreover, all the conditions in the theorems from 3.1 to 3.7 are satisfied. However, $\mathfrak{B}$ is not a commutative ring.

4. MAIN RESULTS ABOUT JORDAN IDEAL

This section examines the interplay of homoderivations and Jordan ideals, analyzing how algebraic constraints on these mappings shape the structure of prime rings. We operate under the framework where $\mathfrak{B}$ is a prime ring with $\text{char}(\mathfrak{B}) \neq 2$, $\mathfrak{J}$ is a non-zero Jordan ideal (not necessarily a subring), and $\mathcal{H}$ is a non-zero homoderivation on $\mathfrak{B}$.

Theorem 4.1. *If $\mathcal{H}$ is zero-power valued on $\mathfrak{J}$ and $[\mathcal{H}(v), \mathcal{H}(\ell)] = [v, \ell]$ for each $v, \ell \in \mathfrak{J}$, then $\mathfrak{B}$ is commutative.*

Proof. By hypotheses, we have

$$(4.1) \quad [\mathcal{H}(v), \mathcal{H}(\ell)] = [v, \ell], \quad \text{for any } v, \ell \in \mathfrak{J}.$$

According to (I) of Lemma 2.5, substituting $v \mapsto 4v\tau^2$, where $\tau \in \mathfrak{J}$, into (4.1), and using (4.1) and $2\tau^2 \in \mathfrak{J}$, it follows that

$$\mathcal{H}(v)[\tau^2, \ell + \mathcal{H}(\ell)] + [v, \ell + \mathcal{H}(\ell)]\mathcal{H}(\tau^2) = 0, \quad \text{for any } v, \ell, \tau \in \mathfrak{J}.$$

Using the fact that $\mathcal{H}$ is also zero-power valued on $\mathfrak{J}$, we further conclude that

$$(4.2) \quad \mathcal{H}(v)[\tau^2, \ell] + [v, \ell]\mathcal{H}(\tau^2) = 0, \quad \text{for any } v, \ell, \tau \in \mathfrak{J}.$$

By substituting $v \mapsto 2rv^2$, where $r \in \mathfrak{B}$, into (4.2), and using (4.2), yields

$$(4.3) \quad \mathcal{H}(r)(v^2 + \mathcal{H}(v^2))[\tau^2, \ell] + [r, \ell]v^2\mathcal{H}(\tau^2) = 0, \quad \text{for any } v, \ell, \tau \in \mathfrak{J}, r \in \mathfrak{B}.$$

By substituting $r \mapsto \ell r$, into (4.3), and using (4.3), yields

$$\mathcal{H}(\ell)(r + \mathcal{H}(r))(v^2 + \mathcal{H}(v^2))[\tau^2, \ell] = 0, \quad \text{or any } v, \ell, \tau \in \mathfrak{J}, r \in \mathfrak{B}.$$

That is,

$$\mathcal{H}(\ell)(r + \mathcal{H}(r))(v + \mathcal{H}(v))^2[\tau^2, \ell] = 0, \quad \text{for any } v, \ell, \tau \in \mathfrak{J}, r \in \mathfrak{B}.$$

In particular,

$$\mathcal{H}(\ell)(j + \mathcal{H}(j))(v + \mathcal{H}(v))^2[\tau^2, \ell] = 0, \quad \text{for any } j, v, \ell, \tau \in \mathfrak{J}.$$

Using the fact that $\mathcal{H}$ is also zero-power valued on $\mathfrak{J}$, we further conclude that

$$\mathcal{H}(\ell)\mathfrak{J}v^2[\tau^2, \ell] = (0), \quad \text{for any } v, \ell, \tau \in \mathfrak{J}.$$

By Lemma 2.1, $\mathfrak{J}$ is union of two additive subgroups $\mathfrak{J}_5 = \{\ell \in \mathfrak{J} : \mathcal{H}(\ell) = 0\}$ and $\mathfrak{J}_6 = \{\ell \in \mathfrak{J} : v^2[\tau^2, \ell] = 0 \text{ for any } v, \tau \in \mathfrak{J}\}$. Therefore, we have two cases either $\mathfrak{J}_5 = \mathfrak{J}$ or $\mathfrak{J}_6 = \mathfrak{J}$. If $\mathfrak{J}_5 = \mathfrak{J}$, then $\mathcal{H} = 0$ on $\mathfrak{B}$, a contradiction. If $\mathfrak{J}_6 = \mathfrak{J}$, then

$$(4.4) \quad v^2[\tau^2, \ell] = 0, \quad \text{for any } v, \ell, \tau \in \mathfrak{J}.$$

According to Lemma 2.7, substituting $\ell \mapsto 2[r, \omega\xi]\ell$, where $r \in \mathfrak{B}$, into (4.4), and using (4.4), yields

$$v^2[r, \omega\xi][\tau^2, \ell] = 0, \quad \text{for any } \omega, \xi, v, \ell, \tau \in \mathfrak{J}, r \in \mathfrak{B}.$$

If $\mathfrak{B}$ is non-commutative, then, by Lemma 2.5, $v^2 = 0$ for all $v \in \mathfrak{J}$ or $[\tau^2, \ell] = 0$ for any $\ell, \tau \in \mathfrak{J}$. Since $\mathfrak{J}$ is non-zero Jordan ideal in prime ring $\mathfrak{B}$, then, by Lemma 2.12, $[\tau^2, \ell] = 0$ for any $\ell, \tau \in \mathfrak{J}$. By Lemma 2.5, $\mathfrak{J} \subseteq \mathcal{Z}(\mathfrak{B})$ and hence, $\mathfrak{B}$ is commutative, a contradiction. Thus, $\mathfrak{B}$ must be commutative. $\square$

Theorem 4.2. *If $\mathcal{H}$ is zero-power valued on $\mathfrak{J}$ and $[\mathcal{H}(v), \mathcal{H}(\ell)] = 0$ for each $v, \ell \in \mathfrak{J}$, then $\mathfrak{B}$ is commutative.*

Proof. By hypotheses, we have

$$(4.5) \quad [\mathcal{H}(v), \mathcal{H}(\ell)] = 0, \quad \text{for any } v, \ell \in \mathfrak{J}.$$

According to (I) of Lemma 2.5, substituting $v \mapsto 4v\tau^2$, where $\tau \in \mathfrak{J}$, into (4.5), and using (4.5), it follows that

$$(4.6) \quad \mathcal{H}(v)[\tau^2, \mathcal{H}(\ell)] + [v, \mathcal{H}(\ell)]\mathcal{H}(\tau^2) = 0, \quad \text{for any } v, \ell, \tau \in \mathfrak{J}.$$

By substituting $v \mapsto 2rv^2$, where $r \in \mathfrak{B}$, into (4.6), and using (4.6), yields

$$(4.7) \quad \mathcal{H}(r)(v^2 + \mathcal{H}(v^2))[\tau^2, \mathcal{H}(\ell)] + [r, \mathcal{H}(\ell)]v^2\mathcal{H}(\tau^2) = 0, \quad \text{for any } v, \ell, \tau \in \mathfrak{J}, r \in \mathfrak{B}.$$

By substituting $r \mapsto \mathcal{H}(\ell)j$, into (4.7), and using (4.7), yields

$$\mathcal{H}^2(\ell)(j + \mathcal{H}(j))(v + \mathcal{H}(v))^2[\tau^2, \mathcal{H}(\ell)] = 0, \quad \text{for any } v, j, \ell, \tau \in \mathfrak{J}, r \in \mathfrak{B}.$$

By virtue of $\mathcal{H}$ being zero-power on $\mathfrak{J}$, we are led to the identity

$$\mathcal{H}^2(\ell)\mathfrak{J}v^2[\tau^2, \mathcal{H}(\ell)] = (0), \quad \text{for any } v, \ell, \tau \in \mathfrak{J}, r \in \mathfrak{B}.$$

By Lemma 2.1, either $\mathcal{H}^2(\ell) = 0$ for all $\ell \in \mathfrak{J}$ or $v^2[\tau^2, \mathcal{H}(j)] = 0$ for any $v, j, \tau \in \mathfrak{J}$.

First case: suppose that $\mathcal{H}^2(\ell) = 0$ for all $\ell \in \mathfrak{J}$. Substituting $\ell \mapsto 2\ell^2v$, where $v \in \mathfrak{J}$, then

$$\mathcal{H}(\ell^2)\mathcal{H}(v) = 0, \quad \text{for any } v, \ell \in \mathfrak{J}.$$

Substituting $v \mapsto 2[r, \omega\tau]v$, where $r \in \mathfrak{B}$, then

$$\mathcal{H}(\ell^2)[r, \omega\tau]\mathcal{H}(v) = 0, \quad \text{for any } v, \ell, \omega, \tau \in \mathfrak{J}.$$

If $\mathfrak{B}$ is non-commutative, then, by (II) of Lemma 2.5, $\mathcal{H}(\ell^2) = 0$ for any $\ell \in \mathfrak{J}$. By (4.6), we get

$$\mathcal{H}(v)[\tau^2, \mathcal{H}(\ell)] = 0, \quad \text{for any } v, \ell, \tau \in \mathfrak{J}.$$

Substituting $v \mapsto 2\ell[r, \omega\tau]$, where $r \in \mathfrak{B}$, then

$$\mathcal{H}(\ell)[r, \omega\tau][\tau^2, \mathcal{H}(\ell)] = 0, \quad \text{for any } \omega, v, \ell, \tau \in \mathfrak{J}, r \in \mathfrak{B}.$$

Again, by (II) and (III) of Lemma 2.5,

$$(4.8) \quad \mathcal{H}(\ell) \in \mathcal{Z}(\mathfrak{B}), \quad \text{for any } \ell \in \mathfrak{J}.$$

Now, since $2\ell^2 = \ell \circ \ell \in \mathfrak{J}$, using (4.8) we get $\mathcal{H}(2\ell^2) \in \mathcal{Z}(\mathfrak{B})$ for any $\ell \in \mathfrak{J}$. This implies $\ell\mathcal{H}(\ell) \in \mathcal{Z}(\mathfrak{B})$ for any $\ell \in \mathfrak{J}$. Hence, for each $\ell \in \mathfrak{J}$, either $\mathcal{H}(\ell) = 0$ or $\ell \in \mathcal{Z}(\mathfrak{B})$. Since $\mathcal{H}$ is a non-zero on $\mathfrak{J}$ (otherwise $\mathcal{H}$ would be zero on $\mathfrak{B}$), then there exists a non-zero element $\xi \in \mathfrak{J}$ such that $\mathcal{H}(\xi) \neq 0$, and hence $\xi \in \mathcal{Z}(\mathfrak{B})$.

Now suppose that $\tau \in \mathfrak{J}$ is such that $\mathcal{H}(\tau) = 0$. Then, $\mathcal{H}(\tau + \xi) = \mathcal{H}(\tau) + \mathcal{H}(\xi) = \mathcal{H}(\xi) \neq 0$. Therefore, $\tau + \xi \in \mathcal{Z}(\mathfrak{B})$, which implies $\tau \in \mathcal{Z}(\mathfrak{B})$. Thus we obtain $\mathfrak{J} \subseteq \mathcal{Z}(\mathfrak{B})$. By Lemma 2.13, this implies that $\mathfrak{B}$ is commutative, which contradicts our assumption that $\mathfrak{B}$ is non-commutative. Hence, the first case leads to the conclusion that $\mathfrak{B}$ is commutative.

Second case: suppose that

$$(4.9) \quad v^2[\tau^2, \mathcal{H}(j)] = 0, \quad \text{for any } v, j, \tau \in \mathfrak{J}.$$

By substituting $v \mapsto v + \mu$, into (4.9), and using (4.9), yields

$$(4.10) \quad (v\mu + \mu v)[\tau^2, \mathcal{H}(j)] = 0, \quad \text{for any } j, v, \mu, \tau \in \mathfrak{J}.$$

By substituting $v \mapsto 4rv^2$, where $r \in \mathfrak{B}$, into (4.10), and using (4.9), yields

$$\mathfrak{B}v^2\mathfrak{J}[\tau^2, \mathcal{H}(j)] = (0), \quad \text{for any } j, v, \tau \in \mathfrak{J}.$$

By Lemma 2.1, $\mathfrak{J} \neq (0)$, and $\mathfrak{B}$ is prime, then

$$[\tau^2, \mathcal{H}(j)] = 0, \quad \text{for any } j, \tau \in \mathfrak{J}.$$

By Lemma 2.5, $\mathcal{H}(j) \in \mathcal{Z}(\mathfrak{B})$ for any $j \in \mathfrak{J}$. Thus, as above in (4.8), $\mathfrak{B}$ is commutative. Consequently, both cases lead to the conclusion that $\mathfrak{B}$ is commutative. $\square$

Theorem 4.3. *If $\mathcal{H}$ is zero-power valued on $\mathfrak{J}$ and $\mathcal{H}([v, \ell]) = 0$ for any $\ell, v \in \mathfrak{J}$, then $\mathfrak{B}$ is commutative.*

Proof. By hypotheses, we have

$$(4.11) \quad \mathcal{H}([v, \ell]) = 0, \quad \text{for any } v, \ell \in \mathfrak{J}.$$

According to (I) of Lemma 2.5, substituting $v \mapsto 4v\tau^2$, where $\tau \in \mathfrak{J}$, into (4.11), and using (4.11), it follows that

$$(4.12) \quad \mathcal{H}(v)[\tau^2, \ell] + [v, \ell]\mathcal{H}(\tau^2) = 0, \quad \text{for any } v, \ell, \tau \in \mathfrak{J}.$$

As in equation (4.2) in Theorem 4.1, equation (4.12) implies that $\mathfrak{B}$ is commutative. $\square$

Theorem 4.4. *Suppose that $\mathcal{H}$ is nilpotent on $\mathfrak{B}$ and $\mathcal{H}(\mathcal{Z}(\mathfrak{B})) \neq (0)$. If $\mathcal{H}([v, \ell]) \in \mathcal{Z}(\mathfrak{B})$ for any $\ell, v \in \mathfrak{J}$, then $\mathfrak{B}$ is commutative.*

Proof. Since $\mathcal{H}(\mathcal{Z}(\mathfrak{B})) \neq (0)$, there exists $0 \neq z \in \mathcal{Z}(\mathfrak{B})$ such that $\mathcal{H}(z) \neq 0$. We will show that $\mathcal{H}(z) \in \mathcal{Z}(\mathfrak{B})$. Because $z \in \mathcal{Z}(\mathfrak{B})$,

$$0 = \mathcal{H}(0) = \mathcal{H}([z, r]) = [\mathcal{H}(z), r + \mathcal{H}(r)], \quad \text{for all } r \in \mathfrak{B}.$$

That is,

$$(4.13) \quad [\mathcal{H}(z), \mathcal{H}(r)] = [r, \mathcal{H}(z)], \quad \text{for all } r \in \mathfrak{B}.$$

Replacing r by $\mathcal{H}(r)$ in (4.13), and using (4.13), we get

$$[\mathcal{H}(z), \mathcal{H}^2(r)] = -[r, \mathcal{H}(z)], \quad \text{for all } r \in \mathfrak{B}.$$

By iterating this process, we get

$$[\mathcal{H}(z), \mathcal{H}^n(r)] = (-1)^{n+1}[r, \mathcal{H}(z)], \quad \text{for all } r \in \mathfrak{B}.$$

Since $\mathcal{H}$ is nilpotent on $\mathfrak{B}$, then

$$[r, \mathcal{H}(z)] = 0, \quad \text{for all } r \in \mathfrak{B}.$$

Therefore, $\mathcal{H}(z) \in \mathcal{Z}(\mathfrak{B})$.

To derive a contradiction, assume that $\mathfrak{J} \not\subseteq \mathcal{Z}(\mathfrak{B})$. By Lemma 2.11, there exists a non-zero ideal $\mathfrak{L}$ of $\mathfrak{B}$ such that $\mathfrak{L} \subseteq \mathfrak{J}$. Since $\mathcal{H}([v, \ell]) \in \mathcal{Z}(\mathfrak{B})$ for any $\ell, v \in \mathfrak{J}$, then

$$(4.14) \quad \mathcal{H}([v, \ell]) \in \mathcal{Z}(\mathfrak{B}), \quad \text{for any } \ell, v \in \mathfrak{L}.$$

Replacing ℓ by ℓz in (4.14), and using $\mathcal{H}(z) \in \mathcal{Z}(\mathfrak{B})$, (4.14), we get

$$[v, \ell]\mathcal{H}(z) \in \mathcal{Z}(\mathfrak{B}), \quad \text{for any } \ell, v \in \mathfrak{L}.$$

However, since $0 \neq \mathcal{H}(z) \in \mathcal{Z}(\mathfrak{B})$,

$$[v, \ell] \in \mathcal{Z}(\mathfrak{B}), \quad \text{for any } \ell, v \in \mathfrak{L}.$$

Therefore, by Lemma 2.10 and Lemma 2.3, $\mathfrak{B}$ is commutative. Hence, $\mathfrak{J} \subseteq \mathcal{Z}(\mathfrak{B})$, a contradiction. Thus, $\mathfrak{B}$ is commutative. $\square$

5. CONCLUSION

In this article, we derive algebraic identities involving homoderivations on prime rings. We further investigate such identities when homoderivations act on a Jordan ideal that is a subring, as in Section 3, or on a Jordan ideal which is not necessarily a subring, as in Section 4. Our results show that if a Jordan ideal satisfies these identities, then it is contained in the center of the prime ring, which consequently implies that the ring is commutative.

6. OPEN QUESTION

Is it possible to relax the hypotheses in Section 3 by assuming that the Jordan ideal is not a subring, and still obtain the same result, namely, the commutativity of the ring?

REFERENCES

- [1] M. K. Abu Naws and R. M. Al-Omry, *On ideals and commutativity of prime rings with generalized derivations*, Eur. J. Pure Appl. Math. **11**(1) (2018), 79–89.
- [2] E. F. Alharfie and N. M. Muthana, *The commutativity of prime rings with homoderivations*, Int. J. Adv. Appl. Sci. **5**(5) (2018), 79–81. <https://doi.org/10.21833/ijaas.2018.05.010>
- [3] S. Belkadi, S. Ali and L. Taoufiq, *On nilpotent homoderivations in prime rings*, Comm. Algebra **51**(9) (2023), 4044–4053. <https://doi.org/10.1080/00927872.2023.2196342>
- [4] A. Boua and E. K. Söğütçü, *Semiprime rings with generalized homoderivations*, Bol. Soc. Parana. Mat. **41** (2023), 1–8. <https://doi.org/10.5269/bspm.62479>
- [5] M. N. Daif and H. E. Bell, *Remarks on derivations on semiprime rings*, Int. J. Math. Math. Sci. **15** (1992), 205–206, <https://doi.org/10.1155/S0161171292000255>.
- [6] M. M. El-Soufi and A. Aboubakr, *Generalized derivations on Jordan ideals in prime rings*, Turk. J. Math. **38** (2014), 233–239.
- [7] M. M. El-Soufi, *Rings with some kinds of mappings*, M.Sc. Thesis, Cairo University, Branch of Fayoum, Cairo, Egypt, 2000.
- [8] A. Engin and N. Aydin, *Homoderivations in prime rings*, Journal of New Theory **43** (2023), 23–34. <https://doi.org/10.53570/jnt.1258402>
- [9] E. Güven, *Homoderivations and their impact on Lie ideals in prime rings*, Natural & Applied Sciences Journal **6**(2) (2023), 41–48. <https://doi.org/10.38061/idunas.1356057>

- [10] S. Huang, *Generalized derivations of prime rings*, Int. J. Math. Math. Sci. **2007**, Article ID 85612, 6 pages.
- [11] A. Y. Hummdi, Z. Bedir, E. K. Sögütcü, Ö. Gölbaşı and N. ur Rehman, *Lie ideals and homoderivations in semiprime rings*, Mathematics **13**(4) (2025), Article ID 548. <https://doi.org/10.3390/math13040548>
- [12] J. Mayne, *Centralizing automorphisms of Lie ideals in prime rings*, Canad. Math. Bull. **35**(4) (1992), 510–514. <https://doi.org/10.4153/CMB-1992-067-0>
- [13] L. Oukhtite and A. Mamouni, *Commutativity theorems for prime rings with generatized derivations on Jordan ideals*, Journal of Taibah University for Science **9** (2015), 314–319.
- [14] G. S. Sandhu and D. Kumar, *A note on derivations and Jordan ideals of prime rings*, AIMS Math. **2**(4) (2017), 580–585.
- [15] L. Taoufiq and S. Belkadi, *On nilpotent homoderivations in semiprime rings*, Bol. Soc. Paran. Mat. **22**(2) (2004), 1–10.
- [16] S. M. A. Zaidi, M. Ashraf and S. Ali, *On Jordan ideals and left (θ, θ) -derivations in prime rings*, Int. J. Math. Math. Sci. **37** (2004), 1957–1964.

¹DEPARTMENT OF MATHEMATICS,
FACULTY OF SCIENCE, FAYOUM UNIVERSITY,
FAYOUM, EGYPT

²DEPARTMENT OF MATHEMATICS,
FACULTY OF SCIENCE, AL-BAHA UNIVERSITY,
AL-BAHA, KINGDOM OF SAUDI ARABIA
Email address: mms06@fayoum.edu.eg & malghamri@bu.edu.sa
ORCID id: <https://orcid.org/0000-0003-0508-1033>