Kragujevac Journal of Mathematics
Volume 50(10) (2026), Pages 1621–1626.

SELECTION PRINCIPLES IN TOPOLOGICAL VECTOR SPACES

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Abstract. This paper investigates bounded, M-bounded, H-bounded, R-bounded topological vector spaces. The most important properties and characteristics of these classes of topological vector spaces are established.

1. Introduction

Recently the theory of selection principles has been intensively developing in topological spaces, in topological groups, in uniform spaces etc.

Significant contributions to the theory of selection principles of choice were made in [1, 3, 6–13] and others.

A good overview of the very extensive literature on selection principles is contained in [3, 10].

Lj.D.R. Kočinac [3] found uniform analogues of the most important properties of the selection principles: uniformly Menger spaces, uniformly Hurewicz spaces, uniformly Rothberger space etc. These properties are considered as types of totally bounded uniform spaces, for example, the uniformly Menger space occupies an intermediate place between totally bounded and ω-bounded spaces. To each selection property of a uniform structure defined above can correspond game.

In this paper we study important properties of the bounded, M-bounded, H-bounded and R-bounded topological vector spaces.

Key words and phrases. Boundedness, M-boundedness, H-boundedness, R-boundedness.
2020 *Mathematics Subject Classification.* Primary: 54E20. Secondary: 54D15.
DOI
Received: October 10, 2024.
Accepted: April 03, 2025.

2. Preliminaries and Denotations

For covers α and β of a set X, we have: $\alpha \wedge \beta = \{A \bigcap B : A \in \alpha, B \in \beta\}$. $\alpha(x) = \bigcup St(\alpha, x)$, $St(\alpha, x) = \{A \in \alpha : A \ni x\}$, $x \in X$, $\alpha(H) = \bigcup St(\alpha, H)$, $St(\alpha, H) = \{A \in \alpha : A \bigcap H \neq \emptyset\}$, $H \subset X$. For covers α and β of the set X, the symbol $\alpha \succ \beta$ means that the cover α is a refinement of the cover β, i.e., for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$ and, the symbol $\alpha * \succ \beta$ means that the cover α is a strongly star refinement of the cover β, i.e., for any $A \in \alpha$ there exists $B \in \beta$ such that $\alpha(A) \subset B$.

Recall that a uniform space (X, U) is called:

(a) *totally bounded*, if each $\alpha \in U$ has a finite set $H \subset X$ such that $\alpha(H) = X$ [2];
(b) *precompact*, if the uniformity U has a base consisting of finite covers [2];
(c) *ω-bounded*, if the uniformity U has a base consisting of countable cover [2, 3];
(d) *uniformly locally compact*, if the uniformity of U contains a uniform cover consisting is compact sets [2];
(e) has the *uniform Menger property*, if for each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset U$ there is a sequence $(\beta_n \mid n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, β_n is a finite subset of α_n and $\bigcup_{n\in\mathbb{N}} \beta_n$ is a cover of X [3];
(f) has the *uniform Hurewicz property* if for each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset U$ there is a sequence $(\beta_n \mid n \in \mathbb{N})$ such that each β_n is a finite subset of α_n and for each $x \in X$ we have $x \in \bigcup \beta_n$ for all but finitely many n [3];
(g) has the *uniform Rothberger property* if for each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset U$ there is a sequence $(A_n \mid n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $A_n \in \alpha_n$ and $\bigcup_{n\in\mathbb{N}} A_n = X$ [3].

Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) onto a uniform space (Y, V). The mapping f is called *precompact*, if for each $\alpha \in U$ there exist a uniform cover $\beta \in V$ and a finite uniform cover $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [2].

For the uniformity U by τ_U we denote the topology generated by the uniformity.

A topological vector space L is said to be

(a) *M-bounded* if for each sequence $(U_n \mid n \in \mathbb{N})$ of 0-neighborhoods in L there exists a sequence $(A_n \mid n \in \mathbb{N})$ of finite subsets of L such that $L = \bigcup_{n\in\mathbb{N}} A_n + U_n$;
(b) *H-bounded* if for each sequence $(U_n \mid n \in \mathbb{N})$ of 0-neighborhoods in L there exists a sequence $(A_n \mid n \in \mathbb{N})$ of finite subsets of L such that each $x \in L$ belongs to all but finitely many $A_n + U_n$;
(c) *R-bounded* if for each sequence $(U_n \mid n \in \mathbb{N})$ of 0-neighborhoods in L there is a sequence $(A_n \mid n \in \mathbb{N})$ of elements of L such that $L = \bigcup_{n\in\mathbb{N}}(x_n + U_n)$.

In other words topological vector space L is said to be M-bounded (H-bounded, R-bounded), if (L, U) a uniformly Menger (uniformly Hurewicz, uniformly Rothberger) space with respect to the uniformity U.

3. Results

Let L be a topological vector space (TVS).

Proposition 3.1. *Any totally bounded TVS L is M-bounded.*

Proof. Let L be a TVS. We will prove that it is M-bounded. Let $(\alpha_{W_n} \mid n \in \mathbb{N}) \subset U$ be an arbitrary sequence of uniformly cover, where W_n is 0-neighborhood in L. Then, for any 0-neighborhood W_n there exist a 0-neighborhood W'_n, such that $W'_n + W'_n \subset W_n$. Clearly, $\beta_{W'_n} * \succ \alpha_{W_n}$. Therefore, for each $n \in \mathbb{N}$ there exists a finite set $M_n \subset L$, such that $\beta_{W'_n}(M_n) = L$. Put $\beta_{W'_n}(M_n) = \bigcup_{i=1}^{k}\{\beta_{W'_n}(x_i) : x_i \in M_n\}$. For each $\beta_{W'_i}(x)$ choose $W_n + x_i \in \alpha_{W_n}$, such that $\beta_{W'_n}(x_i) \subset W_n + x_i$, $i = 1, 2, \ldots, k$. Let $\alpha^0_{W_n} = (W_n + x_i \mid i = 1, 2, \ldots, k)$. Since for each $n \in \mathbb{N}$ the finite family $\alpha^0_{W_n}$ is a cover, then $\bigcup_{n\in\mathbb{N}} \alpha^0_{W_n} = L$. Therefore, TVS L is M-bounded. □

Corollary 3.1. *Any compact TVS L is M-bounded.*

The space of real numbers $\mathbb{R}$ is M-bounded TVS, but it is not totally bounded, i.e., not compact.

Proposition 3.2. *Any M-bounded TVS L is ω-bounded.*

Proof. Let L be a M-bounded TVS. We will prove that it is ω-bounded. It suffices to show that the uniform space (L, U) is a ω-bounded. Let $\alpha_W \in U$ be an arbitrary uniform cover, where W_n is a 0-neighborhood in L. For each $n \in \mathbb{N}$ put $\alpha_{W_n} = \alpha_W$. Then, $(\alpha_{W_n} \mid n \in \mathbb{N})$ is a sequence of uniform covers. Therefore, there exists a sequence of finite subfamilies $(\alpha^0_{W_n} \mid n \in \mathbb{N})$, such that for each $n \in \mathbb{N}$ we have $\alpha^0_{W_n} \subset \alpha_{W_n}$ and $\bigcup_{n\in\mathbb{N}} \alpha^0_{W_n} = X$. Since for each $n \in \mathbb{N}$ the family $\alpha^0_{W_n}$ is finite, then the family $\alpha^0_W = (\alpha_{W_1}, \alpha_{W_2}, \ldots, \alpha_{W_n}, \ldots)$ is subcover of α_W. Therefore, TVS L is ω-bounded. □

Theorem 3.1. *If a TVS L' is a linearly continuous image of a M-bounded (H-bounded, R-bounded) TVS L, then L' is also M-bounded (H-bounded, R-bounded) TVS.*

Proof. Let us consider only the M-bounded case and the remaining cases proceed similarly. Let $f : L \to L'$ be an linearly continuous mapping of the M-bounded TVS L to the TVS L'. Since every linearly continuous mapping is uniformly continuous, the mapping $f : (L, U) \to (L', U')$ is uniformly continuous. Let $(\beta_{V_n} \mid n \in \mathbb{N})$ be an arbitrary sequence of elements of U', where V_n is 0-neighborhoods in L' and put $f^{-1}\beta_{V_n} = \alpha_{f^{-1}V_n}$, $n \in \mathbb{N}$. It is easily seen that we have: if V_n is 0-neighborhoods in L then $f^{-1}V_n$ is a 0-neighborhoods in L. Since (L, U) is uniformly Menger, then there exists a sequence of finite subfamilies $(\alpha^0_{f^{-1}V_n} \mid n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$ $\alpha^0_{f^{-1}V_n} \subset \alpha_{f^{-1}V_n}$ and $\bigcup_{n\in\mathbb{N}} \alpha^0_{f^{-1}V_n} = L$. For each $n \in \mathbb{N}$ put $f\alpha^0_{f^{-1}V_n} = \beta^0_{V_n}$. Then, $f(\bigcup_{n\in\mathbb{N}} \alpha^0_{f^{-1}V_n}) = \bigcup_{n\in\mathbb{N}} f\alpha^0_{f^{-1}V_n} = \bigcup_{n\in\mathbb{N}} \beta^0_{V_n} = L'$, $n \in \mathbb{N}$. This means that (L', U') has indeed the uniform Menger property. Therefore, L' is M-bounded. □

A TVS L is called σ-totally bounded, if it can be represented as the union of countably many totally bounded subspaces.

Proposition 3.3. *Any σ-totally bounded TVS is H-bounded.*

Proof. Let L be a σ-totally bounded TVS. We proof that L is H-bounded. Let $(\alpha_{V_n} \mid n \in \mathbb{N}) \subset U$ be an arbitrary sequence of uniform covers, where V_n is 0-neighborhoods in L and $L = \bigcup L_{n n\in\mathbb{N}}$, where L_n is totally bounded. For each $n \in \mathbb{N}$ put $\alpha_{V_n^{L_n}} = \alpha_{V_n} \wedge \{L_n\}$, $V_n^{L_n} = V_n \bigcap L_n$. Then, $(\alpha_{V_n^{L_n}} \mid n \in \mathbb{N}) \subset U_{L_n}$ is sequence uniform covers of the space (L_n, U_{L_n}) and $V_n^{L_n}$ is 0-neighborhood in L_n. Let $W_n^{L_n}$ be 0-neighborhood such that $W_n^{L_n} + W_n^{L_n} \subset V_n^{L_n}$. Then, $\beta_{W_n^{L_n}} * \succ \alpha_{V_n^{L_n}}$. For each $n \in \mathbb{N}$ choose a finite subset $M_{L_n} \subset L_n$ such that $\beta_{W_n^{L_n}}(M_{L_n}) = \bigcup\{\beta_{W_n^{L_n}}(x_1), \beta_{W_n^{L_n}}(x_2), \ldots, \beta_{W_n^{L_n}}(x_m)\} = L_n$, where $x_i \in M_{L_n}$, $i = 1, 2, \ldots, m$. Therefore, there is $V_n^{L_n} + x_i$ from $\alpha_{V_n^{L_n}}$ such that $\beta_{W_n^{L_n}}(x_i) \subset V_n^{L_n} + x_i$, $i = 1, 2, \ldots, m$. For each $n \in \mathbb{N}$ put $\alpha^0_{V_n^{L_n}} = (V_n^{L_n} + x_i \mid i = 1, 2, \ldots, m)$. It is easy to see that each $\alpha^0_{V_n^{L_n}}$ is finite subfamily of the cover $\alpha_{V_n^{L_n}}$ and each $x \in L$ is an element of $\bigcup \alpha^0_{V_n^{L_n}}$ for all but finitely many n. Thus, L is H-bounded. □

Corollary 3.2. *Any σ-totally bounded TVS is M-bounded.*

Theorem 3.2. *Any countable discrete TVS is a R-bounded.*

Proof. Let L be a countable discrete TVS and $(\alpha_{W_n} \mid n \in \mathbb{N}) \subset U$ be a sequence of uniform covers. Since the TVS L is a countable discrete space, there exists a base $B = (\alpha_W)$ consisting of a countable cover $\alpha_W = (\{0\} + x_n \mid x_n \in L)$ and for each $n \in N$, $\alpha_W \succ \alpha_{W_n}$. Then, we choose one $\{0\} + x_n \in \alpha_W$ such that $\{0\} + x_n \subset W_n + x_n$ and $\bigcup_{n\in\mathbb{N}}(\{0\} + x_n) = L$. Hence, TVS L is a R-bounded. □

Theorem 3.3. *For a locally compact TVS L the following are equivalent:*
(1) *L is M-bounded;*
(2) *L is ω-bounded.*

Proof. (1) $\Rightarrow$ (2) It is evident. So, we have prove only (2) $\Rightarrow$ (1). Let $(\alpha_{W_n} \mid n \in \mathbb{N}) \subset U$ be a sequence uniform covers in U. We apply (2) to the countable uniform cover $\beta_W = (W + x_n \mid n \in \mathbb{N})$ consisting of compact sets. For each $n \in \mathbb{N}$ there exists a finite subfamily $\alpha^0_{W_n} \subset \alpha_{W_n}$ such that $W + x_n \subset \bigcup \alpha^0_{W_n}$. It is easy to see that $\bigcup_{n\in\mathbb{N}} \alpha^0_{W_n} = L$. Consequently, L is a M-bounded. □

Proposition 3.4. *Any locally compact TVS L is strongly paracompact.*

Proof. Let L be a locally compact TVS and W be a compact 0-neighborhood. Then, $\alpha_W = \{W + x : x \in L\}$ is a uniform cover consisting of compact subsets, i.e., a uniform space (L, U) is uniformly locally compact. Since (L, U) is strongly uniformly paracompact [16]. Then, TVS L is locally compact and paracompact, i.e., strongly paracompact. □

Corollary 3.3. *Any locally compact TVS L is complete.*

Proposition 3.5. *The completion of M-bounded (H-bounded, R-bounded) TVS is a M-bounded (H-bounded, R-bounded).*

Proof. Let's consider only the R-bounded case and the remaining cases proceed similarly. Let $\tilde{L}$ be a completion of the R-bounded space L and $(\tilde{\alpha}_{W_n} \mid n \in \mathbb{N}) \subset \tilde{U}$ be an arbitrary sequence. For each $n \in \mathbb{N}$, $\alpha_{W_n} = \tilde{\alpha}_{W_n} \wedge \{L\}$. Then, $(\alpha_{W_n} \mid n \in \mathbb{N})$ is a sequence uniform covers in U. Since L is a R-bounded space, there exists a sequence $(W_n + x \mid n \in \mathbb{N})$ of such that for any $n \in \mathbb{N}$, $W_n + x \in \alpha_{W_n}$ and $\bigcup_{n\in\mathbb{N}}(W_n + x)$ is a cover of the space L. It is easy to see that $\bigcup_{n\in\mathbb{N}}(\tilde{W}_n + \tilde{x}) = \tilde{L}$. Thus, the completion $\tilde{L}$ is R-bounded. □

Let $f : L \to L'$ be a linear continuous mapping of a TVS L onto a TVS L'. The mapping f is called precompact, if the mapping $f : (L, U) \to (L', U')$ of a uniform space (L, U) to uniform space (L', U') is precompact.

Theorem 3.4. *Let $f : L \to L'$ be a precompact mapping. If TVS L' is H-bounded, then a TVS L is also H-bounded.*

Proof. Let $f : L \to L'$ be a precompact mapping of a TVS L onto a TVS L' and $(\alpha_{W_n} \mid n \in \mathbb{N}) \subset U$ be an arbitrary sequence of uniform covers. Then, for any $n \in \mathbb{N}$ there exists a finite cover $\gamma_{W'_n} \in U$ and $\beta_{V_n} \in U'$, such that $f^{-1}\beta_{V_n} \wedge \gamma_{W'_n} \succ \alpha_{W_n}$. Apply to the sequence $(\beta_{V_n} \mid n \in \mathbb{N}) \subset U'$ the fact that the space (L', U') has a uniformly Hurewicz property we find a sequence $(\beta^0_{V_n} \mid n \in \mathbb{N})$ of finite subfamilies such that for each $y \in L'$ we have $y \in \bigcup \beta^0_{V_n}$ for all but finitely many n. Note, that for any $n \in \mathbb{N}$ the family $f^{-1}\beta^0_{V_n} \wedge \gamma_{W'_n}$ is finite and $\bigcup\{f^{-1}\beta^0_{V_n} \wedge \gamma_{W'_n}\} = \bigcup f^{-1}\beta^0_{V_n}$. Next, for any $f^{-1}(V_n + y_i) \bigcap W'_n + x_i \in f^{-1}\beta_0 \wedge \gamma_n$ choose $W_n + x_i \in \alpha_{W_n}$, such that $f^{-1}(V_n + y_i) \bigcap W'_n + x_i \subset W_n + x_i$. Put $\alpha^0_{W_n} = \{W_n + x_1, W_n + x_2, \ldots, W_n + x_i\}$, $f(x_i) = y_i$. It is easy to see that for each $x \in L$ we have $x \in \bigcup \alpha^0_{W_n}$, $f(x) = y$ for all but finitely many n. Therefore, L is H-bounded. □

As it is well known, to each selection principle for topological spaces it is naturally associated the corresponding game and often selection principles can be characterized game-theoretically. In TVS case to each selection principle one can assign also the corresponding game. The game MLG associated to the M-bounded property is defined in the following way. Two players, ONE and TWO, play a round for each positive integer. In the n-th round ONE chooses a 0-neighborhood U_n and TWO responds choosing a finite subset A_n. TWO wins a play $U_1, A_1; U_2, A_2, \ldots$, if $X = \bigcup_{n\in\mathbb{N}} A_n + U_n$, and otherwise ONE wins. The games HLG and RLG associated to the H-bounded and R-bounded properties are defined in the following way.

Acknowledgements. The corrections and excellent suggestions proposed by the anonymous reviewers are gratefully acknowledged.

References

[1] A. V. Arhangel'skii, *Hurewicz spaces, analytic sets and fan tightness of function spaces*, Soviet Math. Doklady **33** (1986), 396–399.

[2] A. A. Borubaev, *Uniform Topology and its Applications*, Bishkek, Ilim, 2021.

[3] Lj. D. R. Kočinac, *Selection principles in uniform spaces*, Note di Mathematica **22** (2003), 127–139.

[4] Lj. D. R. Kočinac and M. Scheepers, *Function spaces and a property of Reznichenko*, Topology Appl. **123** (2002), 135–143.

[5] H.-P.A. Künzi, M. Mrševič, I. L. Reilly and M. K. Vamanamurthy, *Pre-Lindelöf quasi-pseudometric, quasi-uniform spaces*, Mat. Vesnik **46** (1994), 131–135.

[6] M. Menger, *Einige überdeckungssätze der punktmengenlehre*, Sitzungsberischte Abt. 2a, Mathematik, Astronomie, Physik, Meteorolgie und Mechanik (viener Akademie, Wien) **133** (1924), 421–444.

[7] W. Hurewicz, *Über eine verallgemeinerung des Borelschen theorems*, Math. Z. **24** (1925), 401–421.

[8] F. Rothberger, *Eine verschärfung der eigenscafts*, Fund. Math. **30** (1938), 50–55.

[9] M. Sakai, *Property C" and function spaces*, Proc. Amer. Math. Soc. **104** (1988), 917–919. https://doi.org/10.2307/2046816

[10] M. Scheepers, *Combinatorics of open covers I: Ramsey theory*, Topology Appl. **69** (1996), 31–62. https://doi.org/10.1016/0166-8641(95)00067-4

[11] L. Babinkostova, Lj. D. R. Kočinac and M. Scheepers, *Combinatorics of open covers (XI): Menger- and Rothberger-bounded groups*, Topology Appl. **154**(7) (2007), 1269–1280.

[12] B. Tsaban, *Selection principles and special sets of reals*, in: E. Pearl (Ed.), *Open Problems in Topology II*, Elsevier Science, 2007, 91–108. https://doi.org/10.1016/B978-044452208-5/50009-0

[13] A. V. Osipov, *The functional characterizations of the Rothberger and Menger properties*, Topology Appl. **243** (2018), 146–152. https://doi.org/10.1016/j.topol.2018.05.009

[14] M. Machura, S. Shelah and B. Tsaban, *Squares of Menger-bounded groups*, Trans. Amer. Math. Soc. **362**(4) (2010), 1751–1764. https://doi.org/10.1090/S0002-9947-09-05169-1

[15] H. H. Schaefer, *Topological Vector Space*, Springer-Verlag, New York, Heidelberg Berlin, 1986.

[16] B. E. Kanetov, D. E. Kanetova and M. K. Beknazarova, *About uniform analogues of strongly paracompact and Lindelöf spaces*, Transactions Issue Mathematics National Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences **44**(1) (2024), 117–127. https://doi.org/10.30546/2617-7900.44.1.2024.117

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