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MODULAR TOTAL VERTEX IRREGULARITY STRENGTH OF REGULAR GRAPHS

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Abstract. Let G be a graph of order n. For a positive integer k, a labeling $f : V(G) \cup E(G) \to [1, k]$ is called a *modular vertex irregular total k-labeling* if the induced weights $w_f(u) : V(G) \to \mathbb{Z}_n$ defined by $w_f(u) = f(u) + \sum_{ux \in E(G)} f(ux) \pmod n$ is a bijection. The smallest k for which there exists a modular vertex irregular total k-labeling of G is called the modular total vertex irregularity strength of G. In this paper, we determine the modular total vertex irregularity strength of disjoint union of cycles. We also present a new bound of the modular total vertex irregularity strength for regular graphs.

1. Introduction

In the realm of mathematics, graph theory stands as a discipline dedicated to the exploration of abstract structures known as graphs. These graphs comprise two fundamental elements: vertices, often referred to as nodes or points, and edges, which represent the connections between these vertices. Edges can be visualized as lines or links, forming a network that captures the pairwise relations between the objects the vertices represent. Graph theory extends its reach by investigating efficient algorithms or theorems tailored to solve problems arising from graphs. By equipping us with the tools to understand and manipulate these network structures, graph theory finds applications in a multitude of fields, including computer science, physics, social sciences, and beyond.

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Let G be a simple and undirected graph. For a vertex v in G, the notion $\deg(v)$ means the number of vertices adjacent to v in G. A graph is said to be *k-regular* if $\deg(v) = k$ for every vertex v in G. In general, we say a graph is regular if it is k-regular for some integer k. An *r-factor* of a graph G is an r-regular subgraph of G which contains every vertex of G. For a vertex v of degree 2 in a graph G, a *smoothing* on v is a modification of the graph G by deleting vertex v and adding a new edge on two vertices that were adjacent to v. Any other graph notations may be referred to [5].

One of the earliest contribution on graph theory can be seen in the work of Petersen [14] regarding regular graphs.

Theorem 1.1 ([14])**.** *Every regular graph of positive even degree has a 2-factor.*

Let k be an integer. A labeling f is said to be a *vertex irregular total k-labeling* of G if $f : V(G) \cup E(G) \to [1, k]$ is a map such that every two distinct vertices u and v satisfy $w_f(u) \neq w_f(v)$ with $w_f(u) = f(u) + \sum_{ux \in E(G)} f(ux)$. The smallest k such that there exists a vertex irregular total k-labeling of G is called the *total vertex irregularity strength* of G, which is denoted by $\text{tvs}(G)$. This problem was introduced by Bača et al. [3] back in 2007.

Determining the total vertex irregularity strength of a graph poses a fascinating challenge. Graph theorists have devoted significant effort to tackle such problem. Packiam and Kathiresan [13] found the total vertex irregularity strength of disjoint copies of cycle and path of order 3. Indriati et al. [11] determined the total vertex irregularity strength of generalized helm graph and modified prisms. Susilawati et al. [21] considered the total vertex irregularity strength of trees with maximum degree five. Furthermore, the parameter total vertex irregularity strength of a graph has also been determined for zero divisor graph [1], convex polytope graphs [4], dodecahedral modified generalization graph [9], generalized prism graphs [10], certain graphs derived from a star [15], some cubic graphs [16] and trees [8, 19]. Recently, Susanto et al. [20] found counterexamples for a conjecture of total vertex irregularity strength of trees introduced in [8]. Variations of irregularity strength problem can also be seen in [7, 12, 17, 18, 22], and an extensive list of problems related to graph labeling is given in [6].

There are several results given by Bača et al. [3] regarding the total vertex irregularity strength of regular graphs, which can be seen below.

Theorem 1.2 ([3])**.** *Let G be an r-regular graph of order n. Then,*

$$\left\lceil \frac{n+r}{1+r} \right\rceil \leq \text{tvs}(G) \leq n - r + 1.$$

Theorem 1.3 ([3])**.** *Let G be a regular Hamiltonian graph of order n. Then,*

$$\text{tvs}(G) \leq \left\lceil \frac{n+2}{3} \right\rceil.$$

Recently, Ali et al. [2] considered a stronger problem. Let G be a graph of order n and k be a positive integer. A map $f : V(G) \cup E(G) \to [1, k]$ is called *modular* *vertex irregular total k-labeling* if the induced weights $w_f(u) : V(G) \to \mathbb{Z}_n$ defined by $w_f(u) = f(u) + \sum_{ux\in E(G)} f(ux) \pmod n$ is a bijective map. The smallest k such that there exists a modular vertex irregular total k-labeling of G is called *modular total vertex irregularity strength* of G, which is denoted by $\mathrm{mtvs}(G)$.

By definition, it is obvious that every modular vertex irregular total k-labeling is also a vertex irregular total k-labeling. In fact, if the order of G is n, then the following inequality holds

$$\mathrm{tvs}(G) \le \mathrm{mtvs}(G) \le n. \tag{1.1}$$

Ali et al. [2] found a sufficient condition for a graph G to have equal values of $\mathrm{tvs}(G)$ and $\mathrm{mtvs}(G)$, which is written as follows.

Theorem 1.4 ([2])**.** *Let G be a graph with $\mathrm{tvs}(G) = k$. If total vertex weights under a corresponding vertex irregular total k-labeling constitute a set of consecutive integers, then $\mathrm{tvs}(G) = \mathrm{mtvs}(G) = k$.*

By preceding result, they are able to find the modular total vertex irregularity strength of several graphs, such as cycles.

Theorem 1.5 ([2,3])**.** *Let C_n be a cycle with $n \ge 3$ vertices. Then,*

$$\mathrm{tvs}(C_n) = \mathrm{mtvs}(C_n) = \left\lceil \tfrac{n+2}{3} \right\rceil.$$

Consider the modular vertex irregular total $\lceil \frac{n+2}{3} \rceil$-labeling f of a cycle $C_n = v_1 v_2 \dots v_n$ defined in the following way

$$\begin{aligned} f(v_i) &= \begin{cases} \left\lceil \frac{2i-1}{3} \right\rceil, & i \in \left[1, \left\lceil \frac{n}{2} \right\rceil\right], \\ \left\lceil \frac{n+2}{3} \right\rceil, & i = \left\lceil \frac{n}{2} \right\rceil + 1, \\ \left\lceil \frac{2(n-i)+4}{3} \right\rceil, & i \in \left[\left\lceil \frac{n}{2} \right\rceil + 2, n\right], \end{cases} \\ f(v_i v_{i+1}) &= \begin{cases} \left\lceil \frac{2i}{3} \right\rceil, & i \in \left[1, \left\lceil \frac{n}{2} \right\rceil\right], \\ \left\lceil \frac{2(n-i)+3}{3} \right\rceil, & i \in \left[\left\lceil \frac{n}{2} \right\rceil + 1, n-1\right], \end{cases} \\ f(v_n v_1) &= 1. \end{aligned} \tag{1.2}$$

Note that under the labeling f the maximal vertex label is $\lceil \frac{n+2}{3} \rceil$ while the maximal edge label is $\lceil \frac{n+1}{3} \rceil$. The vertex weights are distinct consecutive number from the set $[3, n+2]$.

In the proof of Theorem 1.2, the constructed labeling satisfies the requirement of Theorem 1.4. Therefore, the modular total vertex irregularity strength of a regular graph is bounded as follows.

Corollary 1.1. *Let G be an r-regular graph of order n. Then,*

$$\left\lceil \tfrac{n+r}{1+r} \right\rceil \le \mathrm{mtvs}(G) \le n - r + 1.$$

In this paper, we would like to present a stronger upper bound of mtvs(G) with respect to Corollary 1.1. First, we provide a modular total vertex irregularity strength of any 2-regular graph. Then, we attempt to decompose any regular graphs into disjoint cycles which would yield the desired result. In addition, we are able to conclude a generalization of Theorem 1.3.

2. Main Results

Theorem 2.1. *Let n_i, $i = 1, 2, \dots, p$ be positive integers and let $n = \sum_{i=1}^{p} n_i$. Then,*

$$\operatorname{mtvs}\left(\bigcup_{i=1}^{p} C_{n_i}\right) = \left\lceil \frac{n+2}{3} \right\rceil.$$

Proof. Without loss of generality assume $n_1 \le n_2 \le \cdots \le n_p$. Let $G = \bigcup\limits_{i=1}^{p} C_{n_i}$ and let $G_j = \bigcup_{i=1}^{j} C_{n_i}$ for any integer $j \in [1, p]$. Since G is a 2-regular graph, then mtvs$(G) \ge \lceil \frac{n+2}{3} \rceil$ due to Corollary 1.1.

Let $N_i = \sum_{j=1}^{i} n_j$, where $1 \le i \le p$. We show that there exists a vertex irregular total labeling of G which constitutes consecutive weights from 3 and has the largest label equal to $\left\lceil \frac{n+2}{3} \right\rceil$ by induction on p. It is true for $p = 1$, since the labeling f defined in Equation (1.2) induces consecutive vertex weights from 3 with the largest label of $\left\lceil \frac{N_1+2}{3} \right\rceil$. For G_k, $k \ge 2$, assume there exists a vertex irregular total labeling g of G_k which constitutes consecutive weights from 3 with the largest label $\left\lceil \frac{N_k+2}{3} \right\rceil$. Now we prove that there also exists a vertex irregular total labeling of G_{k+1} which constitutes consecutive weights from 3 with the largest label $\left\lceil \frac{N_{k+1}+2}{3} \right\rceil$.

By the assumption, the largest label of g in G_k is $\left\lceil \frac{N_k+2}{3} \right\rceil$ and the set of induced weights is $[3, N_k + 2]$. We split the problem by cases as follows.

Case 1. $N_k + 2 = 3q + 1$ for some integer q. Thus, $q = \frac{N_k+1}{3}$.

Define a total labeling h of G_{k+1} as follows

$$h(v) = \begin{cases} g(v), & \text{if } v \in V(G_k), \\ f(v) + q - 1, & \text{if } v \in V(C_{n_{k+1}}), \end{cases}$$

$$h(e) = \begin{cases} g(e), & \text{if } e \in E(G_k), \\ f(e) + q, & \text{if } e \in E(C_{n_{k+1}}). \end{cases}$$

Let $N(v)$ denote the open neighborhood of the vertex v. Then, the weights of vertices in G_k are

$$w_h(v) = g(v) + \sum_{u \in N(v)} g(uv) = w_g(v) \in [3, N_k + 2]$$

and the weights of vertices in $C_{n_{k+1}}$ are

$$\begin{aligned} w_h(v) =& (f(v)+q-1) + \sum_{u\in N(v)} (f(uv)+q) = w_f(v) + 3q - 1 \\ \in& [3+(3q-1), (n_{k+1}+2)+(3q-1)] = [3q+2, n_{k+1}+3q+1] \\ =& [N_k+3, N_{k+1}+2]. \end{aligned}$$

Thus, the labeling h induces consecutive vertex weights from 3.

The labels of g are at most $\left\lceil \frac{N_k+2}{3} \right\rceil$. The maximal vertex label in $C_{n_{k+1}}$ is

$$\begin{aligned} \max\{f(v) : v \in V(C_{n_{k+1}})\} + q - 1 &= \left\lceil \tfrac{n_{k+1}+2}{3} \right\rceil + \tfrac{N_k+1}{3} - 1 \\ &= \left\lceil \tfrac{n_{k+1}+2}{3} + \tfrac{N_k+1}{3} - 1 \right\rceil = \left\lceil \tfrac{n_{k+1}+N_k}{3} \right\rceil = \left\lceil \tfrac{N_{k+1}}{3} \right\rceil \end{aligned}$$

and the maximal edge label in $C_{n_{k+1}}$ is

$$\begin{aligned} \max\{f(e) : e \in E(C_{n_{k+1}})\} + q &= \left\lceil \tfrac{n_{k+1}+1}{3} \right\rceil + \tfrac{N_k+1}{3} = \left\lceil \tfrac{n_{k+1}+1}{3} + \tfrac{N_k+1}{3} \right\rceil \\ &= \left\lceil \tfrac{n_{k+1}+N_k+2}{3} \right\rceil = \left\lceil \tfrac{N_{k+1}+2}{3} \right\rceil . \end{aligned}$$

Hence, the labels of h are at most $\left\lceil \frac{N_{k+1}+2}{3} \right\rceil$.

Case 2. $N_k + 2 = 3q + 2$ for some integer q, i.e., $q = \frac{N_k}{3}$.

Similarly, define a total labeling h of G_{k+1} in the following way

$$h(v) = \begin{cases} g(v), & \text{if } v \in V(G_k), \\ f(v)+q, & \text{if } v \in V(C_{n_{k+1}}), \end{cases}$$

$$h(e) = \begin{cases} g(e), & \text{if } e \in E(G_k), \\ f(e)+q, & \text{if } e \in E(C_{n_{k+1}}). \end{cases}$$

Again, the labeling h induces consecutive weights from 3. This follows from the fact that the weights of vertices in G_k are

$$w_h(v) = g(v) + \sum_{u\in N(v)} g(uv) = w_g(v) \in [3, N_k+2]$$

and the weights of vertices in $C_{n_{k+1}}$ are

$$\begin{aligned} w_h(v) =& (f(v)+q) + \sum_{u\in N(v)} (f(uv)+q) = w_f(v) + 3q \\ \in& [3+3q, (n_{k+1}+2)+3q] = [3q+3, n_{k+1}+3q+2] = [N_k+3, N_{k+1}+2]. \end{aligned}$$

The maximal vertex label in $C_{n_{k+1}}$ is

$$\begin{aligned} \max\{f(v) : v \in V(C_{n_{k+1}})\} + q &= \left\lceil \tfrac{n_{k+1}+2}{3} \right\rceil + \tfrac{N_k}{3} = \left\lceil \tfrac{n_{k+1}+2}{3} + \tfrac{N_k}{3} \right\rceil \\ &= \left\lceil \tfrac{n_{k+1}+N_k+2}{3} \right\rceil = \left\lceil \tfrac{N_{k+1}+2}{3} \right\rceil \end{aligned}$$

and the maximal edge label in $C_{n_{k+1}}$ is

$$\max\{f(e) : e \in E(C_{n_{k+1}})\} + q = \left\lceil \frac{n_{k+1}+1}{3} \right\rceil + \frac{N_k}{3} = \left\lceil \frac{n_{k+1}+1}{3} + \frac{N_k}{3} \right\rceil$$
$$= \left\lceil \frac{n_{k+1}+N_k+1}{3} \right\rceil = \left\lceil \frac{N_{k+1}+1}{3} \right\rceil.$$

Hence, the labels of h are at most $\left\lceil \frac{N_{k+1}+2}{3} \right\rceil$.

Case 3. $N_k + 2 = 3q$ for some integer q, hence $q = \frac{N_k+2}{3}$.

Let f' be a labeling of $C_{n_{k+1}}$ constructed as follows. Take a labeling on $C_{n_{k+1}+1}$ from Equation (1.2), and apply the smoothing on the vertex with the weight of 3. Thus the labeling f' is a vertex irregular total $\left\lceil \frac{n_{k+1}+3}{3} \right\rceil$-labeling of $C_{n_{k+1}}$ inducing vertex weights from $[4, n_{k+1} + 3]$. Now, define a total labeling h of G_{k+1} as follows

$$h(v) = \begin{cases} g(v), & \text{if } v \in V(G_k), \\ f'(v) + q - 1, & \text{if } v \in V(C_{n_{k+1}}), \end{cases}$$

$$h(e) = \begin{cases} g(e), & \text{if } e \in E(G_k), \\ f'(e) + q - 1, & \text{if } e \in E(C_{n_{k+1}}). \end{cases}$$

First, we prove that h induces a consecutive weights from 3. The weights of vertices in G_k are

$$w_h(v) = g(v) + \sum_{u \in N(v)} g(uv) = w_g(v) \in [3, N_k + 2]$$

and the weights of vertices in $C_{n_{k+1}}$ are

$$\begin{aligned} w_h(v) =& (f'(v) + q - 1) + \sum_{u \in N(v)} (f'(uv) + q - 1) = w_{f'}(v) + 3q - 3 \\ \in& [4 + (3q - 3), (n_{k+1} + 3) + (3q - 3)] = [3q + 1, n_{k+1} + 3q] \\ =& [N_k + 3, N_{k+1} + 2]. \end{aligned}$$

Thus, the induced vertex weights form the set $[3, N_{k+1} + 2]$.

The labels of G_k are at most $\left\lceil \frac{N_k+2}{3} \right\rceil$ and the largest vertex label in $C_{n_{k+1}}$ is

$$\max\{f'(v) : v \in V(C_{n_{k+1}})\} + q - 1 = \left\lceil \frac{n_{k+1}+3}{3} \right\rceil + \frac{N_k+2}{3} - 1$$
$$= \left\lceil \frac{n_{k+1}+3}{3} + \frac{N_k+2}{3} - 1 \right\rceil = \left\lceil \frac{n_{k+1}+N_k+2}{3} \right\rceil = \left\lceil \frac{N_{k+1}+2}{3} \right\rceil$$

and the maximal edge label in $C_{n_{k+1}}$ is

$$\max\{f'(e) : e \in E(C_{n_{k+1}})\} + q - 1 = \left\lceil \frac{n_{k+1}+2}{3} \right\rceil + \frac{N_k+2}{3} - 1$$
$$= \left\lceil \frac{n_{k+1}+2}{3} + \frac{N_k+2}{3} - 1 \right\rceil = \left\lceil \frac{n_{k+1}+N_k+1}{3} \right\rceil = \left\lceil \frac{N_{k+1}+1}{3} \right\rceil.$$

Hence, the labels under the labeling h are at most $\left\lceil \frac{N_{k+1}+2}{3} \right\rceil$.

In every case, we have the desired modular vertex irregular total labeling h with the largest label of $\left\lceil \frac{N_{k+1}+2}{3} \right\rceil$. This implies that $\text{mtvs}(G) \leq \lceil \frac{n+2}{3} \rceil$. □

For example, consider $G = C_5 \cup C_7 \cup C_{10} \cup C_{11}$. Then, $n = 5 + 7 + 10 + 11 = 33$. Start with the labeling from (1.2) on C_5. Since the largest weight of C_5 is 7, then apply Case 1 from the preceding proof with $q = 2$. This yield that largest weight of $C_5 \cup C_7$ is 14. Similarly, apply Case 2 with $q = 4$ which implies the current largest label is 24. Finally, apply Case 3 with $q = 8$. We conclude that the largest weight and the largest label, respectively of this labeling in G are 35 and 12. Therefore, $\mathrm{mtvs}(G) = \lceil \frac{n+2}{3} \rceil = \lceil \frac{33+2}{3} \rceil = 12$. The labeling of G is depicted in Figure 1.

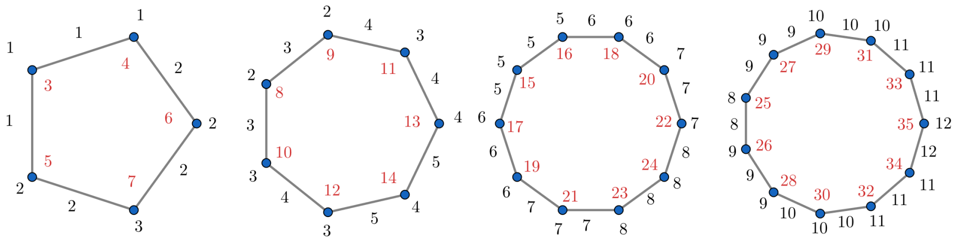

FIGURE 1. The graph $G = C_5 \cup C_7 \cup C_{10} \cup C_{11}$ with $\mathrm{mtvs}(G) = 12$.

We generalize the notion of an r-factor. Let R be a set of positive integers. An R-factor of a graph G is a subgraph H which contains every vertex of G such that every connected component of the R-factor is an r-regular graph, where $r \in R$.

Theorem 2.2. *Let $R = \{r_1, r_2, \dots, r_\ell\}$ be a set of positive integers with $r_1 < r_2 < \cdots < r_\ell$ and let $\mathrm{lcm}(R)$ be the least common multiple of integers in R. Let G be a graph and H be an R-factor of G. Then,*

$$\mathrm{mtvs}(G) \leq \max\left\{\frac{\mathrm{lcm}(R)}{r_1}, \mathrm{mtvs}(G - E(H))\right\}.$$

In particular, if $R = \{r_1\}$, then $\mathrm{mtvs}(G) \leq \mathrm{mtvs}(G - E(H))$.

Proof. Let O_i, $i \in [1, \ell]$ be the union of all connected r_i-regular components of H. Let f^* be a modular vertex irregular total k^*-labeling of $G - E(H)$ with $k^* = \mathrm{mtvs}(G - E(H))$. Define a total labeling f of G as follows

$$f(v) = f^*(v), \quad \text{if } v \in V(G),$$

$$f(e) = \begin{cases} \frac{\mathrm{lcm}(R)}{r_i}, & \text{if } e \in E(O_i),\ i \in [1, \ell], \\ f^*(e), & \text{if } e \in E(G) - E(H). \end{cases}$$

Evidently, the largest label of f is the maximum of k^* and $\frac{\mathrm{lcm}(R)}{r_1}$. Moreover, the weight of a vertex $v \in V(G)$ is $w_f(v) = w_{f^*}(t) + \mathrm{lcm}(R)$. This implies that f is a modular vertex irregular total labeling of G, hence $\mathrm{mtvs}(G) \leq \max\left\{\frac{\mathrm{lcm}(R)}{r_1}, \mathrm{mtvs}(G - E(H))\right\}$.

If $R = \{r_1\}$, then clearly $\mathrm{mtvs}(G) \leq \mathrm{mtvs}(G - E(H))$. □

With preceding theorem, we are able to provide an upper bound of the modular total vertex irregularity strength of some non-regular graphs. As an example, we consider disjoint union of circulant graphs. Let n, m, and $1 \leq a_1 < a_2 < \cdots < a_m \leq \left\lfloor \frac{n}{2} \right\rfloor$ be positive integers. An undirected graph with the set of vertices $V = \{v_0, v_1, \dots, v_{n-1}\}$ and the set of edges $E = \{v_i v_{i+a_j} : 0 \leq i \leq n-1, 1 \leq j \leq m\}$, with arithmetic in the indices being taken modulo n, is called a *circulant graph* and it is denoted by $C_n(a_1, \dots, a_m)$. Let $G = C_8(1,4) \cup C_9(1,4)$ as in Figure 2. Choose H to be the union of all inner edges of G. Since H is a $\{1,2\}$-factor, and $G - E(H) \cong C_8 \cup C_9$ then

$$\begin{aligned} \text{mtvs}(G) &\leq \max\left\{\frac{\text{lcm}(R)}{r_1}, \text{mtvs}(G - E(H))\right\} = \max\{2, \text{mtvs}(C_8 \cup C_9)\} \\ &= \left\lceil \frac{(8+9)+2}{3} \right\rceil = 7. \end{aligned}$$

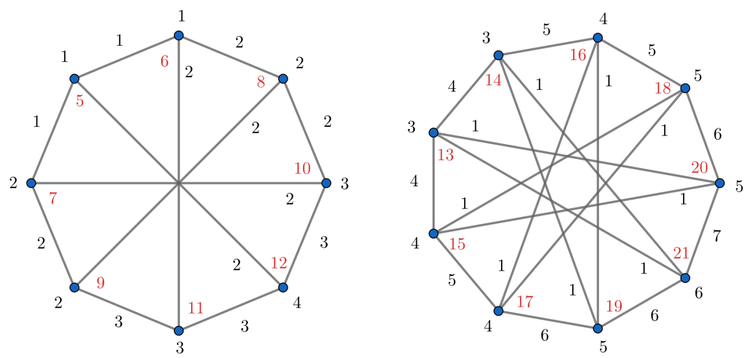

FIGURE 2. A modular vertex irregular total 7-labeling of $C_8(1,4) \cup C_9(1,4)$.

For the previous example we only have an upper bound for mtvs, not the exact value. Moreover, we can improve the obtained labeling. Do the following. In $C_9(1,4)$, decrease every vertex label by 2 and labels of all outer edges by 2. This will decrease the vertex weights by 6. Now, in $C_9(1,4)$ increase labels of all inner edges by 3. This will add 6 to the vertex weights. Thus, at the end all vertex weights remain the same, but the maximal label will be 5. Thus, $\text{mtvs}(C_8(1,4) \cup C_9(1,4)) \leq 5$. Note that the value of the mtvs is possible to be 4. Figure 3 presents the modified mtvs labeling of $C_8(1,4) \cup C_9(1,4)$.

Furthermore, we can use both Theorem 2.1 and Theorem 2.2 to find an upper bound of the modular total vertex irregularity strength of even regular graphs.

Theorem 2.3. *Let G be an even regular graph of order n. Then,*

$$\text{mtvs}(G) \leq \left\lceil \frac{n+2}{3} \right\rceil.$$

Proof. By Theorem 1.1, there exists a 2-factor H in the $2r$-regular graph G. By Theorem 2.2, $\text{mtvs}(G) \leq \text{mtvs}(G - E(H))$. The resulting graph $G - E(H)$ is a $2(r-1)$-regular graph and by Theorem 1.1 it contains a 2-factor. We can remove 2-factors repeatedly $r - 1$ times until the resulting graph will be isomorphic to a union

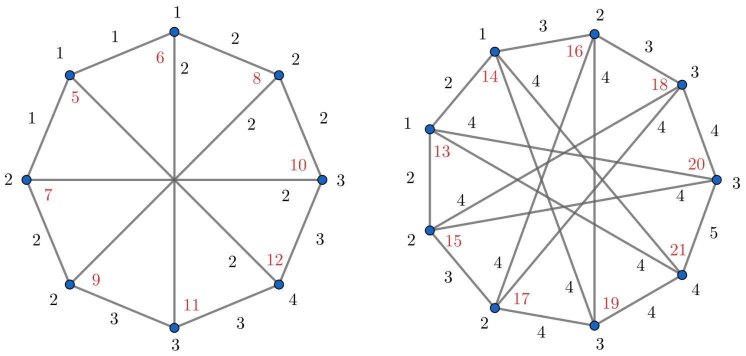

FIGURE 3. A modular vertex irregular total 5-labeling of $C_8(1,4) \cup C_9(1,4)$

of cycles $\bigcup_{i=1}^{p} C_{n_i}$, $\sum_{i=1}^{p} n_i = n$. This implies

$$\operatorname{mtvs}(G) \le \operatorname{mtvs}\left(\bigcup_{i=1}^{p} C_{n_i}\right) = \left\lceil \tfrac{n+2}{3} \right\rceil,$$

due to Theorem 2.1. □

For odd regular graphs satisfying an additional requirement we have a similar result as for even regular graphs.

Corollary 2.1. *Let G be an odd regular graph of order n containing a 1-factor. Then,*

$$\operatorname{mtvs}(G) \le \lceil \tfrac{n+2}{3} \rceil.$$

Proof. Let H be a 1-factor of G. Give the label 1 to every edge of H and combine it with the labeling determined by applying Theorem 2.3 to $G - E(H)$. □

For example, consider the 5-regular graph G which is the union of the complete graph K_6 and the complete bipartite graph $K_{5,5}$ as depicted in Figure 4. Choose the 1-factor H_1 of G denoted by the color green. Similarly, choose the 2-factor H_2 of G denoted by the color orange. It follows that $G - E(H_1 \cup H_2) \cong 2C_3 \cup C_{10}$, implying

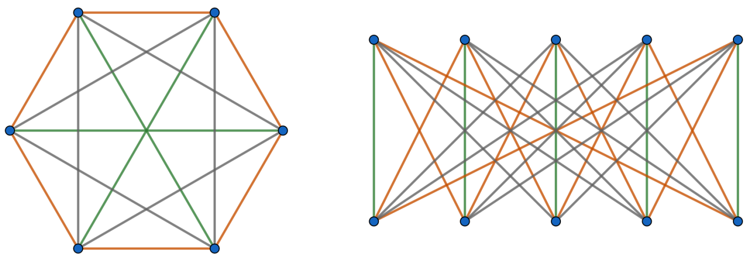

FIGURE 4. A decomposition of $K_6 \cup K_{5,5}$.

$$\text{mtvs}(G) \leq \text{mtvs}(G - E(H_1 \cup H_2)) = \text{mtvs}(2C_3 \cup C_{10}) = \left\lceil \frac{(6+10)+2}{3} \right\rceil = 6.$$

By (1.1), combined with Theorem 2.3 and Corollary 2.1, we have a result for the total vertex irregularity strength of some regular graphs.

Corollary 2.2. *Let G be an even regular graph or an odd regular graph containing a* 1*-factor and let G be of order n. Then,*

$$\text{tvs}(G) \leq \left\lceil \frac{n+2}{3} \right\rceil.$$

This result generalizes Theorem 1.3. Let G be an r-regular Hamiltonian graph of order n. If r is even, then we can apply Corollary 2.2 directly. Meanwhile, if r is odd, then n must be even. Since G is Hamiltonian with even order, then G has a 1-factor. Again, simply apply Corollary 2.2 and we have the desired result.

3. Conclusion

In this paper we investigated the existence of modular vertex irregular total labeling of regular graphs. We determined the precise value of the modular total vertex irregularity strength of any 2-regular graph that proves the sharpness of the presented lower bound. Using the concept of the R-factor, we presented a new upper bound of the modular total vertex irregularity strength for even regular graphs and for odd regular graphs containing a 1-factor.

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