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HALF LIGHTLIKE SUBMANIFOLDS OF A GOLDEN SEMI-RIEMANNIAN MANIFOLD

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Abstract. We present half lightlike submanifolds of a golden semi-Riemannian manifold. We prove that there is no radical anti-invariant half lightlike submanifold of a golden semi-Riemannian manifold. We get results for screen semi-invariant half lightlike submanifolds of a golden semi-Riemannian manifold. We prove the conditions for integrability of distributions on screen semi-invariant half lightlike submanifolds and investigate the geometry of leaves of distributions. Moreover, we study screen conformal half lightlike submanifolds of a golden semi-Riemannian manifold.

1. Introduction

The theory of lightlike submanifolds is a significant topic of research in modern differential geometry. Lightlike submanifolds were developed by Duggal and Bejancu [5] and Duggal and Şahin [9]. Class of lightlike submanifolds of codimension 2 is called half lightlike or coisotropic submanifolds the according to the rank of its radical distribution. This class is composed of two subclasses [6]. Half lightlike submanifold is a special case of the general r-lightlike submanifold such that r = 1 and its geometry is more general form than that of coisotropic submanifold or lightlike hypersurface [5]. Screen semi-invariant half lightlike submanifolds of a semi-Riemannian product manifold were studied in [3]. Real half lightlike submanifolds of an indefinite Kaehler manifold were studied in [17]. Semi-invariant lightlike submanifolds of a semi-Riemannian product manifold were presented in [2].

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Notion of a f-structure which is a (1,1)-tensor field of constant rank on $\tilde{N}$ and satisfies the equality $f^3 + f = 0$. It is a generalization of almost complex and almost contact structures. This notion is presented in [22]. It has been generalized by Goldberg and Yano. They defined a polynomial structure of degree d which is a (1,1)-tensor field f of constant rank on $\tilde{N}$ and satisfies the equation $Q(f) = f_d + a_d f_{d-1} + \cdots + a_2 f + a_1 I = 0$, where $a_1, a_2, \ldots, a_d$ are real numbers and I is the identity tensor of type $(1,1)$ [12]. The number $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$ which is a solution of the equation $x^2 - x - 1 = 0$ represents the golden proportion. The golden proportion has been used in many different areas such as in architecture, music, arts and philosophy. Using the golden proportion, Crasmareanu and Hretcanu defined a golden manifold $\tilde{N}$ by a tensor field $\tilde{P}$ on $\tilde{N}$ satisfies $\tilde{P}^2 = \tilde{P} + I$ in [4]. They also defined golden Riemannian manifolds and studied their submanifolds in [15]. Şahin and Akyol introduced golden maps between golden Riemannian manifolds and showed that such maps are harmonic maps [21]. Gök, Keleş and Kılıç studied some characterizations for any submanifold of a golden Riemannian manifold to be semi-invariant in terms of canonical structures on the submanifold, induced by the golden structure of the ambient manifold [13]. Poyraz and Yaşar introduced lightlike hypersurfaces of a golden semi-Riemannian manifold [20]. Moreover several works in this direction are studied [1, 10, 11].

In this paper, we introduce half lightlike submanifolds of a golden semi-Riemannian manifold. In Section 2, we give basic concepts. In Section 3, we introduce half lightlike submanifolds of a golden semi-Riemannian manifold. We define invariant, screen semi-invariant and radical anti-invariant half lightlike submanifolds. Moreover, we prove that there is no radical anti-invariant half lightlike submanifold of a golden semi-Riemannian manifold. In Section 4, we obtain results for screen semi-invariant half lightlike submanifolds. We prove the conditions for integrability of distributions on screen semi-invariant half lightlike submanifolds and investigate the geometry of leaves of distributions. We also give two examples. We find condition for its Ricci tensor to be symmetric. In Section 5, we investigate screen conformal half lightlike submanifolds of a golden semi-Riemannian manifold.

2. Preliminaries

Let $\tilde{N}$ be an n-dimensional differentiable manifold. If a tensor field $\tilde{P}$ of type $(1,1)$ satisfies the following equation

$$\tilde{P}^2 = \tilde{P} + I, \tag{2.1}$$

then $\tilde{P}$ is called a golden structure on $\tilde{N}$, where I is the identity transformation [14].

Let $(\tilde{N}, \tilde{g})$ be a semi-Riemannian manifold and $\tilde{P}$ be a golden structure on $\tilde{N}$. If $\tilde{P}$ satisfies the following equation

$$\tilde{g}(\tilde{P}X, Y) = \tilde{g}(X, \tilde{P}Y), \tag{2.2}$$

then $(\tilde{N}, \tilde{g}, \tilde{P})$ is called a golden semi-Riemannian manifold [19].

Let $(\tilde{N}, \tilde{g}, \tilde{P})$ be a golden semi-Riemannian manifold. Then the equation (2.2) is equivalent to

$$\tilde{g}(\tilde{P}X, \tilde{P}Y) = \tilde{g}(\tilde{P}X, Y) + \tilde{g}(X, Y), \tag{2.3}$$

for any $X, Y \in \Gamma(T\tilde{N})$.

If F is an almost product structure on $\tilde{N}$, then

$$\tilde{P} = \frac{1}{2}(I + \sqrt{5}F)$$

is a golden structure on $\tilde{N}$. Conversely, if $\tilde{P}$ is a golden structure on $\tilde{N}$, then

$$F = \frac{1}{\sqrt{5}}(2\tilde{P} - I)$$

is an almost product structure on $\tilde{N}$ [4].

Let $\acute{N}_p$ and $\acute{N}_q$ be real space forms with constant sectional curvatures c_p and c_q, respectively. Then similar calculations of semi-Riemannian product real space form (see [23]), one obtains the Riemannian curvature tensor $\tilde{R}$ of a locally golden product space form $(\tilde{N} = \acute{N}_p(c_p) \times \acute{N}_q(c_q), \tilde{g}, \tilde{P})$ as the following

$$\begin{aligned} \tilde{R}(X,Y)Z = &\left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right)\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y + \tilde{g}(\tilde{P}Y,Z)\tilde{P}X \\ &- \tilde{g}(\tilde{P}X,Z)\tilde{P}Y\} + \left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right)\{\tilde{g}(\tilde{P}Y,Z)X \\ &- \tilde{g}(\tilde{P}X,Z)Y + \tilde{g}(Y,Z)\tilde{P}X - \tilde{g}(X,Z)\tilde{P}Y\}. \end{aligned} \tag{2.4}$$

Let $(\tilde{N}, \tilde{g})$ be an $(n+3)$-dimensional semi-Riemannian manifold of index $q \geq 1$ and $\acute{N}$ be a lightlike submanifold of codimension 2 of $\tilde{N}$. Then the radical distribution $Rad(T\acute{N}) = T\acute{N} \cap T\acute{N}^{\perp}$ of $\acute{N}$ is a vector subbundle of the tangent bundle $T\acute{N}$ and the normal bundle $T\acute{N}^{\perp}$ of rank 1 or 2. If $rank(Rad(T\acute{N})) = 1$, then $\acute{N}$ is called half lightlike submanifold of $\tilde{N}$. Then there exist complementary non-degenerate distributions $S(T\acute{N})$ and $S(T\acute{N}^{\perp})$ of $Rad(T\acute{N})$ in $T\acute{N}$ and $T\acute{N}^{\perp}$, which are called the screen and the screen transversal distribution on $\acute{N}$ respectively. Thus, we have

$$T\acute{N} = Rad(T\acute{N}) \perp S(T\acute{N}),\ T\acute{N}^{\perp} = Rad(T\acute{N}) \perp S(T\acute{N}^{\perp}). \tag{2.5}$$

Choose $L \in \Gamma(S(T\acute{N}^{\perp}))$ as a unit vector field with $\tilde{g}\,(L, L) = \epsilon = \pm 1$. Consider the orthogonal complementary distribution $S(T\acute{N})^{\perp}$ to $S(T\acute{N})$ in $T\tilde{N}$. Then ξ and L belong to $\Gamma(S(T\acute{N})^{\perp})$. Thus, we obtain

$$S(T\acute{N})^{\perp} = S(T\acute{N}^{\perp}) \perp S(T\acute{N}^{\perp})^{\perp},$$

where $S(T\acute{N}^{\perp})^{\perp}$ is the orthogonal complementary to $S(T\acute{N}^{\perp})$ in $S(T\acute{N})^{\perp}$. For any null section $\xi \in Rad(T\acute{N})$ on a coordinate neighborhood $U \subset \acute{N}$, there exists a uniquely determined null vector field $N \in \Gamma(ltr(T\acute{N}))$ satisfying

$$\tilde{g}\,(N, \xi) = 1, \quad \tilde{g}\,(N, N) = \tilde{g}(N, X) = \tilde{g}(N, L) = 0, \quad \text{for all } X \in \Gamma(T\acute{N}).$$

We call N, $ltr(T\acute{N})$ and $tr(T\acute{N}) = S(T\acute{N}^{\perp}) \perp ltr(T\acute{N})$ the lightlike transversal vector field, lightlike transversal vector bundle and transversal vector bundle of $\acute{N}$ with respect to $S(T\acute{N})$, respectively. Hence we have

$$\begin{aligned} T\tilde{N} &= T\acute{N} \oplus tr(T\acute{N}) \\ &= \{Rad(T\acute{N}) \oplus ltr(T\acute{N})\} \perp S(T\acute{N}) \perp S(T\acute{N}^{\perp}). \end{aligned} \tag{2.6}$$

Let $\tilde{\nabla}$ be the Levi-Civita connection of $\tilde{N}$. Using (2.6) we define the projection morphism $Q : \Gamma(T\acute{N}) \to \Gamma(S(T\acute{N}))$. Hence we derive

$$\tilde{\nabla}_X Y = \nabla_X Y + D_1(X,Y)N + D_2(X,Y)L, \tag{2.7}$$

$$\tilde{\nabla}_X U = -A_U X + \nabla^t_X U, \tag{2.8}$$

$$\tilde{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.9}$$

$$\tilde{\nabla}_X L = -A_L X + \psi(X)N, \tag{2.10}$$

$$\nabla_X QY = \nabla^*_X QY + E(X,QY)\xi, \tag{2.11}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.12}$$

for any $X, Y \in \Gamma(T\acute{N})$, $\xi \in \Gamma(Rad(T\acute{N}))$, $U \in \Gamma(tr(T\acute{N}))$, $N \in \Gamma(ltr(T\acute{N}))$ and $L \in \Gamma(S(T\acute{N}^{\perp}))$. Then ∇ and ∇^* are called induced linear connections on $T\acute{N}$ and $S(T\acute{N})$ respectively, D_1 and D_2 are called the local second fundamental forms of $\acute{N}$, C is called the local second fundamental form on $S(T\acute{N})$. A_N, A^*_ξ and A_L are called linear operators on $T\acute{N}$. Also τ, ρ and ψ are called 1-forms on $T\acute{N}$. Since the connection $\tilde{\nabla}$ of $\tilde{N}$ is torsion-free, ∇ of $\acute{N}$ is also torsion-free and D_1 and D_2 are symmetric on $T\acute{N}$. D_1 and D_2 satisfy

$$D_1(X,\xi) = 0, \quad D_2(X,\xi) = -\epsilon\psi(X), \tag{2.13}$$

for all $X \in \Gamma(T\acute{N})$.

The induced connection ∇ of $\acute{N}$ is not metric and satisfies

$$(\nabla_X g)(Y,Z) = D_1(X,Y)\eta(Z) + D_1(X,Z)\eta(Y),$$

for any $X, Y, Z \in \Gamma(T\acute{N})$, where η is a $1-$form defined by

$$\eta(X) = \tilde{g}(X,N),$$

for all $X \in \Gamma(T\acute{N})$. Therefore, one obtains

$$D_1(X,Y) = g(A^*_\xi X, Y), \quad g(A^*_\xi X, N) = 0, \tag{2.14}$$

$$E(X,QY) = g(A_N X, QY), \quad g(A_N X, N) = 0, \tag{2.15}$$

$$\epsilon D_2(X,QY) = g(A_L X, QY), \quad g(A_L X, N) = \epsilon\rho(X), \tag{2.16}$$

$$\epsilon D_2(X,Y) = g(A_L X, Y) - \psi(X)\eta(Y), \tag{2.17}$$

for all $X, Y \in \Gamma(T\acute{N})$. By (2.14) and (2.15), A^*_ξ and A_N are $\Gamma(S(T\acute{N}))$-valued shape operators related to D_1 and E, respectively and $A^*_\xi\xi = 0$.

Using (2.7), (2.12) and (2.13), one derives

$$\tilde{\nabla}_X \xi = -A^*_\xi X - \tau(X)\xi - \epsilon\psi(X)L, \tag{2.18}$$

for any $X \in \Gamma(T\acute{N})$.

Definition 2.1. A half lightlike submanifold $(\acute{N}, g)$ of a semi-Riemannian manifold $(\tilde{N}, \tilde{g})$ is said to be irrotational [18] if $\tilde{\nabla}_X \xi \in \Gamma(T\acute{N})$ for any $X \in \Gamma(T\acute{N})$. From (2.13) and (2.18), definition of irrotational is equivalent to the condition $\psi(X) = 0$, that is, $D_2(X, \xi) = 0$ for any $X \in \Gamma(T\acute{N})$.

Definition 2.2. A half lightlike submanifold $(\acute{N}, g)$ of a semi-Riemannian manifold $(\tilde{N}, \tilde{g})$ is called totally umbilical in $\tilde{N}$, if there is a smooth vector field $H \in \Gamma(tr(T\acute{N}))$ on any coordinate neighborhood U such that

$$h(X, Y) = Hg(X, Y),$$

for any $X, Y \in \Gamma(T\acute{N})$, where

$$h(X, Y) = D_1(X, Y)N + D_2(X, Y)L \tag{2.19}$$

is the global second fundamental form tensor of $\acute{N}$. In case $H = 0$ on U, we say that $\acute{N}$ is totally geodesic [6].

It is easy to see that $\acute{N}$ is totally umbilical iff, on each coordinate neighborhood U, there exist smooth vector functions λ and δ such that

$$D_1(X, Y) = \lambda g(X, Y), D_2(X, Y) = \delta g(X, Y), \tag{2.20}$$

for any $X, Y \in \Gamma(T\acute{N})$.

Definition 2.3. We say that the screen distribution $S(T\acute{N})$ of $\acute{N}$ is totally umbilical [6] in $\acute{N}$ if there is a smooth function γ on any coordinate neighborhood $U \subset \acute{N}$ such that

$$E(X, QY) = \gamma g(X, Y), \tag{2.21}$$

for any $X, Y \in \Gamma(T\acute{N})$. If $\gamma = 0$ on U, then we say that $S(T\acute{N})$ is totally geodesic in $\acute{N}$.

We indicate by $\tilde{R}$, R and R^* the curvature tensors of $\tilde{\nabla}$, ∇ and ∇^*, respectively. From (2.7)–(2.12), we derive the Gauss-Codazzi equations for $\acute{N}$ and $S(T\acute{N})$:

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)Z, QW) =& g(R(X, Y)Z, QW) \\ &+ D_1(X, Z)E(Y, QW) - D_1(Y, Z)E(X, QW) \\ &+ \epsilon \{D_2(X, Z)D_2(Y, QW) - D_2(Y, Z)D_2(X, QW)\}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)Z, \xi) =& (\nabla_X D_1)(Y, Z) - (\nabla_Y D_1)(X, Z) \\ &+ \tau(X)D_1(Y, Z) - \tau(Y)D_1(X, Z) \\ &+ \psi(X)D_2(Y, Z) - \psi(Y)D_2(X, Z), \end{aligned} \tag{2.23}$$

$$\tilde{g}(\tilde{R}(X,Y)Z,N) = g(R(X,Y)Z,N) + \epsilon\left\{\rho(Y)D_2(X,Z) - \rho(X)D_2(Y,Z)\right\}, \tag{2.24}$$

$$\tilde{g}(\tilde{R}(X,Y)\xi,N) = g(A^*_\xi X, A_N Y) - g(A^*_\xi Y, A_N X) - 2d\tau(X,Y) + \rho(X)\psi(Y) - \rho(Y)\psi(X), \tag{2.25}$$

$$g(R(X,Y)QZ,QW) = g(R^*(X,Y)Z,QW) + D_1(Y,QW)E(X,QZ) - D_1(X,QW)E(Y,QZ), \tag{2.26}$$

$$\tilde{g}(R(X,Y)QZ,N) = (\nabla_X E)(Y,QZ) - (\nabla_Y E)(X,QZ) + \tau(Y)E(X,QZ) - \tau(X)E(Y,QZ), \tag{2.27}$$

for any $X,Y,Z \in \Gamma(T\acute{N})$.

3. Half Lightlike Submanifolds of a Golden Semi-Riemannian Manifold

Let $(\tilde{N},\tilde{g},\tilde{P})$ be a golden semi-Riemannian manifold and $\acute{N}$ be a half lightlike submanifold of $\tilde{N}$. For any $X \in \Gamma(T\acute{N})$, $N \in \Gamma(ltr(T\acute{N}))$ and $L \in \Gamma(S(T\acute{N}^\perp))$, we can write

$$\tilde{P}X = PX + \theta_1(X)N + \theta_2(X)L, \tag{3.1}$$

$$\tilde{P}N = U + \theta_1(N)N + \theta_2(N)L, \tag{3.2}$$

$$\tilde{P}L = W + \theta_1(L)N + \theta_2(L)L, \tag{3.3}$$

where $PX, U, W \in \Gamma(T\acute{N})$ and θ_1 and θ_2 are 1-forms defined by

$$\theta_1(\cdot) = g(\cdot, \tilde{P}\xi), \quad \theta_2(\cdot) = \epsilon g(\cdot, \tilde{P}L).$$

Lemma 3.1. *Let $\acute{N}$ be a half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N},\tilde{g},\tilde{P})$. Then, we have*

$$P^2X = PX + X - \theta_1(X)U - \theta_2(X)W, \tag{3.4}$$

$$\theta_1(PX) = \theta_1(X)(1-\theta_1(N)) - \theta_2(X)\theta_1(L), \tag{3.5}$$

$$\theta_2(PX) = \theta_2(X)(1-\theta_2(L)) - \theta_1(X)\theta_2(N), \tag{3.6}$$

$$PU = U(1-\theta_1(N)) - \theta_2(N)W, \tag{3.7}$$

$$\theta_1(U) = 1 + \theta_1(N) - (\theta_1(N))^2 - \theta_2(N)\theta_1(L), \tag{3.8}$$

$$\theta_2(U) = \theta_2(N)(1-\theta_1(N)) - \theta_2(N)\theta_2(L), \tag{3.9}$$

$$PW = (1-\theta_2(L))W - \theta_1(L)U, \tag{3.10}$$

$$\theta_1(W) = \theta_1(L)(1-\theta_1(N)-\theta_2(L)), \tag{3.11}$$

$$\theta_2(W) = 1 + \theta_2(L) - (\theta_2(L))^2 - \theta_1(L)\theta_2(N), \tag{3.12}$$

$$g(PX,Y) - g(X,PY) = (-\theta_1 \otimes \eta + \eta \otimes \theta_1)(X,Y), \tag{3.13}$$

$$g(PX,PY) = g(PX,Y) + g(X,Y) + \theta_1(X)\eta(Y) - \eta(PX)\theta_1(Y) \tag{3.14}$$

$$- \theta_1(X)\eta(PY) - \epsilon\theta_2(X)\theta_2(Y),$$

for any $X, Y \in \Gamma(T\acute{N})$.

Proof. Applying $\tilde{P}$ to (3.1), using (2.1) and taking tangential, lightlike transversal and screen transversal parts of the resulting equation, we derive (3.4), (3.5) and (3.6). Similarly, applying $\tilde{P}$ to (3.2) and (3.3), using (2.1), we get (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12). Using (2.2), (2.3) and (3.1), we obtain (3.13) and (3.14). □

Lemma 3.2. *Let* $\acute{N}$ *be a half lightlike submanifold of a golden semi-Riemannian manifold* $(\tilde{N}, \tilde{g}, \tilde{P})$ *with* $\tilde{\nabla}\tilde{P} = 0$. *Then, we have*

$$(\nabla_X P)Y = \theta_1(Y)A_N X + \theta_2(Y)A_L X + D_1(X,Y)U + D_2(X,Y)W, \tag{3.15}$$

$$\begin{aligned}(\nabla_X \theta_1)Y =& - D_1(X, PY) - \tau(X)\theta_1(Y) - \phi(X)\theta_2(Y)\\ &+ D_1(X,Y)\theta_1(N) + D_2(X,Y)\theta_1(L),\end{aligned} \tag{3.16}$$

$$\begin{aligned}(\nabla_X \theta_2)Y =& - D_2(X, PY) - \rho(X)\theta_1(Y) + D_1(X,Y)\theta_2(N)\\ &+ D_2(X,Y)\theta_2(L),\end{aligned} \tag{3.17}$$

$$\nabla_X U = - PA_N X + \tau(X)U + \rho(X)W + \theta_1(N)A_N X + \theta_2(N)A_L X, \tag{3.18}$$

$$D_1(X,U) = - X(\theta_1(N)) - \phi(X)\theta_2(N) - \theta_1(A_N X) + \rho(X)\theta_1(L), \tag{3.19}$$

$$\begin{aligned}D_2(X,U) =& - X(\theta_2(N)) - \rho(X)\theta_1(N) - \theta_2(A_N X) + \tau(X)\theta_2(N)\\ &+ \rho(X)\theta_2(L),\end{aligned} \tag{3.20}$$

$$\nabla_X W = - PA_L X + \theta_1(L)A_N X + \theta_2(L)A_L X + \phi(X)U, \tag{3.21}$$

$$\begin{aligned}D_1(X,W) =& - \tau(X)\theta_1(L) - \phi(X)\theta_2(L) - \theta_1(A_L X) + \phi(X)\theta_1(N)\\ &- X(\theta_1(L)),\end{aligned} \tag{3.22}$$

$$D_2(X,W) = - \rho(X)\theta_1(L) - X(\theta_2(L)) - \theta_2(A_L X) + \phi(X)\theta_2(N), \tag{3.23}$$

for any $X, Y \in \Gamma(T\acute{N})$.

Proof. Since $\tilde{\nabla}\tilde{P} = 0$, we obtain $\tilde{\nabla}_X\tilde{P}Y = \tilde{P}\tilde{\nabla}_X Y$ for any $X, Y \in \Gamma(T\acute{N})$. Taking tangential, lightlike transversal and screen transversal parts of the resulting equation, we get (3.15), (3.16) and (3.17). Similarly, replacing Y with N and L respectively we obtain (3.18), (3.19), (3.20), (3.21), (3.22) and (3.23). □

Throughout this paper, we assume that $\tilde{\nabla}\tilde{P} = 0$.

Definition 3.1. Let $\acute{N}$ be a half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$.

i) We say that $\acute{N}$ is an invariant half lightlike submanifold if $\tilde{P}(T\acute{N}) = T\acute{N}$.
ii) We say that $\acute{N}$ is a screen semi-invariant half lightlike submanifold if $\tilde{P}(Rad(T\acute{N})) \subset S(T\acute{N})$ and $\tilde{P}(ltr(T\acute{N})) \subset S(T\acute{N})$.
iii) We say that $\acute{N}$ is a radical anti-invariant half lightlike submanifold if $\tilde{P}(Rad(T\acute{N})) = ltr(T\acute{N})$.

Theorem 3.1. *Let $\acute{N}$ be a half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then the following assertions are equivalent.*

i) *$\acute{N}$ is invariant.*
ii) *θ_1 and θ_2 vanish on $\acute{N}$.*
iii) *P is a golden structure on $\acute{N}$.*

Proof. $\acute{N}$ is invariant if and only if $\tilde{P}X = PX$ for any $X \in \Gamma(T\acute{N})$. Then $\theta_1(X) = \theta_2(X) = 0$ and we obtain i)$\Leftrightarrow$ii).

θ_1 and θ_2 vanish on $\acute{N}$ if and only if $\tilde{P}X = PX$ for any $X \in \Gamma(T\acute{N})$. Then $P^2X = PX + X$ and $g(PX, Y) = g(X, PY)$ for any $X, Y \in \Gamma(T\acute{N})$. Thus, P is a golden structure on $\acute{N}$ and we get ii)$\Leftrightarrow$iii). □

Theorem 3.2. *There is no radical anti-invariant half lightlike submanifold of a golden semi-Riemannian manifold.*

Proof. Suppose on the contrary that $\acute{N}$ is a radical anti-invariant half lightlike submanifold of a golden semi-Riemannian manifold $\tilde{N}$. By the definition of radical anti-invariant for $\xi \in \Gamma(Rad(T\acute{N}))$, $\tilde{P}\xi \in \Gamma(ltr(T\acute{N}))$. Using (2.3), we obtain

$$\begin{aligned} \tilde{g}(\tilde{P}\xi, \tilde{P}\xi) &= \tilde{g}(\tilde{P}\xi, \xi) + \tilde{g}(\xi, \xi), \\ 0 &= \tilde{g}(\tilde{P}\xi, \xi) + 0. \end{aligned}$$

Thus, $\tilde{g}(\tilde{P}\xi, \xi) = 0$ and $\tilde{P}\xi \notin \Gamma(ltr(T\acute{N}))$ which is a contradiction. □

4. Screen Semi-invariant Half Lightlike Submanifolds of a Golden Semi-Riemannian Manifold

Let $(\acute{N}, g, S(T\acute{N}))$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. If we take $L_1 = \tilde{P}(Rad(T\acute{N}))$, $L_2 = \tilde{P}(ltr(T\acute{N}))$ and $L_3 = \tilde{P}(S(T\acute{N}^\perp))$, then we can write

$$S(T\acute{N}) = L_0 \perp \{L_1 \oplus L_2\} \perp L_3,$$

where L_0 is a $(n-3)$-dimensional distribution. Therefore, we have

$$T\tilde{N} = L_0 \perp \{L_1 \oplus L_2\} \perp L_3 \perp \{Rad(T\acute{N}) \oplus ltr(T\acute{N})\} \perp S(T\acute{N}^\perp).$$

If we set

$$L = L_0 \perp Rad(T\acute{N}) \perp \tilde{P}(Rad(T\acute{N})) \quad \text{and} \quad L^\perp = L_2 \perp L_3,$$

we can write

$$T\acute{N} = L \oplus L^\perp, \quad T\tilde{N} = \{L \oplus L^\perp\} \oplus ltr(T\acute{N}) \perp S(T\acute{N}^\perp). \tag{4.1}$$

Let U, V and W be vector fields defined by

$$U = \tilde{P}N, \quad V = \tilde{P}\xi, \quad W = \tilde{P}L. \tag{4.2}$$

From Lemma 3.1, Lemma 3.2, differentiating (4.2) with X and using Gauss-Weingarten formulas we obtain

$$P^2X = PX + X - \theta_1(X)U - \theta_2(X)W, \tag{4.3}$$

$$\theta_1(PX) = \theta_1(X), \quad \theta_2(PX) = \theta_2(X), \quad PU = U, \quad PW = W, \tag{4.4}$$

$$\theta_1(U) = 1, \quad \theta_2(U) = 0, \quad \theta_1(W) = 0, \quad \theta_2(W) = 1, \tag{4.5}$$

$$g(PX, Y) - g(X, PY) = (-\theta_1 \otimes \eta + \eta \otimes \theta_1)(X, Y), \tag{4.6}$$

$$\begin{aligned} g(PX, PY) =& g(PX, Y) + g(X, Y) + \theta_1(X)\eta(Y) - \eta(PX)\theta_1(Y) \\ & - \theta_1(X)\eta(PY) - \epsilon\theta_2(X)\theta_2(Y), \end{aligned} \tag{4.7}$$

$$\begin{aligned} (\nabla_X P)Y =& \theta_1(Y)A_N X + \theta_2(Y)A_L X + D_1(X, Y)U \\ & + D_2(X, Y)W, \end{aligned} \tag{4.8}$$

$$(\nabla_X \theta_1)Y = - D_1(X, PY) - \tau(X)\theta_1(Y) - \psi(X)\theta_2(Y), \tag{4.9}$$

$$(\nabla_X \theta_2)Y = - D_2(X, PY) - \rho(X)\theta_1(Y), \tag{4.10}$$

$$\nabla_X U = - PA_N X + \tau(X)U + \rho(X)W, \tag{4.11}$$

$$\nabla_X V = - PA^*_\xi X - \tau(X)V - \epsilon\psi(X)W, \tag{4.12}$$

$$\nabla_X W = - PA_L X + \psi(X)U, \tag{4.13}$$

$$D_1(X, U) = - E(X, V), \quad D_1(X, W) = -\epsilon D_2(X, V), \tag{4.14}$$

$$\epsilon D_2(X, U) = - E(X, W),$$

$$D_1(X, V) = E(X, U) = D_2(X, W) = 0, \tag{4.15}$$

for any $X, Y \in \Gamma(T\acute{N})$.

Corollary 4.1. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then, we have*

$$D_1(X, V) = 0,$$

that is, vector field V degenerates local second fundamental form of $\acute{N}$.

Corollary 4.2. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then, there is no L_2-component of A^*_ξ.*

Proof. From (2.14)-1 and (4.15), we get $D_1(X, V) = g(A^*_\xi X, V) = 0$. Thus, the proof is completed. □

Corollary 4.3. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then, there is no L_1-component of A_N.*

Proof. From (2.15)-1 and (4.15), we obtain $E(X, U) = g(A_N X, U) = 0$, which proves the assertion. □

Corollary 4.4. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then, there is no L_3-component of A_L.*

Proof. From (2.16)-1 and (4.15), we have $D_2(X,W) = g(A_L X, W) = 0$. Thus, the proof is completed. □

Proposition 4.1. *The distribution L_0 and L are invariant distributions with respect to $\tilde{P}$.*

Example 4.1. Let $(\tilde{N} = \mathbb{R}^7_3, \tilde{g})$ be a 7-dimensional semi-Euclidean space with signature $(-,+,-,+,+,+,-)$ and $(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$ be the standard coordinate system of $\mathbb{R}^7_3$. If we define a mapping $\tilde{P}$ by $\tilde{P}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = (\phi x_1, \phi x_2, (1-\phi)x_3, (1-\phi)x_4, \phi x_5, (1-\phi)x_6, \phi x_7)$, then $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\tilde{N}$. Let $\acute{N}$ be a half lightlike submanifold in $\tilde{N}$ given by the equations

$$\begin{aligned} &x_1 = t_1 + \phi t_2 - \frac{\phi}{2(2+\phi)} t_3, \quad x_2 = t_1 + \phi t_2 + \frac{\phi}{2(2+\phi)} t_3, \\ &x_3 = \phi t_1 - t_2 + \frac{1}{2(2+\phi)} t_3, \quad x_4 = \phi t_1 - t_2 - \frac{1}{2(2+\phi)} t_3, \\ &x_5 = \sqrt{2}\phi t_4 + t_5, \quad x_6 = -t_4, \quad x_7 = \phi t_4 + \sqrt{2} t_5, \end{aligned}$$

where t_i, $1 \le i \le 5$, are real parameters. Thus, $T\acute{N} = \mathrm{Span}\{U_1, U_2, U_3, U_4, U_5\}$, where

$$\begin{aligned} &U_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \phi \frac{\partial}{\partial x_3} + \phi \frac{\partial}{\partial x_4}, \quad U_2 = \phi \frac{\partial}{\partial x_1} + \phi \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}, \\ &U_3 = -\frac{1}{2(2+\phi)} \left(\phi \frac{\partial}{\partial x_1} - \phi \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} \right), \quad U_4 = \sqrt{2}\phi \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6} + \phi \frac{\partial}{\partial x_7}, \\ &U_5 = \frac{\partial}{\partial x_5} + \sqrt{2} \frac{\partial}{\partial x_7}. \end{aligned}$$

We easily check that the vector U_1 is a degenerate vector, $\acute{N}$ is a 1-lightlike submanifold of $\tilde{N}$. We set $\xi = U_1$, then we have $Rad(T\acute{N}) = \mathrm{Span}\{\xi\}$ and $S(T\acute{N}) = \mathrm{Span}\{U_2, U_3, U_4, U_5\}$. We can easily obtain

$$ltr(T\acute{N}) = \mathrm{Span}\left\{ N = -\frac{1}{2(2+\phi)} \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \phi \frac{\partial}{\partial x_3} - \phi \frac{\partial}{\partial x_4} \right) \right\}$$

and

$$S(T\acute{N}^{\perp}) = \mathrm{Span}\left\{ L = \sqrt{2} \frac{\partial}{\partial x_5} + \phi \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7} \right\}.$$

Thus, $\acute{N}$ is a half lightlike submanifold of $\tilde{N}$. We also get

$$\tilde{P}\xi = U_2, \quad \tilde{P}N = U_3, \quad \tilde{P}L = U_4.$$

If we set $L_0 = \mathrm{Span}\{U_5\}$, $L_1 = \mathrm{Span}\{U_2\}$, $L_2 = \mathrm{Span}\{U_3\}$, $L_3 = \mathrm{Span}\{U_4\}$, then $\acute{N}$ is a screen semi-invariant half lightlike submanifold of $\tilde{N}$.

Example 4.2. Let $(\tilde{N} = \mathbb{R}^8_2, \tilde{g})$ be a 8-dimensional semi-Euclidean space with signature $(+,+,-,+,-,+,+,+)$ and $(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)$ be the standard coordinate system of $\mathbb{R}^8_2$. If we define a mapping $\tilde{P}$ by $\tilde{P}(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) =$

$(\phi x_1, \phi x_2, \phi x_3, \phi x_4, (1-\phi)x_5, \phi x_6, (1-\phi)x_7, (1-\phi)x_8)$, then $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\tilde{N}$. Let $\acute{N}$ be a half lightlike submanifold in $\tilde{N}$ given by the equations

$$\begin{aligned}
&x_1 = t_1 + t_4 + \phi t_5 + t_6, \quad x_2 = -t_2 + t_4 + \phi t_5, \\
&x_3 = \frac{1}{\sqrt{2}} t_1 + \frac{1}{\sqrt{2}} t_2 + \sqrt{2} t_4 + \sqrt{2}\phi t_5 + \frac{1}{\sqrt{2}} t_6, \quad x_4 = \frac{1}{2}\log(1 + (t_1 - t_2)^2), \\
&x_5 = (1-\phi)t_2 + \phi t_4 - t_5, \quad x_6 = \phi t_3, \quad x_7 = -(1-\phi)t_2 + \phi t_4 - t_5, \quad x_8 = t_3,
\end{aligned}$$

where t_i, $1 \leq i \leq 6$, are real parameters. Thus, $T\acute{N} = \mathrm{Span}\{U_1, U_2, U_3, U_4, U_5, U_6\}$, where

$$\begin{aligned}
U_1 &= \frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3} + \frac{(t_1 - t_2)}{(1 + (t_1 - t_2)^2)}\frac{\partial}{\partial x_4}, \\
U_2 &= -\frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3} - \frac{(t_1 - t_2)}{(1 + (t_1 - t_2)^2)}\frac{\partial}{\partial x_4} + (1-\phi)\frac{\partial}{\partial x_5} - (1-\phi)\frac{\partial}{\partial x_7}, \\
U_3 &= \phi\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_8}, \quad U_4 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_3} + \phi\frac{\partial}{\partial x_5} + \phi\frac{\partial}{\partial x_7}, \\
U_5 &= \phi\frac{\partial}{\partial x_1} + \phi\frac{\partial}{\partial x_2} + \sqrt{2}\phi\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_7}, \quad U_6 = \frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3}.
\end{aligned}$$

It follows that $Rad(T\acute{N}) = \mathrm{Span}\{U_4\}$ and $S(T\acute{N}) = \mathrm{Span}\{W_1 = U_1, W_2 = U_5, W_3 = -\frac{\phi}{2(2+\phi)}(U_1 + U_2), W_4 = U_3, W_5 = U_6\}$. By direct calculations we obtain

$$ltr(T\acute{N}) = \mathrm{Span}\left\{N = -\frac{1}{2(2+\phi)}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_3} + \phi\frac{\partial}{\partial x_5} - \phi\frac{\partial}{\partial x_7}\right)\right\}$$

and

$$S(T\acute{N}^{\perp}) = \mathrm{Span}\left\{L = \frac{\partial}{\partial x_6} - \phi\frac{\partial}{\partial x_8}\right\}.$$

Thus, $\acute{N}$ is a half lightlike submanifold of $\tilde{N}$. We also get

$$\tilde{P}\xi = W_2, \quad \tilde{P}N = W_3, \quad \tilde{P}L = W_4.$$

If we set $L_0 = \mathrm{Span}\{W_1, W_5\}$, $L_1 = \mathrm{Span}\{W_2\}$, $L_2 = \mathrm{Span}\{W_3\}$, $L_3 = \mathrm{Span}\{W_4\}$, then $\acute{N}$ is a screen semi-invariant half lightlike submanifold of $\tilde{N}$.

Theorem 4.1. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then L_0 is integrable if and only if*

$$\begin{aligned}
D_1(\tilde{P}X, \tilde{P}Y) &= D_1(\tilde{P}X, Y) + D_1(X, Y), \\
E(\tilde{P}X, \tilde{P}Y) &= E(Y, \tilde{P}X) + E(Y, X), \\
D_2(\tilde{P}X, \tilde{P}Y) &= D_2(\tilde{P}X, Y) + D_2(X, Y), \\
E(\tilde{P}X, Y) &= E(Y, \tilde{P}X),
\end{aligned}$$

for any $X, Y \in \Gamma(L_0)$.

Proof. Since L_0 is invariant, if $X \in \Gamma(L_0)$, then $\tilde{P}X \in \Gamma(L_0)$. The distribution L_0 is integrable if and only if

$$\theta_1([\tilde{P}X, Y]) = \theta_3([\tilde{P}X, Y]) = \theta_2([\tilde{P}X, Y]) = \eta([\tilde{P}X, Y]) = 0,$$

for any $X, Y \in \Gamma(L_0)$, where θ_3 is 1-form defined by

$$\theta_3(X) = g(X, \tilde{P}N).$$

Then from (2.2), (2.3), (2.7) and (2.11) we derive

$$\theta_1([\tilde{P}X, Y]) = D_1(\tilde{P}X, \tilde{P}Y) - D_1(Y, \tilde{P}X) - D_1(Y, X), \tag{4.16}$$

$$\theta_3([\tilde{P}X, Y]) = E(\tilde{P}X, \tilde{P}Y) - E(Y, \tilde{P}X) - E(Y, X), \tag{4.17}$$

$$\theta_2([\tilde{P}X, Y]) = \epsilon D_2(\tilde{P}X, \tilde{P}Y) - \epsilon D_2(Y, \tilde{P}X) - \epsilon D_2(Y, X), \tag{4.18}$$

$$\eta([\tilde{P}X, Y]) = E(\tilde{P}X, Y) - E(Y, \tilde{P}X). \tag{4.19}$$

From (4.16), (4.17), (4.18) and (4.19) we derive our theorem. □

Theorem 4.2. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then the distribution L is integrable if and only if*

$$D_1(\tilde{P}X, \tilde{P}Y) = D_1(\tilde{P}X, Y) + D_1(X, Y),$$
$$D_2(\tilde{P}X, \tilde{P}Y) = D_2(\tilde{P}X, Y) + D_2(X, Y),$$

for any $X, Y \in \Gamma(L)$.

Proof. L is integrable if and only if

$$\theta_1([\tilde{P}X, Y]) = \theta_2([\tilde{P}X, Y]) = 0,$$

for any $X, Y \in \Gamma(L)$. Then using (4.16) and (4.18) we obtain our assertion. □

Theorem 4.3. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. If L_0 is integrable, then leaves of L_0 have a golden structure.*

Proof. Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold and $\acute{N}'$ be a leaf of L_0. Then for any $p \in \acute{N}'$ we obtain $T_p\acute{N}' = (L_0)_p$. Since $X \in \Gamma(L_0)$, then $\theta_1(X) = \theta_2(X) = 0$. Therefore, from (3.1) we get $\tilde{P}X = PX$.

Letting $P' = P_{|_{L_0}}$, we say that P' defines an $(1,1)$-tensor field on $\acute{N}'$ because L_0 is $\tilde{P}$-invariant. For any $X \in \Gamma(L_0)$ we derive$P'^2X = P'X + X$, which proves the assertion. □

From Theorem 4.3 we derive Corollary 4.5.

Corollary 4.5. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. If L is integrable, then the leaves of L have a golden structure.*

Theorem 4.4. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then the following assertions are equivalent.*

i) *The distribution L is parallel.*
ii) $D_1(X, \tilde{P}Y) = D_2(X, \tilde{P}Y) = 0$ *for any* $X, Y \in \Gamma(L)$.
iii) $(\nabla_X P)Y = 0$ *for any* $X, Y \in \Gamma(L)$.

Proof. Using (4.1)-1, L is parallel if and only if $\theta_1(\nabla_X Y) = \theta_2(\nabla_X Y) = 0$, for any $X, Y \in \Gamma(L)$. Then from (2.7), we derive

$$\theta_1(\nabla_X Y) = D_1(X, \tilde{P}Y),$$
$$\theta_2(\nabla_X Y) = D_2(X, \tilde{P}Y).$$

Thus, we derive i)⇔ii). For any $Y \in \Gamma(L)$, then $\theta_1(Y) = \theta_2(Y) = 0$. From (4.8), we derive

$$(\nabla_X P)Y = D_1(X, Y)U + D_2(X, Y)W.$$

Hence, we have ii)⇔iii). □

Theorem 4.5. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then $\acute{N}$ is totally geodesic if and only if*

$$(\nabla_X P)Y = 0, \tag{4.20}$$
$$(\nabla_X P)U = A_N X, \tag{4.21}$$
$$(\nabla_X P)W = A_L X, \tag{4.22}$$

for any $X \in \Gamma(T\acute{N})$ and $Y \in \Gamma(L)$.

Proof. Let $\acute{N}$ be totally geodesic. For any $Y \in \Gamma(L)$, we have $\theta_1(Y) = \theta_2(Y) = 0$ and thus, from (4.8), we get $(\nabla_X P)Y = 0$. Similarly, letting $Y = U$ in (4.8), we get $(\nabla_X P)U = A_N X$. Similarly, letting $Y = W$ in (4.8), we get $(\nabla_X P)W = A_L X$.

Conversely, we suppose that the conditions (4.20), (4.21) and (4.22) hold. If $Y \in \Gamma(T\acute{N})$, using (4.1)-1, we can write $Y = Y_l + fU + hW$ for any $Y \in \Gamma(T\acute{N})$. Thus, we obtain

$$D_1(X, Y) = D_1(X, Y_l) + fD_1(X, U) + hD_1(X, W), \tag{4.23}$$
$$D_2(X, Y) = D_2(X, Y_l) + fD_2(X, U) + hD_2(X, W). \tag{4.24}$$

Using (4.20) and replacing Y by Y_l in (4.8), we find $D_1(X, Y_l)U + D_2(X, Y_l)W = -\theta_1(Y_l)A_N X - \theta_2(Y_l)A_L X = 0$. From this fact we get $D_1(X, Y_l) = D_2(X, Y_l) = 0$. Using (4.21) and replacing Y by U in (4.8), we derive $D_1(X, U)U + D_2(X, U)W = 0$. From this we obtain $D_1(X, U) = D_2(X, U) = 0$. From (4.15) we have $D_2(X, W) = 0$. Moreover, replacing Y by W in (4.8), using (4.15) and (4.22) we derive $D_1(X, W)U = 0$, which implies $D_1(X, W) = 0$. Considering (4.23) and (4.24) we obtain $D_1 = D_2 = 0$. Hence, the claim holds. □

Definition 4.1. Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then $\acute{N}$ is mixed totally geodesic if and only if

$$D_1(X, Y) = D_2(X, Y) = 0,$$

for any $X \in \Gamma(L)$ and $Y \in \Gamma(L^{\perp})$.

Theorem 4.6. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then the following assertions are equivalent.*

i) *$\acute{N}$ is mixed totally geodesic.*
ii) *$\nabla_Y \tilde{P}X$ has no component in $\Gamma(L^{\perp})$ for any $X \in \Gamma(L)$ and $Y \in \Gamma(L^{\perp})$.*
iii) *$(\nabla_Y P)X = 0$ for any $X \in \Gamma(L)$ and $Y \in \Gamma(L^{\perp})$.*

Proof. $\acute{N}$ is mixed totally geodesic if and only if for any $X \in \Gamma(L)$, $Y \in \Gamma(L^{\perp})$, $D_1(X, Y) = D_2(X, Y) = 0$. Since D_1 and D_2 are symmetric and using (2.3) and (2.7) we get

$$D_1(X, Y) = D_1(Y, X) = g(\nabla_Y \tilde{P}X, \tilde{P}\xi) - D_1(Y, \tilde{P}X) \tag{4.25}$$

and

$$\epsilon D_2(X, Y) = \epsilon D_2(Y, X) = g(\nabla_Y \tilde{P}X, \tilde{P}L) - \epsilon D_2(Y, \tilde{P}X). \tag{4.26}$$

From (4.25) and (4.26) we derive $D_1(Y, X) + D_1(Y, \tilde{P}X) = D_1(Y, \tilde{P}^2 X) = g(\nabla_Y \tilde{P}X, \tilde{P}\xi)$ and $\epsilon D_2(Y, X) + \epsilon D_2(Y, \tilde{P}X) = \epsilon D_2(Y, \tilde{P}^2 X) = g(\nabla_Y \tilde{P}X, \tilde{P}L)$, respectively. Thus, we have i)$\Leftrightarrow$ii). For any $X \in \Gamma(L)$, $Y \in \Gamma(L^{\perp})$, $\theta_1(X) = \theta_2(X) = 0$ and from (4.8) we derive

$$(\nabla_Y P)X = D_1(Y, X)U + D_2(Y, X)W.$$

Thus, we have i)$\Leftrightarrow$iii). □

Theorem 4.7. *Let $\acute{N}$ be a totally umbilical screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then $\acute{N}$ is totally geodesic.*

Proof. Let $\acute{N}$ be a totally umbilical screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $\tilde{N}$. From (4.14)-2, we have $D_1(X, W) = -\epsilon D_2(X, V)$. Since $\acute{N}$ is totally umbilical, using (2.20) we derive $\lambda g(X, W) = -\epsilon\delta g(X, V)$. Replacing X by U and W in this equation, respectively, we obtain $\lambda = \delta = 0$. Thus, the proof is completed. □

Theorem 4.8. *Let $\acute{N}$ be a totally umbilical screen semi-invariant half lightlike submanifold of a locally golden product space form $(\tilde{N} = \acute{N}_p(c_p) \times \acute{N}_q(c_q), \tilde{g}, \tilde{P})$. Then we have $c_p = c_q = 0$.*

Proof. From (2.4) we get

$$\tilde{g}(\tilde{R}(X, Y)Z, \xi) = \left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right)\left\{g(\tilde{P}Y, Z)\theta_1(X) - g(\tilde{P}X, Z)\theta_1(Y)\right\}$$

$$+\left(-\frac{(1-\phi)c+\phi c_q}{4}\right)\{g(Y,Z)\theta_1(X)-g(X,Z)\theta_1(Y)\}, \tag{4.27}$$

for any $X, Y, Z \in \Gamma(T\acute{N})$. Using Theorem 4.7 in (2.23) we derive $\tilde{g}(\tilde{R}(X,Y)Z,\xi)=0$. Moreover, letting $X=U$, $Y=\xi$, $Z=U$ in (4.27), we obtain

$$\frac{(1-\phi)c_p-\phi c_q}{2\sqrt{5}}=0. \tag{4.28}$$

Similarly, if we let $X=U$, $Y=V$, $Z=U$ in (4.27), we get

$$\left(-\frac{(1-\phi)c_p-\phi c_q}{2\sqrt{5}}\right)+\left(-\frac{(1-\phi)c_p+\phi c_q}{4}\right)=0. \tag{4.29}$$

From (4.28) and (4.29), we obtain $c_p=c_q=0$. Thus, the proof is completed. □

Theorem 4.9. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N},\tilde{g},\tilde{P})$. If $S(T\acute{N})$ is totally umbilical, then $S(T\acute{N})$ is totally geodesic.*

Proof. Let $S(T\acute{N})$ be a totally umbilical. From (2.21) and (4.15) we get

$$E(X,U)=\gamma g(X,U)=0,$$

for any $X\in\Gamma(T\acute{N})$. Letting $X=V$ in last equation, we obtain $\gamma=0$, i.e., $E=0$. Thus, the proof is completed. □

Theorem 4.10. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N},\tilde{g},\tilde{P})$. Then we have the following assertions.*

i) *If P is parallel with respect to ∇ on $\acute{N}$, then $\rho(X)=\psi(X)=D_2(X,U)=0$,*

$$E(X,Z)\theta_1(Y)+\epsilon D_2(X,Z)\theta_2(Y)+D_1(X,Y)\theta_3(Z)+D_2(X,Y)\theta_2(Z)=0 \tag{4.30}$$

and

$$D_1(X,Y)=-E(X,V)\theta_1(Y)-\epsilon D_2(X,V)\theta_2(Y), \tag{4.31}$$

$$D_2(X,Y)=-E(X,W)\theta_1(Y), \tag{4.32}$$

for any $X,Y\in\Gamma(T\acute{N})$ and $Z\in\Gamma(S(T\acute{N}))$.

ii) *If V is parallel with respect to ∇ on $\acute{N}$, then $\tau(X)=0$,*

$$A^*_\xi X=\theta_2(A^*_\xi X)W \quad \textit{and} \quad \theta_2(A^*_\xi X)=-\epsilon\psi(X),$$

for any $X\in\Gamma(T\acute{N})$.

iii) *If U is parallel with respect to ∇ on $\acute{N}$, then*

$$A_NX=\theta_1(A_NX)U+\theta_2(A_NX)W,\quad \theta_1(A_NX)=\tau(X) \quad \textit{and} \quad \theta_2(A_NX)=\rho(X),$$

for any $X\in\Gamma(T\acute{N})$.

iv) *If W is parallel with respect to ∇ on $\acute{N}$, then $\rho(X)=0$,*

$$A_LX=\psi(X)U \quad \textit{and} \quad \theta_1(A_LX)=\psi(X),$$

for any $X \in \Gamma(T\acute{N})$.

Moreover, if all of V*,* U *and* W *are parallel with respect to* ∇ *on* $\acute{N}$*, then* $S(T\acute{N})$ *is totally geodesic in* $\acute{N}$ *and* $\tau = \rho = 0$.

Proof. Let P be parallel with respect to ∇. Then taking the scalar product with for any $Z \in \Gamma(T\acute{N})$ and V in (4.8) we obtain (4.30) and (4.31), respectively. Taking the scalar product with W in (4.8) and using (4.15) we derive (4.32). Moreover, taking the scalar product with for any N in (4.8) we get

$$\epsilon\rho(X)\theta_2(Y) = 0. \tag{4.33}$$

Taking $Y = W$ in (4.33) we get $\rho(X) = 0$. Similarly, taking the scalar product with for any U in (4.8) and using (4.15) we get $D_2(X,U) = 0$. Moreover letting $Y = \xi$ in (4.32) we get $\psi(X) = 0$.

If V is parallel with respect to ∇ on $\acute{N}$, then from (4.12) we obtain

$$-\tilde{P}A^*_\xi X + \theta_1(A^*_\xi X)N + \theta_2(A^*_\xi X)L - \tau(X)V - \epsilon\psi(X)W = 0.$$

Using (2.14)-1 and (4.15), we derive $D_1(X,V) = \theta_1(A^*_\xi X) = 0$. Thus, we get

$$-\tilde{P}A^*_\xi X + \theta_2(A^*_\xi X)L - \tau(X)V - \epsilon\psi(X)W = 0, \tag{4.34}$$

for any $X \in \Gamma(T\acute{N})$. Applying $\tilde{P}$ to (4.34) and from (2.1), (3.1) and (4.2), we derive

$$\begin{aligned} &- PA^*_\xi X - A^*_\xi X - \tau(X)V + (\theta_2(A^*_\xi X) - \epsilon\psi(X))W - \tau(X)\xi - \theta_2(A^*_\xi X)L \\ &- \epsilon\psi(X)L = 0, \end{aligned} \tag{4.35}$$

for any $X \in \Gamma(T\acute{N})$. Then subtracting (4.12) from (4.35) and taking tangential and normal part of the resulting equation, we get ii). Similarly, by using (2.1), (3.1), (4.2), (4.11) and (4.13), we have iii) and iv).

Suppose that all of V, U and W are parallel with respect to ∇ on $\acute{N}$. Then from iii) we have $A_N X = \theta_1(A_N X)U + \theta_2(A_N X)W$. From ii) and iii) we get $\theta_1(A_N X) = \tau(X) = 0$ and from iii) and iv) we obtain $\rho(X) = \theta_2(A_N X) = 0$. Thus, $A_N = 0$, that is, $S(T\acute{N})$ is totally geodesic in $\acute{N}$. □

From Theorem 4.10 i) we have Corollary 4.6.

Corollary 4.6. *Let* $\acute{N}$ *be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold* $(\tilde{N}, \tilde{g}, \tilde{P})$*. If* P *is parallel with respect to* ∇ *on* $\acute{N}$*, then* $\acute{N}$ *is irrotational.*

Theorem 4.11. *Let* $\acute{N}$ *be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold* $(\tilde{N}, \tilde{g}, \tilde{P})$*. If* P *and* V *are parallel with respect to* ∇ *on* $\acute{N}$*, then* $\acute{N}$ *is totally geodesic in* $\tilde{N}$ *and the* 1*-forms* ρ*,* ψ *and* τ *vanish.*

Proof. Suppose that P and V are parallel with respect to ∇ on $\acute{N}$. Then from Theorem 4.10 i) and ii) we have $\rho(X) = \psi(X) = \tau(X) = 0$ and $A^*_\xi X = -\epsilon\psi(X)W$. From this fact, we get $A^*_\xi = 0$, i.e., $D_1 = 0$.

For any $Y \in \Gamma(T\acute{N})$, we have (4.24). Using (4.32) with $Y = Y_l \in \Gamma(L)$, we find $D_2(X, Y_l) = -E(X, W)\theta_1(Y_l) = 0$. From this fact we get $D_2(X, Y_l) = 0$. From (4.15) and Theorem 4.10 i) we have $D_2(X, U) = D_2(X, W) = 0$. Using (4.24) we obtain $D_2 = 0$. Thus, we get $D = 0$, which completes the proof. □

Theorem 4.12. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. If P is parallel with respect to ∇ on $\acute{N}$, then L and $L^{\perp}$ are parallel and integrable distributions with respect to ∇ and $\acute{N}$ is locally a product manifold $\acute{N}_1 \times \acute{N}_2$, where $\acute{N}_1$ and $\acute{N}_2$ are leaves of L and $L^{\perp}$, respectively.*

Proof. Since $\tilde{\nabla}$ is a metric connection, from (2.2), (2.3), (2.7) and (2.18), we derive

$$\begin{aligned} &g(\nabla_X \xi, V) = D_1(X, V), \quad g(\nabla_X V, V) = 0, \quad g(\nabla_X Y, V) = D_1(X, \tilde{P}Y), \\ &g(\nabla_X \xi, W) = \epsilon D_2(X, V), \quad g(\nabla_X V, W) = \epsilon D_2(X, V) - \psi(X), \\ &g(\nabla_X Y, W) = \epsilon D_2(X, \tilde{P}Y), \end{aligned} \tag{4.36}$$

for any $X \in \Gamma(L)$ and $Y \in \Gamma(L_0)$.

Since $\tilde{\nabla}$ is a metric connection, using (2.2), (2.3), (2.7), (2.9) and (2.11), we derive

$$\begin{aligned} &g(\nabla_Z W, N) = -\epsilon D_2(Z, U), \quad g(\nabla_Z W, U) = -\epsilon D_2(Z, U) - \epsilon\rho(Z), \\ &g(\nabla_Z W, Y) = -\epsilon D_2(Z, \tilde{P}Y), \quad g(\nabla_Z U, N) = E(Z, U), \\ &g(\nabla_Z U, U) = 0, \quad g(\nabla_Z U, Y) = -E(Z, \tilde{P}Y), \end{aligned} \tag{4.37}$$

for any $Z \in \Gamma(L^{\perp})$ and $Y \in \Gamma(L_0)$.

From (4.15) we have $D_1(X, V) = 0$. Letting $Y = V$ in equation (4.32) we obtain $D_2(X, V) = 0$ for any $X \in \Gamma(T\acute{N})$. If we replace Y by $\tilde{P}Y \in \Gamma(L_0)$ in equation (4.31) and (4.32) then we derive $D_1(X, \tilde{P}Y) = D_2(X, \tilde{P}Y) = 0$. Also, from (4.15) and Theorem 4.10 i) we have $E(X, U) = \rho(X) = \psi(X) = D_2(X, U) = 0$ for any $X \in \Gamma(T\acute{N})$. Replacing X, Y, Z by $Z \in \Gamma(L^{\perp}), U, \tilde{P}Y \in \Gamma(L_0)$, respectively, in equation (4.30) and if $D_2(X, U) = 0$ is used in this equation, we get $E(Z, \tilde{P}Y) = 0$. Thus, we prove our theorem. □

Theorem 4.13. *Let $\acute{N}$ be a totally umbilical screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then L is a parallel and integrable distribution with respect to ∇ and $\acute{N}$ is locally a product manifold $R_u \times R_w \times \acute{N}_1$, where R_u and R_w are null and non-null curves tangent to $\tilde{P}(ltr(T\acute{N}))$ and $\tilde{P}(S(T\acute{N}^{\perp}))$, respectively, and $\acute{N}_1$ is a leaf of L.*

Proof. Suppose that $\acute{N}$ is totally umbilical, then $\acute{N}$ is totally geodesic and $D_1 = D_2 = \psi = 0$. All terms of (4.36) are zero. Hence, L is a parallel and integrable distribution with respect to ∇. Moreover, $\tilde{P}(ltr(T\acute{N}))$ and $\tilde{P}(S(T\acute{N}^{\perp}))$ are integrable distributions. Hence, the proof is completed. □

Theorem 4.14. *Let $\acute{N}$ be a half lightlike submanifold of a semi-Remannian manifold $(\tilde{N}, \tilde{g})$. Then the screen transversal distribution $S(T\acute{N}^{\perp})$ is parallel with respect to $\tilde{\nabla}$ if and only if $A_L = 0$ on $\Gamma(T\acute{N})$* [16].

Theorem 4.15. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a locally golden product space form $(\tilde{N} = \acute{N}_p(c_p) \times \acute{N}_q(c_q), \tilde{g}, \tilde{P})$ with a parallel screen transversal distribution. If $S(T\acute{N})$ is totally umbilicial, then $c_p = c_q = 0$.*

Proof. Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a locally golden product space form $(\tilde{N} = \acute{N}_p(c_p) \times \acute{N}_q(c_q), \tilde{g}, \tilde{P})$, $c_p, c_q \neq 0$, with a parallel screen transversal distribution. From (2.4) we derive

$$\begin{aligned}\tilde{g}(\tilde{R}(\xi, Y)QZ, N) = &\left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right)\{\tilde{g}(Y, QZ) - \theta_1(QZ)\theta_3(Y)\}\\ &+\left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right)\{\tilde{g}(\tilde{P}Y, QZ) - \theta_1(QZ)\eta(Y)\}.\end{aligned} \tag{4.38}$$

Since $S(T\acute{N})$ is totally umbilicial and screen transversal distribution is parallel, using Theorem 4.9 and Theorem 4.14 in (2.24) and (2.27) we obtain

$$\tilde{g}(\tilde{R}(X, Y)QZ, N) = 0. \tag{4.39}$$

If we put $Y = V$, $Z = U$ in (4.38), we obtain

$$\frac{(1-\phi)c_p + \phi c_q}{2\sqrt{5}} = 0. \tag{4.40}$$

Similarly, if we put $Y = U$, $Z = V$ in (4.38), we get

$$\left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right) + \left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right) = 0. \tag{4.41}$$

From (4.40) and (4.41), we obtain $c_p = c_q = 0$, which proves the assertion. □

The induced Ricci type tensor $R^{(0,2)}$ of $\acute{N}$ is defined by

$$R^{(0,2)} = trace\{Z \to R(Z, X)Y\},$$

for any $X, Y, Z \in \Gamma(T\acute{N})$, where

$$R^{(0,2)}(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(E_i, X)Y, E_i) + \bar{g}(R(\xi, X)Y, N),$$

for the quasi-orthonormal frame $\{E_1, \ldots, E_n, \xi\}$ of $T_p\acute{N}$ and where $\epsilon_i = g(E_i, E_i)$ is the sign of E_i. Generally, the induced Ricci type tensor R is not symmetric [5–7]. A tensor field $R^{(0,2)}$ of lightlike submanifold M is called its induced Ricci tensor if it is symmetric. A symmetric $R^{(0,2)}$ tensor will be indicated by Ric.

If $\tilde{N} = \acute{N}_p(c_p) \times \acute{N}_q(c_q)$ is a locally golden product space form, then we have

$$R^{(0,2)}(X, Y) = \left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right)\{(n-1)\tilde{g}(X, Y) - \theta_1(Y)\theta_3(X)$$

$$
\begin{aligned}
&+ ((tr\tilde{P}) - 1 + \theta_3(\xi))\tilde{g}(\tilde{P}X, Y)\} \\
&+ \left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right)\{(n-1)\tilde{g}(\tilde{P}X, Y) \\
&+ ((tr\tilde{P}) + \theta_3(\xi))\tilde{g}(X, Y) - \theta_1(Y)\eta(X)\} \\
&+ D_1(X, Y)trA_N + D_2(X, Y)trA_L - g(A_N X, A^*_\xi Y) \\
&- \epsilon g(A_L X, A_L Y) + \rho(X)\psi(Y).
\end{aligned} \tag{4.42}
$$

From (4.42), we have

$$
\begin{aligned}
R^{(0,2)}(X, Y) - R^{(0,2)}(Y, X) = &\left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right)(\theta_1(X)\theta_3(Y) - \theta_1(Y)\theta_3(X)) \\
&+ \left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right)(\theta_1(X)\eta(Y) - \theta_1(Y)\eta(X)) \\
&+ g(A^*_\xi X, A_N Y) - g(A^*_\xi Y, A_N X) \\
&+ \rho(X)\psi(Y) - \rho(Y)\psi(X).
\end{aligned} \tag{4.43}
$$

From (2.4) and (2.25) we get

$$
\begin{aligned}
2d\tau(X, Y) = &- \frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}(\theta_1(X)\theta_3(Y) - \theta_1(Y)\theta_3(X)) \\
&+ \left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right)(\theta_1(X)\eta(Y) - \theta_1(Y)\eta(X)) \\
&+ g(A^*_\xi X, A_N Y) - g(A^*_\xi Y, A_N X) + \rho(X)\psi(Y) - \rho(Y)\psi(X).
\end{aligned} \tag{4.44}
$$

Thus, from (4.43) and (4.44) we obtain

$$
R^{(0,2)}(X, Y) - R^{(0,2)}(Y, X) = 2d\tau(X, Y). \tag{4.45}
$$

From (4.45), we obtain Theorem 4.16.

Theorem 4.16. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a locally golden product space form $(\tilde{N} = \acute{N}_p(c_p) \times \acute{N}_q(c_q), \tilde{g}, \tilde{P})$. Then, $R^{(0,2)}$ is a symmetric if and only if τ is closed.*

5. Screen Conformal Screen Semi-invariant Half Lightlike Submanifolds Of A Golden Semi-Riemannian Manifold

A half lightlike submanifold $(\acute{N}, g, S(T\acute{N}))$ of a semi-Riemannian manifold $(\tilde{N}, \tilde{g})$ is screen conformal if the shape operators A_N and A^*_ξ of $\acute{N}$ and $S(T\acute{N})$, respectively are related by $A_N = \varphi A^*_\xi$, or equivalently

$$
E(X, QY) = \varphi D_1(X, Y), \tag{5.1}
$$

for all $X, Y \in \Gamma(T\acute{N})$, where φ is a non-vanishing smooth function on a neighborhood U in $\acute{N}$. In particular, if φ is a non-zero constant, then $\acute{N}$ is called screen homothetic [8].

Remark 5.1. If $\acute{N}$ is a screen conformal half lightlike submanifold, then E is symmetric on $\Gamma(S(T\acute{N}))$. Thus, $S(T\acute{N})$ is integrable distribution and $\acute{N}$ is locally a product manifold $R_\xi \times \acute{N}^*$ where R_ξ is a null curve tangent to $Rad(T\acute{N})$ and $\acute{N}^*$ is a leaf of $S(T\acute{N})$ [5].

Theorem 5.1. *Let $\acute{N}$ be a screen conformal totally umbilical screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then, $\acute{N}$ and $S(T\acute{N})$ are totally geodesic.*

Proof. Let $\acute{N}$ be a screen conformal totally umbilical screen semi-invariant half lightlike submanifold of $\tilde{N}$. Then from Theorem 4.7 we have $D_1 = D_2 = 0$. Since $\acute{N}$ is screen conformal, $E(X, QY) = \varphi D_1(X, Y) = 0$, which proves the assertion. □

Theorem 5.2. *Let $\acute{N}$ be a screen conformal screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. If P is parallel with respect to ∇ on $\acute{N}$, then $\acute{N}$ and $S(T\acute{N})$ are totally geodesic in $\tilde{N}$ and $\rho = \psi = 0$.*

Proof. Suppose that P is parallel with respect to ∇ on $\acute{N}$. For any $Y \in \Gamma(T\acute{N})$, we have (4.23) and (4.24). Replacing Y by Y_l in (4.31) and (4.32), we find $D_1(X, Y_l) = D_2(X, Y_l) = 0$. From (4.15) and (5.1) we have $E(X, U) = \varphi D_1(X, U) = 0$. Taking $Y = V$ in (4.32) we get $D_2(X, V) = 0$ and from (4.14)-2 we obtain $D_1(X, W) = -\epsilon D_2(X, V) = 0$. From (4.15) and Theorem 4.10 i) we have $D_2(X, U) = D_2(X, W) = 0$. Considering (4.23) and (4.24) we obtain $D_1 = D_2 = 0$. Since $\acute{N}$ is conformal, $E = 0$. Also, from Theorem 4.10 i) $\rho(X) = \psi(X) = 0$, which proves the assertion. □

Theorem 5.3. *Let $\acute{N}$ be a screen conformal totally umbilical screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. If U or W is parallel with respect to ∇ on $\acute{N}$, then L and $L^\perp$ are parallel and integrable distribution with respect to ∇ and $\acute{N}$ is locally a product manifold $\acute{N}_1 \times \acute{N}_2$, where $\acute{N}_1$ is a leaf of L and $\acute{N}_2$ is a leaf of $L^\perp$.*

Proof. Let $\acute{N}$ be totally umbilical. From Theorem 5.2 $\acute{N}$ and $S(T\acute{N})$ are totally geodesic and all terms of (4.36) and (4.37) are zero except $\rho(Z)$. Since $S(T\acute{N})$ is totally geodesic and U is parallel, using Theorem 4.10 iii) we obtain $\theta_2(A_N X) = \epsilon E(X, W) = \rho(X) = 0$.

If W is parallel, from Theorem 4.10 (iv), we have $\rho(X) = 0$. Hence L and $L^\perp$ are parallel and integrable distributions on $\acute{N}$. This completes the proof. □

From (2.19), (4.14), (4.15) and (5.1), we have

$$\begin{aligned} h(X, U) =& D_1(X, U)N + D_2(X, U)L = -E(X, V)N - \epsilon E(X, W)L \\ =& -\varphi D_1(X, V)N - \epsilon\varphi D_1(X, W)L \\ =& \varphi D_1(X, V)N + \varphi D_2(X, V)L = \varphi h(X, V), \end{aligned}$$

for any $X \in \Gamma(T\acute{N})$. Thus, we have

$$h(X, U - \varphi V) = 0. \tag{5.2}$$

Since $\{U, V\}$ is a basis for $\Gamma(\tilde{P}(Rad(T\acute{N})) \oplus \tilde{P}(ltr(T\acute{N})))$, $\{\omega_1, \omega_2\}$ is an orthogonal basis of $\Gamma(\tilde{P}(Rad(T\acute{N})) \oplus \tilde{P}(ltr(T\acute{N})))$, where

$$\omega_1 = U - \varphi V, \quad \omega_2 = U + \varphi V.$$

Let $R(\omega_1) = \text{Span}\{\omega_1\}$. Then $S(\omega_1) = L_0 \perp \text{Span}\{\omega_2, W\}$ is a complementary vector subbundle to $R(\omega_1)$ in $S(T\acute{N})$. Thus, we have

$$S(T\acute{N}) = R(\omega_1) \perp S(\omega_1). \tag{5.3}$$

Theorem 5.4. *Let $\acute{N}$ be a screen conformal totally umbilical screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$ such that $S(T\acute{N}^{\perp})$ is parallel distribution with respect to $\tilde{\nabla}$. Then the non-null vector field ω_1 is parallel with respect to ∇ if and only if the 1-forms ρ, τ and ψ vanish and $\acute{N}$ is screen homothetic.*

Proof. Since P is linear and using (4.11), (4.12), $A_N = \varphi A^*_\xi$, we obtain

$$\nabla_X \omega_1 = \tau(X)U + (\varphi\tau(X) - X[\varphi])V + (\rho(X) + \epsilon\varphi\psi(X))W,$$

for any $X \in \Gamma(T\acute{N})$. Thus, we say that ω_1 is parallel if and only if

$$\tau(X)U + (\varphi\tau(X) - X[\varphi])V + (\rho(X) + \epsilon\varphi\psi(X))W = 0.$$

If we take the scalar product with U, V and W, respectively, we obtain $\tau(X) = \varphi\tau(X) - X[\varphi] = \rho(X) + \epsilon\varphi\psi(X) = 0$. Since $\tau(X) = 0$, then $X[\varphi] = 0$, i.e., $\acute{N}$ is screen homothetic. If $S(T\acute{N}^{\perp})$ is parallel, then we obtain $\rho(X) = 0$. Thus, $\psi(X) = 0$. □

Theorem 5.5. *Let $\acute{N}$ be a screen conformal screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$ such that $S(T\acute{N}^{\perp})$ is parallel distribution with respect to $\tilde{\nabla}$. If ω_1 is parallel with respect ∇, then $\acute{N}$ is locally a product manifold $R_\xi \times R_{\omega_1} \times \acute{N}_1$, where R_ξ is null curve tangent to $T\acute{N}^{\perp}$, R_{ω_1} is non-null geodesic tangent to $R(\omega_1)$ and $\acute{N}_1$ is a leaf of $S(\omega_1)$. Also, $\acute{N}$ is screen homothetic.*

Proof. For any $X \in \Gamma(S(\omega_1))$ and $Y \in \Gamma(L_0)$, we have

$$\begin{aligned} g(\nabla_X Y, \omega_1) &= g(\tilde{\nabla}_X Y, \omega_1) = -g(Y, \tilde{\nabla}_X \omega_1) = -g(Y, \nabla_X \omega_1) = 0, \\ g(\nabla_X \omega_2, \omega_1) &= g(\tilde{\nabla}_X \omega_2, \omega_1) = -g(\omega_2, \nabla_X \omega_1) = X[\varphi] - 2\varphi\tau(X), \\ g(\nabla_X W, \omega_1) &= g(\tilde{\nabla}_X W, \omega_1) = -g(W, \nabla_X \omega_1) = -\rho(X) - \epsilon\varphi\psi(X). \end{aligned} \tag{5.4}$$

From Theorem 5.4, the 1-forms ρ, τ and ψ vanish and $\acute{N}$ is screen homothetic. Then all equations in (5.4) are zero. Thus, the distribution $S(\omega_1)$ is a parallel and integrable distribution. Using this fact and Remark 5.1, we derive our assertion. □

Theorem 5.6. *Let $\acute{N}$ be a screen conformal screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$ such that $S(T\acute{N}^{\perp})$ is parallel distribution with respect to $\tilde{\nabla}$. If ω_1 is parallel with respect to ∇ on $\acute{N}$, then $\acute{N}$ is locally a product manifold $R_{\omega_1} \times \acute{N}_1$, where R_{ω_1} is non-null geodesic tangent to $R(\omega_1)$ and $\acute{N}_1$ is a leaf of $G(\omega_1) = L_0 \perp \mathrm{Span}\{\xi, \omega_2, W\}$ respectively. Furthermore, $\acute{N}$ is screen homothetic.*

Proof. From (2.5) and (5.3), we derive $T\acute{N} = R(\omega_1) \oplus_{orth} G(\omega_1)$. For any $X \in \Gamma(G(\omega_1))$ and $Y \in \Gamma(L_0)$, we derive

$$\begin{aligned} g(\nabla_X Y, \omega_1) &= g(\tilde{\nabla}_X Y, \omega_1) = -g(Y, \tilde{\nabla}_X \omega_1) = -g(Y, \nabla_X \omega_1) = 0, \\ g(\nabla_X \xi, \omega_1) &= g(\tilde{\nabla}_X \xi, \omega_1) = -g(\xi, \tilde{\nabla}_X \omega_1) = -D_1(X, \omega_1) = 0, \\ g(\nabla_X \omega_2, \omega_1) &= g(\tilde{\nabla}_X \omega_2, \omega_1) = -g(\omega_2, \nabla_X \omega_1) = X[\varphi] - 2\varphi\tau(X), \\ g(\nabla_X W, \omega_1) &= g(\tilde{\nabla}_X W, \omega_1) = -g(W, \nabla_X \omega_1) = -\rho(X) - \epsilon\varphi\psi(X). \end{aligned} \tag{5.5}$$

From Theorem 5.4, all equations in (5.5) is zero. Thus, distribution $G(\omega_1)$ is a parallel and integrable. Thus, we derive our assertion. □

Theorem 5.7. *Let $\acute{N}$ be a screen conformal screen semi-invariant half lightlike submanifold of a locally golden product space form $(\tilde{N} = \acute{N}_p(c_p) \times \acute{N}_q(c_q), \tilde{g}, \tilde{P})$. Then, we have $c_p = (\phi + 1)c_q$.*

Proof. From (2.4) and (2.23), we derive

$$\begin{aligned} &(\nabla_X D_1)(Y, Z) - (\nabla_Y D_1)(X, Z) + \tau(X) D_1(Y, Z) - \tau(Y) D_1(X, Z) \\ &+ \psi(X) D_2(Y, Z) - \psi(Y) D_2(X, Z) \\ &= \left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right)\{\tilde{g}(\tilde{P}Y, Z)\theta_1(X) - \tilde{g}(\tilde{P}X, Z)\theta_1(Y)\} \\ &+ \left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right)\{\tilde{g}(Y, Z)\theta_1(X) - \tilde{g}(X, Z)\theta_1(Y)\}, \end{aligned} \tag{5.6}$$

for any $X, Y, Z \in \Gamma(T\acute{N})$. Using (2.4), (2.24), (2.27) and (5.1) we get

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)QZ, N) = &\left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right)\{\tilde{g}(Y, QZ)\eta(X) - \tilde{g}(X, QZ)\eta(Y) \\ &+ \tilde{g}(\tilde{P}Y, QZ)\theta_3(X) - \tilde{g}(\tilde{P}X, QZ)\theta_3(Y)\} \\ &+ \left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right)\{\tilde{g}(\tilde{P}Y, QZ)\eta(X) \\ &- \tilde{g}(\tilde{P}X, QZ)\eta(Y) + \tilde{g}(Y, QZ)\theta_3(X) - \tilde{g}(X, QZ)\theta_3(Y)\} \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)QZ, N) = &\varphi((\nabla_X D_1)(Y, Z) - (\nabla_Y D_1)(X, Z)) + \varphi\tau(Y) D_1(X, QZ) \\ &- \varphi\tau(X) D_1(Y, QZ) + X[\varphi] D_1(Y, QZ) - Y[\varphi] D_1(X, QZ) \end{aligned}$$

$$+\epsilon\left\{\rho(Y)D_2(X,QZ)-\rho(X)D_2(Y,QZ)\right\}. \tag{5.8}$$

Thus, from (5.6), (5.7) and (5.8), we derive

$$\begin{aligned}
&\left(-\frac{(1-\phi)c_p-\phi c_q}{2\sqrt{5}}\right)\{\varphi\tilde{g}(\tilde{P}Y,QZ)\theta_1(X)-\varphi\tilde{g}(\tilde{P}X,QZ)\theta_1(Y)\\
&\quad-\tilde{g}(Y,QZ)\eta(X)+\tilde{g}(X,QZ)\eta(Y)-\tilde{g}(\tilde{P}Y,QZ)\theta_3(X)+\tilde{g}(\tilde{P}X,QZ)\theta_3(Y)\}\\
&\quad+\left(-\frac{(1-\phi)c_p+\phi c_q}{4}\right)\{\varphi\tilde{g}(Y,QZ)\theta_1(X)-\varphi\tilde{g}(X,QZ)\theta_1(Y)\\
&\quad-\tilde{g}(\tilde{P}Y,QZ)\eta(X)+\tilde{g}(\tilde{P}X,QZ)\eta(Y)-\tilde{g}(Y,QZ)\theta_3(X)+\tilde{g}(X,QZ)\theta_3(Y)\}\\
&=\left[-X\left[\varphi\right]+2\varphi\tau(X)\right]D_1(Y,QZ)+\left[Y\left[\varphi\right]-2\varphi\tau(Y)\right]D_1(X,QZ)\\
&\quad-\left[\varphi\psi(Y)+\epsilon\rho(Y)\right]D_2(X,QZ)+\left[\varphi\psi(X)+\epsilon\rho(X)\right]D_2(Y,QZ).
\end{aligned}\tag{5.9}$$

Replacing QZ by ω_1 in (5.9) and using (5.2), we obtain

$$\left(-\frac{(1-\phi)c_p+\phi c_q}{4}\right)\{-\varphi\theta_1(X)\eta(Y)+\varphi\theta_1(Y)\eta(X)+\theta_3(X)\eta(Y)-\theta_3(Y)\eta(X)\}=0.$$

Letting $X=V$, $Y=\xi$ in last equation we get

$$\left(-\frac{(1-\phi)c_p+\phi c_q}{4}\right)=0.$$

From this, we see that $c_p=(\phi+1)c_q$, which completes the proof. □

Corollary 5.1. *There is no screen conformal screen semi-invariant half lightlike submanifold of a locally golden product space form* $(\tilde{N}=\acute{N}_p(c_p)\times\acute{N}_q(c_q),\tilde{g},\tilde{P})$ *with* $c_p\neq(\phi+1)c_q$.

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