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SOME PROPERTIES OF RANGE OPERATORS ON LCA GROUPS

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Abstract. In this paper, we study the structure of shift preserving operators acting on shift-invariant spaces in $L^2(G)$, where G is a locally compact Abelian group. We generalize some results related to shift-preserving operator and its associated range operator from $L^2(\mathbb{R}^d)$ to $L^2(G)$. We investigate the matrix structure of range operator $R(\xi)$ on range function J associated to shift-invariant space V, in the case of a locally compact Abelian group G. We also focus on some properties like as normal and unitary operator for range operator on $L^2(G)$. We show that shift preserving operator U is invertible if and only if fiber of corresponding range operator R is invertible and investigate the measurability of inverse $R^{-1}(\xi)$ of range operator on $L^2(G)$.

1. Introduction

Many authors such as Aldroubi, Benedetto, Bownik, de Boor, De Vore, Li, Ron, Rzeszotnik, Shen, Weiss and Wilson have studied shift-invariant subspaces of $L^2(\mathbb{R}^n)$ cf. [2, 3, 5, 8–10, 24–27]. The theory of shift-invariant spaces plays an important role in many areas such as theory of wavelets, Gabor systems, multi-resolution analysis, frames, approximation theory etc. Shift-invariant spaces of $L^2(\mathbb{R}^n)$ are the spaces which are invariant under integer translations. After that, the structural properties of shift-invariant spaces are studied by R. A. Kamyabi Gol and R. Raisi Tousi [18–22], by C. Cabrelli and V. Paternostro [12], and by M. Bownik and K. A. Ross [4], in locally compact Abelian groups. The locally compact Abelian group framework has several advantages because it has a valid theory for the classical groups such as $\mathbb{Z}^d$, $\mathbb{T}^d$ and $\mathbb{Z}_n$ (see [13–17]).

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Let G be locally compact Abelian (which will be abbreviates as ”LCA“) group with a Haar measure m_G. We shall use the constructions and notations from [18–21], associated to LCA groups. The dual group of G is denoted by $\widehat{G}$. Let L be a uniform lattice in G. A subspace V of $L^2(G)$ is shift-invariant if it is invariant under translation operations, that is $T_kV \subseteq V$, where $T_kf(x) = f(k^{-1}x)$ for all $x \in G$, $f \in V$ and $k \in L$. For any function $f \in L^1(G)$, its Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(\xi) = \int_G f(x)\bar{\xi}(x)dm_G(x),$$

where $\xi \in \widehat{G}$ is a character on G. If L is a uniform lattice L in G, then a fundamental domain is defined by a measurable set S_L in G such that every element $x \in G$ can be uniquely represented as $x = kz$, where $k \in L$ and $z \in S_L$. There always exists a fundamental domain for a uniform lattice in a LCA group, see [23]. Let $\Phi \subseteq L^2(G)$ be a countable set of functions, then

$$S(\Phi) = \overline{\text{Span}}\{T_k\phi : \phi \in \Phi, k \in L\}$$

is a shift-invariant space generated by Φ. If the set of generators Φ is finite, then the space $S(\Phi)$ is called finitely generated shift-invariant space. A range function J [3,8,9,19] is associated to each shift-invariant space V, which represents that the space V as a measurable field of closed subspaces of $\ell^2(L^\perp)$, where $L^\perp$ is annihilator of L. These subspaces are called the fiber spaces. There is an isometric isomorphism T (see [19]) between shift-invariant space V and its associated range function J. A bounded linear operator $U : L^2(G) \to L^2(G)$ is called a shift-preserving operator with respect to uniform lattice L, if $UT_k = T_kU$ for all $k \in L$. Every shift-preserving operator U has a corresponding range operator $R(\xi)$.

Our paper is organised as follows. Section 2 includes some background results on LCA groups. In Section 3, we prove our main results. The eigenvalues of a shift-preserving operator are named as s-eigenvalues and eigenspaces as s-eigenspaces. We show that when $\dim J(\xi) < +\infty$, then, the operator $R(\xi)$ can be represented by a matrix with measurable entries for a.e. (almost every) $\xi \in S_{L^\perp}$. We see that the invertibility of shift preserving operator U can be deduced from invertibility of its fibers $R(\xi)$ and $\xi \mapsto R^{-1}(\xi)$ is measurable for a.e. $\xi \in S_{L^\perp}$, where $R^{-1}(\xi)$ is the inverse of range operator associated to shift-preserving operator U. We have taken the ideas of our main results from the paper [1]. We proved similar results for locally compact abelian groups.

2. Background on LCA Groups

First we provide the background and notations on the LCA group, which we will use later in our main results.

Let G be an LCA group and $\widehat{G}$ is dual of G, elements of $\widehat{G}$ are usually denoted by ξ (called characters on G). Throughout this paper we assume that G is a second countable LCA group. A subgroup L of LCA group G is called uniform lattice if it is discrete and co-compact (i.e., G/L is compact). Let $L^{\perp}$ is annihilator of L in $\widehat{G}$, where

$$L^{\perp} = \{\xi \in \widehat{G} : \xi(L) = \{1\}\}.$$

Then, "the identities $L^{\perp} = \widehat{G/L}$ and $\widehat{G}/L^{\perp}$, together with the fact that a locally compact Abelian group is compact if and only if its dual group is discrete, imply that the subgroup $L^{\perp}$ is a uniform lattice in $\widehat{G}$", see [18].

Definition 2.1 ([18])**.** Let G be a LCA group and L be a uniform lattice in G. A closed subspace $V \subseteq L^2(G)$ is called a shift-invariant space if $f \in V$ implies that $T_k f \in V$ for any $k \in L$, where T_k is translation operator defined by $T_k f(x) = f(k^{-1}x)$ for all $x \in G$.

Let $\phi \in L^2(G)$, then $V_\phi = \overline{\text{Span}}\{T_k\phi : k \in L\}$ is called the principal shift-invariant space generated by ϕ.

The following proposition (cf. [18, Proposition 2.2]) characterizes the elements in a principal shift-invariant subspace V_ϕ of $L^2(G)$ in terms of their Fourier transforms.

Proposition 2.1 ([18])**.** *Let $\phi \in L^2(G)$, then $f \in V_\phi$ if and only if $\widehat{f}(\xi) = r(\xi)\widehat{\phi}(\xi)$ for some $r \in L^2\left(\widehat{L}, \omega_\phi\right)$, which is given by $r(\xi) = \sum_{i=1}^{n} a_i \overline{\xi}(k_i)$, $a_i \in \mathbb{C}$, where $L^2\left(\widehat{L}, \omega_\phi\right)$ is the space of all functions $r : \widehat{L} \to \mathbb{C}$ satisfying*

$$\int\limits_{\widehat{L}} |r(\xi)|^2 \omega_\phi(\xi) d\xi < +\infty$$

and $\omega_\phi(\xi) = \sum_{\eta \in L^{\perp}} \left|\hat{\phi}(\xi\eta)\right|^2$.

The following proposition gives necessary and sufficient condition for the shifts of function ϕ to be an orthonormal system in space $L^2(G)$.

Proposition 2.2 ([18])**.** *Suppose $\phi \in L^2(G)$. Then $\{T_k\phi : k \in L\}$ is an orthonormal system in $L^2(G)$ if and only if $\omega_\phi = 1$ a.e. on $\widehat{G}$.*

Remark 2.1 ([18]). If V_ϕ is a principal shift-invariant space, and ω_ϕ is as in Proposition 2.1, then the spectrum of $V_\phi = \operatorname{supp}(\omega_\phi)$, where $\operatorname{supp}(\omega_\phi)$ denote the support of ω_ϕ. That is, $\Omega_\phi = \{\xi \in \widehat{G} : \omega_\phi(\xi) \neq 0\}$. Also, when the set $\{T_k\phi : k \in L\}$ is an orthonormal system for $L^2(G)$, then $\Omega_\phi = \widehat{G}$.

Definition 2.2 ([18])**.** A subset X of a Hilbert space H is called a frame for H if there exist two numbers $0 < A \leq B < +\infty$ which satisfy the following inequality

$$A\|h\|^2 \leq \sum_{\eta_0 \in X} |\langle h, \eta_0 \rangle|^2 \leq B\|h\|^2, \quad h \in H.$$

If $A = B = 1$, then X is called Parseval frame.

The next theorem (cf. [18, Theorem 3.6]) shows that for every principal shift-invariant space V_ϕ, shifts of its generator ϕ form a Parseval frame.

Theorem 2.1 ([18]). *Let $\phi \in L^2(G)$. Then the set $\{T_k\phi : k \in L\}$ forms a Parseval frame for space V_ϕ if and only if*

$$\omega_\phi = \chi_{\Omega_\phi} \quad a.e. \text{ on } \widehat{G},$$

and in this case ϕ is said to be a Parseval frame generator for the space V_ϕ.

The following proposition shows that the spaces $L^2(G)$ and $L^2\left(S_{L^\perp}, \ell^2(L^\perp)\right)$ are isometrically isomorphic to each other.

Proposition 2.3 ([19]). *The mapping $T : L^2(G) \to L^2\left(S_{L^\perp}, \ell^2(L^\perp)\right)$ defined by $Tf(\xi) = \left(\widehat{f}(\xi\eta)\right)_{\eta\in L^\perp}$ is an isometric isomorphism between $L^2(G)$ and $L^2\left(S_{L^\perp}, \ell^2(L^\perp)\right)$, where $L^2\left(S_{L^\perp}, \ell^2(L^\perp)\right)$ is the space of square integrable functions $f : S_{L^\perp} \to \ell^2(L^\perp)$ with inner product defined by*

$$\langle f, g\rangle = \int\limits_{S_{L^\perp}} \langle f(\xi), g(\xi)\rangle_{\ell^2\in L^\perp} d(\xi),$$

and $S_{L^\perp}$ is fundamental domain of $L^\perp$ in space $\widehat{G}$ with measure $d\xi$ on it.

Definition 2.3 ([19]). Let G be a LCA group and L be an uniform lattice in G. A range function associated to a shift-invariant space V is a mapping

$$J : S_{L^\perp} \to \{\text{closed subspaces of } \ell^2(L^\perp)\}.$$

"The range function J is called measurable if associated orthogonal projections $P(\xi) : \ell^2(L^\perp) \to J(\xi)$ are measurable that is $\xi \mapsto \langle P(\xi)a, b\rangle$ is measurable for each $a, b \in \ell^2(L^\perp)$" [19].

The shift-invariant space V can be defined in terms of a measurable range function J as follows

$$V = \{f \in L^2(G) : Tf(\xi) \in J(\xi) \text{ for a.e. } \xi \in S_{L^\perp}\}.$$

There is a one to one correspondence between V and J under the convention that the range functions are identified if they are equal a.e.

The following theorem (cf. [19, Theorem 4.1]) allows to reduce the problem of checking whether the shifts of a generator ϕ form a frame in a subspace of the space $L^2(G)$, to analyzing the elements in the subspaces of $\ell^2(L^\perp)$, which are parameterized by the base space $S_{L^\perp}$.

Theorem 2.2 ([19]). *Let G be a second countable LCA group, L be a uniform lattice in group G, $S_{L^\perp}$ be a fundamental domain for annihilator $L^\perp$ in the dual group $\widehat{G}$ of G, $\Phi \subseteq L^2(G)$ is a countable set and T is the mapping defined in Proposition* 2.3*. Then the set $\{T_k\phi : \phi \in \Phi, k \in L\}$ forms a frame for $S(\phi)$ with frame bounds A and B if and only if the set $\{T\phi(\xi) \; : \phi \in \Phi\}$ forms a frame for $J(\xi)$ with same frame bounds A and B for a.e. $\xi \in S_{L^\perp}$ (same result holds for fundamental frame and Riesz family).*

Definition 2.4 ([19])**.** A bounded linear operator $U : L^2(G) \to L^2(G)$ is called shift preserving if $UT_k = T_kU$ for all $k \in L$, where $T_kf(x) = f(k^{-1}x)$ for all $x \in G$.

Definition 2.5 ([21])**.** A range operator R on a range functon J is a mapping defined as

$$R : S_{L^\perp} \to \{\text{bounded linear operator on closed subspaces of } \ell^2(L^\perp)\},$$

that is, the domain of $R(\xi)$ is $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$. Range operator R is said to be measurable if the mapping $\xi \mapsto \langle R(\xi)P(\xi)a, b\rangle$ is measurable for all $a, b \in l^2(L^\perp)$.

The following theorem (cf. [21, Theorem 6.1]) gives a characterization of shift preserving operators in terms of its range operators.

Theorem 2.3 ([21])**.** *Suppose V is a shift-invariant space in $L^2(G)$ and J is its associated range function. Then for every shift preserving operator $U : V \to L^2(G)$, there exists a measurable range operator R on range function J such that*

$$(T \circ U) f(\xi) = R(\xi) (Tf(\xi)), \tag{2.1}$$

for a.e. $\xi \in S_{L^\perp}$ and for all $f \in V$, where T is an isometric isomorphism between the spaces $L^2(G)$ and $L^2\left(S_{L^\perp}, \ell^2(L^\perp)\right)$. Conversely, for a measurable range operator R on J satisfying the condition $\operatorname{ess\,sup}_{\xi \in S_{L^\perp}} \|R(\xi)\| < +\infty$, there exists a bounded shift preserving operator $U : V \to L^2(G)$, such that equation (2.1) holds. There is a one-to-one correspondence between U and R, under the convention that the range operators are identified if they are equal a.e.

The following proposition (cf. [20, Proposition 2.2]) characterized all Parseval frame generators of shift-invariant space $S(\phi)$ as follows.

Proposition 2.4 ([20])**.** *Let $\phi \in L^2(G)$. Then ϕ is a Parseval frame generator of the space $S(\phi)$ if and only if $\|T\phi(\xi)\|^2_{l^2(L^\perp)} = \sum_{\eta \in L^\perp} |\widehat{\phi}(\xi\eta)|^2 = \chi_{\Omega_\phi}(\xi)$ for a.e. $\xi \in S_{L^\perp}$, where $\Omega_\phi = \{\xi \in S_{L^\perp} : T\phi(\xi) \neq 0\}$.*

Remark 2.2 ([20])*.* $TT_k\phi(\xi) = \widehat{T_k\phi}(\xi\eta) = \overline{\xi}(k)\widehat{\phi}(\xi\eta) = M_k(\xi)T\phi(\xi)$, where $\eta \in L^\perp$ and $M_k \in L^2(S_{L^\perp})$ is defined by $M_k(\xi) = \overline{\xi}(k)$, $\xi \in S_{L^\perp}$.

The next theorem (cf. [12, Theorem 4.11]) shows that a shift-invariant space V of $L^2(G)$ can be orthogonally decomposed in the sum of principal shift-invariant spaces having some additional properties.

Theorem 2.4 ([12])**.** *Let V be a shift-invariant space in space $L^2(G)$. Then V can be decomposed as an orthonormal sum*

$$V = \bigoplus_{i \in \mathbb{N}} S(\phi_i),$$

where each function ϕ_i is a Parseval frame generator for space $S(\phi_i)$ and $\sigma(S(\phi_{i+1})) \subseteq \sigma(S(\phi_i))$ for all $i \in \mathbb{N}$. Moreover, $\dim J_{S(\phi_i)}(\xi) = \|T\phi_i(\xi)\|$ for all $i \in \mathbb{N}$ and $\sum_{i \in \mathbb{N}} \|T\phi_i(\xi)\| = \dim J_V(\xi)$ for a.e. $\xi \in S_{L^\perp}$.

The following proposition (cf. [20, Proposition 2.4]) shows that the orthogonality of $S(\phi_1)$ and $S(\phi_2)$ depends upon the relation of their generators ϕ_1 and ϕ_2 in $L^2(G)$.

Proposition 2.5 ([20])**.** *The shift-invariant spaces $S(\phi_1)$ and $S(\phi_2)$ are orthogonal if and only if the following condition holds*

$$\sum_{\eta\in L^{\perp}} \widehat{\phi}_1(\xi\eta)\overline{\widehat{\phi}_2}(\xi\eta) = 0 \quad a.e.\ \xi \in \widehat{G}.$$

3. Main Results

Some of the properties of a shift preserving operator U which are related to the properties of its fibers are already proved in [20] like the following theorem.

Theorem 3.1 ([20])**.** *Suppose V is a shift-invariant space in $L^2(G)$. Let J be range function associated to V and $U : V \to V$ be a shift preserving operator with its corresponding range operator R. Then the following statements hold.*

(a) *If U is compact, then $R(\xi)$ is compact for a.e. $\xi \in S_{L^\perp}$.*
(b) *The operator U is an isometry if and only if the operator $R(\xi)$ is an isometry for a.e. $\xi \in S_{L^\perp}$.*
(c) *The adjoint operator $U^* : V \to V$ of U is also a shift preserving operator and its corresponding range operator R^* is given by $R^*(\xi) = R(\xi)^*$ for a.e. $\xi \in S_{L^\perp}$.*
(d) *The operator U is self adjoint if and only if the operator $R(\xi)$ is self adjoint.*

In this section, we prove the similar results for normal and unitary operators. We begin with the following.

3.1. **s-eigenvalue and s-eigenspace** ([1])**.** Let $a = \{a_k\}_{k\in L} \in l^2(L)$, define $\Lambda_a : L^2(G) \to L^2(G)$ as

$$\Lambda_a = \sum_{k\in L} a_k T_k.$$

Then operator Λ_a is well defined and bounded if and only if the spectrum of the sequence a is bounded.

Remark 3.1. If $a \in l^2(L)$ is of bounded spectrum, that is $\widehat{a} \in L^\infty(S_{L^\perp})$, then

$$T(\Lambda_a f)(\xi) = T\left(\sum_{k\in L} a_k T_k f(\xi)\right) = \sum_{k\in L} a_k T T_k f(\xi) = \sum_{k\in L} a_k M_k(\xi) T f(\xi),$$

that is

$$T(\Lambda_a f)(\xi) = \sum_{k\in L} a_k \overline{\xi}(k) T f(\xi).$$

Thus, $T(\Lambda_a f)(\xi) = \widehat{a}(\xi) T f(\xi)$ for each $f \in L^2(G)$ and for a.e. $\xi \in S_{L^\perp}$.

We see that the operator $\Lambda_a : V \to V$ is a shift preserving operator if V is shift-invariant space, and its corresponding range operator is given by $R_a(\xi) = \widehat{a}(\xi) I$ for a.e. $\xi \in S_{L^\perp}$, where I is the identity operator on space $J(\xi)$.

If the set $E(\Phi) = \{T_k\phi : \phi \in \Phi, k \in L\}$ forms a frame for space V, where $\Phi \subseteq L^2(G)$ is a countable set, then every function $f \in V$ can be expressed as

$$f = \sum_{\alpha\in I}\sum_{k\in L} b_\alpha(k) T_k \phi_\alpha, \quad b_\alpha \in l^2(L^\perp),$$

where I is index set and $\Phi = (\phi_\alpha)_{\alpha\in I} \subseteq L^2(G)$. Then

$$\widehat{\Lambda_a f}(\xi) = \left(\sum_{k\in L}\widehat{a_k T_k f}(\xi)\right) = \sum_{k\in L} a_k \widehat{T_k f}(\xi) = \sum_{k\in L} a_k \overline{\xi}(k)\widehat{f}(\xi) = \widehat{a}(\xi)\widehat{f}(\xi),$$

implies

$$\widehat{\Lambda_a f}(\xi) = \widehat{a}(\xi)\sum_{\alpha\in I}\widehat{b_\alpha}(\xi)\widehat{\phi_\alpha}(\xi),$$

$$\because f(x) = \sum_{\alpha\in I}\sum_{k\in L} b_\alpha(k) T_k\phi_\alpha(x) = \sum_{\alpha\in I}\sum_{k\in L} b_\alpha(k)\phi_\alpha(k^{-1}x).$$

Then $\widehat{f}(\xi) = \sum_{\alpha\in I}\sum_{k\in L} b_\alpha(k)\overline{\xi}(k_\alpha)\widehat{\phi}_\alpha(\xi) = \sum_{\alpha\in I}\widehat{b}_\alpha(\xi)\widehat{\phi}_\alpha(\xi)$. Therefore, we have

$$\Lambda_a f = \sum_{\alpha\in I}\sum_{k\in L}(a * b_\alpha)(k) T_k \phi_\alpha.$$

Definition 3.1 ([1]). Let V be a shift-invariant space and $U : V \to V$ be a bounded shift-preserving operator. Let $a \in l^2(L)$ be a sequence with bounded spectrum. Then, Λ_a is called an s-eigenvalue of operator U if the following condition holds

$$V_a = \ker(U - \Lambda_a) \neq \{0\},$$

and V_a is called the s-eigenspace corresponding to s-eigenvalue Λ_a. s-eigenspace V_a is always a shift-invariant subspace of V with respect to operator U, that is $UV_a \subseteq V_a$ and for each $f \in V_a$, we have $Uf = \Lambda_a f$.

The next result establishes a relation between the s-eigenvalues of shift preserving operator U and eigenvalues of corresponding range operator of U.

Proposition 3.1. *Let V be a shift-invariant space and J be its associated range function J with $\dim J(\xi) < +\infty$ for a.e. $\xi \in S_{L^\perp}$ and $U : V \to V$ be a bounded shift-preserving operator with corresponding range operator R and $a \in \ell^2(L^\perp)$ is a sequence with bounded spectrum. If Λ_a is an s-eigenvalue of operator U, then $\Lambda_a(\xi) = \widehat{a}(\xi)$ is an eigenvalue of corresponding range operator $R(\xi)$ for a.e. $\xi \in \Omega_{\phi_a} = \sigma(V_a)$, the spectrum of V_a.*

Proof. Let G be a second countable LCA group and let V be a shift-invariant subspace of $L^2(G)$. By using Theorem 2.4, there exists a family of functions $\{\phi_n\} \subseteq L^2(G)$ such that

$$V = \bigoplus_{n=1}^{+\infty} S(\phi_n),$$

where each ϕ_n is a Parseval frame generator of $S(\phi_n)$ for every $n \in \mathbb{N}$.

Since every principal shift-invariant space $S(\phi)$ has a Parseval frame generator ϕ, by using Proposition 2.4, a function ϕ is a Parseval frame generator of space $S(\phi)$ if and only if $\|T\phi(\xi)\|^2_{l^2(L^\perp)} = \sum_{\eta\in L^\perp} |\widehat{\phi}(\xi\eta)|^2 = \chi_{\Omega_\phi}(\xi)$ for a.e. $\xi \in S_{L^\perp}$, where $\Omega_\phi = \sigma(S(\phi)) = \{\xi \in S_{L^\perp} : T\phi(\xi) \neq 0\}$ is spectrum of $S(\phi)$. This implies that

$$\text{supp}(\|T\phi(\xi)\|^2_{l^2(L^\perp)}) = \Omega_\phi.$$

Now, since s-eigenspace V_a is a shift-invariant subspace of V. So, using the above results for V_a, there exists Parseval frame generator $\phi_a \in V_a$ such that the following condition holds

$$\text{supp}(\|T\phi_a(\xi)\|^2_{l^2(L^\perp)}) = \Omega_{\phi_a} = \sigma(V_a).$$

So,

$$T\phi_a(\xi) \neq 0, \quad \text{for a.e. } \xi \in \Omega_{\phi_a}. \tag{3.1}$$

Also, since $T(\Lambda_a\phi_a)(\xi) = \widehat{a}(\xi)T\phi_a(\xi)$, this implies that $T(U\phi_a)(\xi) = \lambda_a(\xi)T\phi_a(\xi)$, that is, $R(\xi)T\phi_a(\xi) = \lambda_a(\xi)T\phi_a(\xi)$, that is, $(R(\xi) - \lambda_a(\xi)I)T\phi_a(\xi) = 0$. So, $T\phi_a(\xi) \in \ker(R(\xi) - \lambda_a(\xi)I)$. Thus, using equation (3.1), for a.e. $\xi \in \Omega_{\phi_a}$

$$\ker(R(\xi) - \lambda_a(\xi)I) \neq \{0\}.$$

Hence, $\lambda_a(\xi) = \widehat{a}(\xi)$ is an eigenvalue of $R(\xi)$ for a.e. $\xi \in \Omega_{\phi_a} = \sigma(V_a)$. ☐

Remark 3.2. ([20, Remark 3.2]). Let V be a shift-invariant space in $L^2(G)$. Suppose $V = \bigoplus_{n=1}^{\infty} S(\phi_n)$ is orthonormal decomposition of V, where each $(\phi_n)_{n\in\mathbb{N}}$ is a Parseval frame generator of space $S(\phi_n)$. Then

(a) the set $\{T_k\phi_n : k \in L, n \in \mathbb{N}\}$ forms a Parseval frame for space V;
(b) $\{T\phi_n(\xi) : \ n \in \mathbb{N}\} - \{0\}$ is an orthonormal basis for $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$;
(c) for $\phi_n \neq 0$, $n \in \mathbb{N}$ and $k \in L$, we have

$$\|T_k\phi_n\|_2^2 = \|\phi_n\|_2^2 = \|T\phi_n\|^2_{L\left(S_{L^\perp}, l^2(L^\perp)\right)} = \int_{S_{L^\perp}} \|T\phi_n(\xi)\|^2_{l^2(L^\perp)} d\xi = 1;$$

(d) $\{T_k\phi_n : k \in L, n \in \mathbb{N}\}$ is an orthonormal basis for space V.

The following lemma will be a key working for our main results. It will be used at many places in this article.

Lemma 3.1. *Let V be a shift-invariant space in $L^2(G)$ with associated range function J such that* $\dim J(\xi) < +\infty$ *for a.e. $\xi \in S_{L^\perp}$. Then, there are disjoint measurable sets $\{A_n\}_{n\in\mathbb{N}_0}$ and functions $\{\phi_i\}_{i\in\mathbb{N}}$ in $L^2(G)$ such that $S_{L^\perp} = \cup_{n\in\mathbb{N}_0} A_n$ and the following statements hold:*

(i) set $\{T_k\phi_i : i \in \mathbb{N}, k \in L\}$ forms a Parseval frame for space V;
(ii) $T\phi_i(\xi) = 0$ for a.e. $\xi \in A_n$ and $i > n$;
(iii) $\{T\phi_1(\xi), T\phi_2(\xi), \ldots, T\phi_n(\xi)\}$ is an orthonormal basis of space $J(\xi)$ for a.e. $\xi \in A_n$;
(iv) $\dim J(\xi) = n$ *for a.e. $\xi \in A_n$.*

Proof. Since V is a shift-invariant subspace of $L^2(G)$, therefore using Theorem 2.4 there exist functions $\{\phi_i\}_{i\in\mathbb{N}}$ in space $L^2(G)$ satisfying

$$V = \bigoplus_{i\in\mathbb{N}} S(\phi_i), \tag{3.2}$$

where for each i, function ϕ_i is a Parseval frame generator of the space $S(\phi_i)$, $i \in \mathbb{N}$. This shows that the set $\{T_k\phi_i : i \in \mathbb{N},\ k \in L\}$ forms a Parseval frame for shift-invariant space V. So, by using Theorem 2.2, $\{T\phi_i(\xi) : i \in \mathbb{N}\}$ also forms a Parseval frame for space $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$.

Now, define $A_0 = S_{L^\perp}\backslash\sigma(V)$ and $A_n = \sigma\left(S(\phi_n)\right)\backslash\sigma\left(S(\phi_{n+1})\right)$, for $n > 0$. These sets A_n are pairwise disjoint as $\sigma\left(S(\phi_{i+1})\right)$ is contained in $\sigma\left(S(\phi_i)\right)$ for all $n \in \mathbb{N}$. Also, it is given that $\dim J(\xi) < +\infty$. This implies that $\sum_{i\in\mathbb{N}} \|T\phi_i(\xi)\| < +\infty$ for a.e. $\xi \in S_{L^\perp}$. Since $\sigma\left(S(\phi_i)\right) = \{\xi \in S_{L^\perp} : T\phi_i(\xi) \neq 0\}$, so, if $\xi \in \cap_{i\in\mathbb{N}}\sigma\left(S(\phi_i)\right)$, then this implies that $T\phi_i(\xi) \neq 0$ for all $i \in \mathbb{N}$ and therefore, $\sum_{i\in\mathbb{N}} \|T\phi_i(\xi)\|$ is not finite, which is not possible. Thus, $\cap_{i\in\mathbb{N}}\sigma\left(S(\phi_i)\right) = \emptyset$, this implies $\cup_{n\in\mathbb{N}_0} A_n = S_{L^\perp}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Now, by definition of A_n, $\xi \in A_n$ implies $\xi \in \sigma\left(S(\phi_n)\right)$ and $\xi \notin \sigma\left(S(\phi_{n+1})\right) \supseteq \sigma\left(S(\phi_{n+2})\right) \supseteq \cdots$. Thus, $T\phi_i(\xi) = 0$ for $i > n$ and a.e. $\xi \in A_n$.

From equation (3.2), spaces $S(\phi_i)$ and $S(\phi_j)$ are orthogonal to each other for all $i \neq j$. So, using Proposition 2.5, we have

$$\sum_{\eta\in L^\perp} \widehat{\phi}_i(\xi\eta)\overline{\widehat{\phi}_j}(\xi\eta) = 0, \quad \text{for } i \neq j \ \text{ and for a.e. } \xi \in S_{L^\perp} \subset \widehat{G},$$

this implies

$$\langle T\phi_i(\xi), T\phi_j(\xi)\rangle = 0, \quad \text{for } i \neq j \text{ and for a.e. } \xi \in S_{L^\perp}.$$

So, $\{T\phi_i(\xi) : i \in \mathbb{N}\}$ is an orthogonal set. Also, by Proposition 2.4 for $\xi \in A_n$, we have $\|T\phi_i(\xi)\|^2_{l^2(L^\perp)} = 1$. Thus, $\{T\phi_i(\xi) : i \in \mathbb{N}\}$ is an orthonormal set in $J(\xi)$. Hence, $\{T\phi_1(\xi), T\phi_2(\xi), \ldots, T\phi_n(\xi)\}$ is an orthonormal basis of space $J(\xi)$ for a.e. $\xi \in A_n$ and $\dim J(\xi) = n$ for a.e. $\xi \in A_n$. This completes the proof. □

The following remark is taken from paper [20] which is useful in our main results while using the properties related to Parseval frame of the shift-invariant space V.

Remark 3.3. If A is a measurable set such that $A \subseteq S_{L^\perp}$ and $\dim J(\xi) = n$ a.e. $\xi \in S_{L^\perp}$, then $A \subseteq A_n$, $n \in \mathbb{N}$ or $A_n = \{\xi \in S_{L^\perp} : \dim J(\xi) = n\}$.

In the next proposition, we see that $R(\xi)$ can be written in the form of a matrix when $\dim J(\xi) < +\infty$ for a.e. $\xi \in S_{L^\perp}$.

Proposition 3.2. *Let V be a shift-invariant space in $L^2(G)$ and J be the associated range function J with* $\dim J(\xi) < +\infty$ *for a.e. $\xi \in S_{L^\perp}$ and $U : V \to V$ be a shift preserving operator and R is its associated range operator. Then, $R(\xi)$ has a matrix representation for a.e. $\xi \in S_{L^\perp}$. If* $\dim J(\xi) = n$ *for a.e. $\xi \in A$, where $A \subseteq S_{L^\perp}$ is a* *measurable set, then the $n \times n$ matrix representation of $R(\xi)$ is given by*

$$R(\xi) = \begin{bmatrix} r_{1,1}(\xi) & r_{1,2}(\xi) & \cdots & r_{1,n}(\xi) \\ r_{2,1}(\xi) & r_{2,2}(\xi) & \cdots & r_{2,n}(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ r_{n,1}(\xi) & r_{n,2}(\xi) & \cdots & r_{n,n}(\xi) \end{bmatrix}, \tag{3.3}$$

where $\{r_{i,j}\}_{i,j=1}^{n}$ are measurable bounded functions defined on A.

Proof. Since $\dim J(\xi) < +\infty$, then by using Lemma 3.1, there exist functions $\{\phi_i\}_{\in \mathbb{N}} \subseteq L^2(G)$ and a family of disjoint measurable sets $\{A_n\}_{n \in \mathbb{N}_0}$ satisfying $S_{L^\perp} = \cup_{n \in \mathbb{N}_0} A_n$ and $\{T\phi_1(\xi), T\phi_2(\xi), \dots, T\phi_n(\xi)\}$ is an orthonormal basis of $J(\xi)$, for a fixed $n \in \mathbb{N}$. Since the domain of $R(\xi)$ is $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$. So, $R(\xi)$ has a matrix representation given in (3.3) with respect to orthonormal basis of space $J(\xi)$.

Now, we show that entries of matrix given in (3.3) are measurable and bounded. Since the set $\{T_k\phi_i : i \in \mathbb{N}, k \in L\}$ is a Parseval frame for space V, then for every $j \in \mathbb{N}$, we get

$$U\phi_j = \sum_{i \in \mathbb{N}} \sum_{k \in L} d_i^j(k) T_k \phi_i, \tag{3.4}$$

where $d_i^j(k) \in l^2(\mathbb{N} \times L)$, $i, j, \in \mathbb{N}$ and $k \in L$. This implies that

$$\begin{aligned} TU\phi_j(\xi) &= T\left(\sum_{i \in \mathbb{N}} \sum_{k \in L} d_i^j(k) T_k \phi_i(\xi)\right) = \sum_{i \in \mathbb{N}} \sum_{k \in L} d_i^j(k) T T_k \phi_i(\xi) \\ &= \sum_{i \in \mathbb{N}} \sum_{k \in L} d_i^j(k) M_k(\xi) T\phi_i(\xi). \end{aligned}$$

Now, $T\phi_i(\xi) = 0$ for $i > n$ as a.e. $\xi \in A_n$ (Lemma 3.1). So,

$$TU\phi_j(\xi) = \sum_{i=1}^{n} \sum_{k \in L} d_i^j(k) M_k(\xi) T\phi_i(\xi).$$

Let $r_{i,j}(\xi) = \sum_{k \in L} d_i^j(k) M_k(\xi)$, $1 \le i, j \le n$. Then

$$TU\phi_j(\xi) = \sum_{i=1}^{n} r_{i,j}(\xi) T\phi_i(\xi). \tag{3.5}$$

Since, $M_k \in L^2(S_{L^\perp})$, this implies that M_k is square integrable with respect to the Haar measure defined on G. So, the functions $r_{i,j}(\xi)$ defined above are measurable for a.e. $\xi \in A_n$.

Let $[R](\xi)$ denote the matrix form of operator $R(\xi)$ relative to basis $\{T\phi_1(\xi), T\phi_2(\xi), \dots, T\phi_n(\xi)\}$ for a.e. $\xi \in A_n$. Then

$$([R](\xi))_{i,j} = (R(\xi)T\phi_j(\xi))_i = (TU\phi_j(\xi))_i = r_{i,j}(\xi).$$

So, the matrix $[R](\xi)$ can be described in terms of measurable entries $\{r_{i,j}(\xi)\}_{i,j=1}^{n}$, $\xi \in A_n$ for fixed $n \in \mathbb{N}$. Now using the fact that the function T is an isometry, we have

$$|r_{i,j}(\xi)| = |\,(TU\phi_j(\xi))_i\,| \leq \|\,(TU\phi_j(\xi))_i\,\| = \|\,(U\phi_j(\xi))_i\,\| \leq \|U\|.$$

This implies that $\{r_{i,j}(\xi)\}_{i,j=1}^n$ are bounded functions for a.e. $\xi \in A_n$, because U is a bounded operator. Since, $A \subseteq S_{L^\perp}$ is measurable, where $\dim J(\xi) = n$ for a.e. $\xi \in A$, then $A \subseteq A_n$. Hence, the proposition also holds for A. $\square$

Remark 3.4. The entries $r_{i,j}(\xi)$ in matrix $[R](\xi)$, may not be L-periodic in case of LCA groups. It is explained in the following example.

Example 3.1. Let $G = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the circle group. Then, its dual group is $\mathbb{Z}$, i.e., $\widehat{\mathbb{T}} = \mathbb{Z}$ and in this case $\xi(x) = x^\xi$ where $\xi \in \mathbb{Z}$ and $x \in \mathbb{T}$. So,

$$r_{i,j}(\xi) = \sum_{k \in L} d_i^j(k)\overline{\xi}(k) = \sum_{k \in L} d_i^j(k) k^{\overline{\xi}},$$

where $L = \{z \in \mathbb{T} : z = x + iy, x, y \in \mathbb{Q}\}$ is a discrete subgroup of $\mathbb{T}$. In this case, the entries $\{r_{i,j}\}_{i,j=1}^n$ of matrix $[R](\xi)$ are not L-periodic. That is, $r_{i,j}(\xi k_1) \neq r_{i,j}(\xi)$ for any $k_1 \in L$, as $k^{\overline{\xi k_1}} \neq k^{\overline{\xi}}$ in general for any $k_1 \in L$.

In the next result, we prove that under certain conditions the inverse of a range operator is measurable.

Proposition 3.3. *Let $R(\xi) : J(\xi) \to J(\xi)$ be a measurable range operator, where J is corresponding range function J satisfying* $\dim J(\xi) < +\infty$ *for a.e. $\xi \in S_{L^\perp}$. Then $\xi \mapsto (R(\xi))^{-1}$, $\xi \in S_{L^\perp}$, is a measurable range operator.*

Proof. Since $\dim J(\xi) < +\infty$, therefore, using Lemma 3.1, there exist a family of functions $\{\phi_i\}_{i \in \mathbb{N}} \subseteq L^2(G)$ and measurable sets $\{A_n\}_{n \in \mathbb{N}_0}$. So, to prove $\xi \mapsto (R(\xi))^{-1}$, $\xi \in S_{L^\perp}$, is measurable range operator, it is sufficient to prove that function $\xi \mapsto (R(\xi))^{-1}$, $\xi \in A_n$, is measurable for each $n \in \mathbb{N}_0$. The result trivially holds for A_0.

Now let $n \geq 1$, then to prove $\xi \mapsto (R(\xi))^{-1}$ is measurable, we need to show that $\langle (R(\xi))^{-1} P_{J(\xi)} u, v\rangle$ is measurable for a.e. $\xi \in S_{L^\perp}$ and for all $u, v \in l^2(L^\perp)$. Since, the set $\{T\phi_1(\xi), T\phi_2(\xi), \ldots, T\phi_n(\xi)\}$ is an orthonormal basis of space $J(\xi)$ for a.e. $\xi \in A_n$. So, it is sufficient to prove that $\langle (R(\xi))^{-1} T\phi_i(\xi), T\phi_j(\xi)\rangle$ is measurable for every $i, j = 1, 2, \ldots, n$.

Let $\{e_1, e_2, \ldots, e_n\}$ denote the canonical basis of $\mathbb{C}^n$. Define $u(\xi) : J(\xi) \to \mathbb{C}^n$ as

$$u(\xi)(T\phi_i(\xi)) = e_i \quad \text{a.e. } \xi \in A_n.$$

Here, the operator $u(\xi)$ is change of basis operator from $\{T\phi_1(\xi), T\phi_2(\xi), \ldots, T\phi_n(\xi)\}$ to $\{e_1, e_2, \ldots, e_n\}$. Then

$$([R](\xi))^{-1} = u(\xi)(R(\xi))^{-1}(u(\xi))^{-1},$$

where $[R](\xi)$ denotes the matrix of range operator $R(\xi)$ relative to basis $\{T\phi_1(\xi), T\phi_2(\xi), \ldots, T\phi_n(\xi)\}$. Now

$$\langle (R(\xi))^{-1} T\phi_i(\xi), T\phi_j(\xi)\rangle = \langle (u(\xi))^{-1}([R](\xi))^{-1} u(\xi) T\phi_i(\xi), T\phi_j(\xi)\rangle = \langle (R(\xi))^{-1} e_i, e_j\rangle.$$

Let entries of matrix $([R](\xi))^{-1}$ are $s_{i,j}(\xi)$, which can be obtained as

$$s_{i,j}(\xi) = (-1)^{i+j} \frac{\det\left(([R](\xi))_{ij}\right)}{\det([R](\xi))}, \tag{3.6}$$

where $([R](\xi))_{ij}$ is the minor matrix which is obtained after removing i-th row and j-th column from matrix $[R](\xi)$. Since the entries of the matrix $[R](\xi)$ are measurable, therefore measurablity is preserved under these operations. This implies that $([R](\xi))^{-1}$ has measurable entries. So, $\xi \mapsto \langle (R(\xi))^{-1}e_i, e_j\rangle$ is measurable for a.e. $\xi \in A_n$ and therefore $\xi \mapsto (R(\xi))^{-1}$ is measurable for a.e. $\xi \in A_n$ for all $n \in \mathbb{N}_0$. Hence, $\xi \mapsto (R(\xi))^{-1}$ is measurable for a.e. $\xi \in S_{L^\perp}$. □

In the following theorem, we show the relation between invertiblity of a shift preserving operator and invertiblity of its fibers with $\dim J(\xi) < +\infty$ for a.e. $\xi \in S_{L^\perp}$.

Theorem 3.2. *Let V be a shift-invariant space and J be its range function with $\dim J(\xi) < +\infty$ a.e. $\xi \in S_{L^\perp}$ and $U : V \to V$ be a shift preserving operator with corresponding range operator R. Then, the following statements are true.*

(a) *The inverse U^{-1} of U is also a shift preserving operator, if operator U is invertible.*

(b) *The shift preserving operator U is invertible if and only if operator $R(\xi)$ is invertible for a.e. $\xi \in S_{L^\perp}$ and there exists a constant $K > 0$ such that $R(\xi)$ is uniformly bounded from below by K. In that case, range operator of U^{-1} is denoted by R^{-1}, and $(R(\xi))^{-1} = R^{-1}(\xi)$ for a.e. $\xi \in S_{L^\perp}$.*

Proof. (a) Let U be invertible shift preserving operator, then U^{-1} is a bounded operator. Now, for each function $f \in V$ and $k \in L$ we get

$$U^{-1}T_k f = U^{-1}T_k UU^{-1}f = U^{-1}UT_k U^{-1}f = T_k U^{-1}f.$$

This implies that $U^{-1}T_k = T_k U^{-1}$. So, U^{-1} is also shift preserving operator.

(b) First suppose that U is invertible. Let U^{-1} is inverse of U and R^{-1} be corresponding range operator of U^{-1}. We prove that $\xi \mapsto R^{-1}(\xi)R(\xi)$ and $\xi \mapsto R(\xi)R^{-1}(\xi)$ are measurable and uniformly bounded range operators on J for a.e. $\xi \in S_{L^\perp}$. Then it is enough to show that the mapping $\xi \mapsto \langle R^{-1}(\xi)R(\xi)P_{(J(\xi))^{-1}J(\xi)}u, v\rangle$ is measurable for all $u, v \in l^2(L^\perp)$. Let $u(\xi)$ be change of basis operator defined in Proposition 3.3. Then

$$([R](\xi))^{-1} = u(\xi)(R(\xi))^{-1}(u(\xi))^{-1}.$$

This implies that $(R(\xi))^{-1} = (u(\xi))^{-1}([R](\xi))^{-1}u(\xi)$ and $R(\xi) = (u(\xi))^{-1}[R](\xi)u(\xi)$. Since, $\{T\phi_1(\xi), T\phi_2(\xi), \dots, T\phi_n(\xi)\}$ is an orthonormal basis for $J(\xi)$ a.e. $\xi \in A_n$. Then, it is sufficient to prove that $\xi \mapsto \langle R^{-1}(\xi)R(\xi)T\phi_i(\xi), T\phi_j(\xi)\rangle$ is measurable for a.e. $\xi \in A_n$. Let $([R](\xi))^{-1}[R](\xi)$ denotes matrix form of $(R^{-1}(\xi))R(\xi)$, then

$$([R](\xi))^{-1}[R](\xi) = u(\xi)R^{-1}(\xi)R(\xi)(u(\xi))^{-1}.$$

So,

$$\langle R^{-1}(\xi)R(\xi)T\phi_i(\xi), T\phi_j(\xi)\rangle = \langle (u(\xi))^{-1}([R](\xi))^{-1}[R](\xi)u(\xi)T\phi_i(\xi), T\phi_j(\xi)\rangle$$

$$= \langle ([R](\xi))^{-1}[R](\xi)e_i, e_j\rangle.$$

Since $[R](\xi)$ is matrix corresponding to operator $R(\xi)$ with measurable entries $\{r_{i,j}\}_{i,j=1}^n$ and the matrix $([R](\xi))^{-1}$ has measurable entries $\{s_{i,j}\}_{i,j=1}^n$ defined in (3.6). Then, the matrix $([R](\xi))^{-1}[R](\xi)$ has measurable entries $\{t_{i,j}\}_{i,j=1}^n$, where

$$t_{i,j} = \sum_{k=1}^{n} s_{i,j} r_{k,j}, \quad \text{for } i, j = 1, 2, \ldots, n.$$

This shows that $\xi \mapsto \langle ([R](\xi))^{-1}[R](\xi)e_i, e_j\rangle$ is a measurable function for a.e. $\xi \in A_n$. Thus, $\xi \mapsto R^{-1}(\xi)R(\xi)$ is measurable for a.e. $\xi \in A_n$. Similarly, $\xi \mapsto R(\xi)R^{-1}(\xi)$ is measurable for a.e. $\xi \in A_n$. Also, the operators $R^{-1}(\xi)R(\xi)$ and $R(\xi)R^{-1}(\xi)$ are uniformly bounded range operators on J and their corresponding shift preserving operators are $U^{-1}U$ and UU^{-1}, respectively. So,

$$R^{-1}(\xi)R(\xi) = R(\xi)R^{-1}(\xi) = I,$$

where I denotes the identity range operator on space $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$. Thus, $R^{-1}(\xi)$ is inverse of $R(\xi)$. Next we prove that the operator $R(\xi)$ is uniformly bounded below.

Since R^{-1} is range operator corresponding to shift preserving operator U^{-1}. Therefore,

$$\operatorname{ess\,sup}_{\xi \in S_{L^\perp}} \|R^{-1}(\xi)\| \leq \|U^{-1}\|.$$

Thus, $R^{-1}(\xi)$ is bounded uniformly from above by $\|U^{-1}\|$ as U^{-1} is bounded. This implies that $R(\xi)$ is bounded uniformly from below.

Converse, assume that the operator $R(\xi)$ is invertible for a.e. $\xi \in S_{L^\perp}$ and uniformly bounded by a constant $K > 0$. Then $\xi \mapsto (R(\xi))^{-1}$ is uniformly bounded below by constant K and it is a measurable range operator on J. So, there is a corresponding shift preserving operator $\tilde{U}$ such that for every $f \in V$ and a.e. $\xi \in S_{L^\perp}$, we have

$$(R(\xi))^{-1}Tf(\xi) = T\tilde{U}f(\xi).$$

Now $Tf(\xi) = R^{-1}(\xi)R(\xi)Tf(\xi) = T\tilde{U}Uf(\xi)$ and $Tf(\xi) = R(\xi)R^{-1}(\xi) = TU\tilde{U}f(\xi)$. Thus, $U\tilde{U} = \tilde{U}U = I$. Hence, U is invertible and $U^{-1} = \tilde{U}$. This completes the proof. ☐

Theorem 3.3. *Let V be a shift-invariant space in space $L^2(G)$, where G is a LCA group and let J be the range function associated to V. Let $U : V \to V$ be a bounded shift preserving operator with corresponding range operator R on $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$. Then the following conditions hold.*

(1) *The operator U is normal if and only if $R(\xi)$ is a normal operator for a.e. $\xi \in S_{L^\perp}$.*

(2) *The operator U is unitary if and only if $R(\xi)$ is unitary for a.e. $\xi \in S_{L^\perp}$. (where $S_{L^\perp}$ is a fundamental domain for $L^\perp$ in $\widehat{G}$).*

Proof. Since $R(\xi)$ is a range operator on $J(\xi)$ corresponding to shift preserving operator U. So,

$$(T \circ U)(f(\xi)) = R(\xi)(T(f(\xi)) \quad \text{a.e. } \xi \in S_{L^\perp}, \tag{3.7}$$

where $T : L^2(G) \to L^2(S_{L^\perp}, l^2(L^\perp))$ is fiberization mapping which is defined by $Tf(\xi) = (\widehat{f}(\xi\eta)_{\eta \in L^\perp}$. Also, we know that if U is shift preserving operator then its adjoint operator $U^* : V \to V$ is also a shift-preserving operator, and the associated range operator is R^* which is given by $R^*(\xi) = (R(\xi))^*$ for a.e. $\xi \in S_{L^\perp}$ (by Proposition 3.5 in [20]).

(1) First we show that the range operators corresponding to U^*U and UU^* are R^*R and RR^*, respectively given by $R^*(\xi)R(\xi) = (R(\xi))^*(R(\xi))$ and $R(\xi)R(\xi)^* = (R(\xi))(R(\xi))^*$ for a.e. $\xi \in S_{L^\perp}$. Note that the operators R^*R and RR^* given by $R^*(\xi)R(\xi) = (R(\xi))^*(R(\xi))$ and $R(\xi)R(\xi)^* = (R(\xi))(R(\xi))^*$ for a.e. $\xi \in S_{L^\perp}$ are measurable. Also, $\operatorname{ess\,sup}_{\xi \in S_{L^\perp}} \|R^*(\xi)R(\xi)\| < +\infty$ and $\operatorname{ess\,sup}_{\xi \in S_{L^\perp}} \|R(\xi)R^*(\xi)\| < +\infty$. Then, by Theorem 2.3, there exist shift preserving operators W_1 and W_2 on V which satisfies

$$(T \circ W_1)(f(\xi)) = R^*(\xi)R(\xi)(T(f(\xi))$$

and

$$(T \circ W_2)(f(\xi)) = R(\xi)R^*(\xi)(T(f(\xi)),$$

for a.e. $\xi \in S_{L^\perp}$ and for all $f \in V$. Now for all functions $f, g \in V$, consider

$$\begin{aligned}
\langle U^*Uf, g\rangle &= \langle Uf, Ug\rangle, \\
&= \langle T \circ Uf, T \circ Ug\rangle, \\
&= \int_{S_{L^\perp}} \langle T \circ Uf(\xi), T \circ Ug(\xi)\rangle d\xi, \\
&= \int_{S_{L^\perp}} \langle R(\xi)(Tf(\xi)), R(\xi)(Tg(\xi))\rangle d\xi, \\
&= \int_{S_{L^\perp}} \langle R^*(\xi)R(\xi)(Tf(\xi)), Tg(\xi)\rangle d\xi, \\
&= \int_{S_{L^\perp}} \langle (ToW_1)f(\xi), Tg(\xi)\rangle d\xi,
\end{aligned}$$

$$\langle U^*Uf, g\rangle = \langle T \circ W_1 f, Tg\rangle. \tag{3.8}$$

This implies that $\langle U^*Uf, g\rangle = \langle W_1 f, Tg\rangle$ for all $f, g \in V$. So, $U^*U = W_1$. That is,

$$(T \circ (U^*U))f(\xi) = R(\xi)(Tf(\xi)), \quad \text{for a.e. } \xi \in S_{L^\perp} \text{ and for all } f \in V. \tag{3.9}$$

Similarly,

$$(T \circ (UU^*))f(\xi) = R(\xi)(Tf(\xi)), \quad \text{for a.e. } \xi \in S_{L^\perp} \text{ and for all } f \in V. \tag{3.10}$$

Now suppose that U is a normal operator, so $U^*U = UU^*$, then from (3.9) and (3.10)

$$R^*(\xi)R(\xi) = R(\xi)R^*(\xi) \quad \text{a.e. } \xi \in S_{L^\perp}.$$

This implies that $R(\xi)$ is a normal operator for a.e. $\xi \in S_{L^\perp}$.

(2) Using Theorem 27, if U is invertible then U^{-1} is also a shift preserving operator and the corresponding range operator R^{-1} is given by $R^{-1}(\xi) = (R(\xi))^{-1}$ for a.e. $\xi \in S_{L^\perp}$. Also the adjoint operator U^* is a shift-preserving and the corresponding range operator is R^* given by $R^*(\xi) = (R(\xi))^*$ for a.e. $\xi \in S_{L^\perp}$. So, we have $(T \circ U^{-1})f(\xi) = R^{-1}(\xi)(Tf(\xi))$ and $(T \circ U^*)f(\xi) = R^*(\xi)(Tf(\xi))$ for a.e. $\xi \in S_{L^\perp}$ and for all $f \in V$. Since, U is unitary this implies that $U^* = U^{-1}$. Then, we get

$$R^{-1}(\xi) = R^*(\xi) \quad \text{a.e. } \xi \in S_{L^\perp}.$$

Thus, $R(\xi)$ is unitary for a.e. $\xi \in S_{L^\perp}$. □

4. Conclusion

In this paper, we used the concept of s-eigenvalue and s-eigenspace to see the relation between the eigenvalues of a shift preserving operator and the corresponding range operator on LCA group. We also characterized the matrix structure of range operator, in the finite dimensional case. We got the conditions which ensure that invertibility of shift preserving operator implies the invertibility of the fiber of the corresponding range operator and vice versa. In the end, we got some conditions which show that a shift preserving operator and the fiber of corresponding range operator both share the same properties like as unitary, normal, isometry, self adjoint etc.

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