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ON THE EXISTENCE AND UNIQUENESS OF FUZZY MILD SOLUTION OF FRACTIONAL EVOLUTION EQUATIONS

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Abstract. In this paper, the nonlocal Cauchy problem is discussed for the fuzzy fractional evolution equations in an arbitrary Banach space for order $q \in (1,2)$ and the criteria on the existence and uniqueness of mild fuzzy solutions are obtained by using Schauder's fixed point theorem. An example to illustrate the applications of main results is also given.

1. Introduction

Fuzzy set theory has been attracting increasing interest in recent years as it is widely used in several fields such as mechanics, electrical engineering, signal processing, etc. As a result, in recent decades, fuzzy set theory has become a hot and current topic and has received much attention from researchers (see for instance [16,17]).

Note that Kaleva [11] discussed the properties of differentiable fuzzy set-valued mappings by means of the concept of H-differentiability due to Puri and Ralescu [12], gave the existence and uniqueness theorem for a solution of the fuzzy differential equation

$$u'(t) = f(t, u(t)), \quad u(0) = u_0, \tag{1.1}$$

when $f : I \times E^n \to E^n$ satisfies the Lipschitz condition.

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In [13] Bhaskar Dubey and Raju K. George studied the linear time-invariant systems with fuzzy initial condition

$$u'(t) = Au(t) + Bc(t), \quad u(t_0) = u_0, \tag{1.2}$$

where $c(t) \in (E^1)^m$ a control and A, B, are $n \times n, n \times m$ real matrices, respectively, $t_0 \geq 0$.

In [14] Nguyen Thi Kim Son demonstrate the efficiency of theoretical results by studying the existence of fuzzy mild solutions of nonlinear fuzzy fractional evolution equations

$$\begin{cases} {}_{gH}^{C}\mathcal{D}^q x(t) = Ax(t) + f(t, x(t)), \quad t \in [0, a], \\ x(0) = \varphi_0, \end{cases} \tag{1.3}$$

where ${}_{gH}^{C}\mathcal{D}^q$ is the fuzzy Caputo fractional derivative of order $q \in (0,1)$, and A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{T}$ the set of all triangular fuzzy numbers.

Building on this work, we have opted for the fuzzy Caputo derivative to prove the existence and uniqueness of the soft solution of the fuzzy initial value problem of the fractional evolution equation of order $q \in (1,2)$

$$\begin{cases} {}_{gH}^{C}\mathcal{D}^q x(t) = Ax(t) + f\left(t, x(t), {}_{gH}^{C}\mathcal{D}^{q-1} x(t)\right), \quad t \in [0, a], 1 < q < 2, \\ x(0) = x_0, \\ {}_{gH}^{C}\mathcal{D}^{q-i} x(0) = x_i, \quad i = 1, \dots, |q|, \end{cases} \tag{1.4}$$

where A is a linear operator and f is a continuous function.

The purpose of this study is to develop an original kind of fuzzy initial value problem of the fractional evolution equation of order $q \in (1,2)$ utilizing fuzzy Caputo derivative of order $q \in (1,2)$, and also to show the existence and uniqueness of its mild solutions.

The following is a breakdown of the paper's structure. After this Introduction we give Preliminaries which will be used throughout this paper, Fractional Integral Equation can be found in Section 3, The existence of mild solutions of the Cauchy problem for fractional evolution equations is studied in Section 4. In Section 5,we provide an example to present the applications of the results obtained in the abstract.

2. Preliminaries

In this part we recall some basic notions that will be useful in the rest of our article.

2.1. The metric space E^1.

Definition 2.1. A fuzzy number is a fuzzy set $x : \mathbb{R} \to [0,1]$ that satisfies the following conditions:

1. x is normal, i.e., there is a $t_0 \in \mathbb{R}$ such that $x(t_0) = 1$;
2. x is a fuzzy convex set;

3. x is upper semi-continuous;
4. x closure of $\{t \in \mathbb{R} : x(t) > 0\}$ is compact.

We denote by E^1 the space of all fuzzy numbers on $\mathbb{R}$.

$$E^1 = \{x : \mathbb{R} \to [0,1] : x \text{ satisfies 1–4. below}\}.$$

For all $\alpha \in (0,1]$ the α-cut of an element of E^1 is defined by

$$x^\alpha = \{t \in \mathbb{R} : x(t) \geq \alpha\}.$$

By the former parcels we can write

$$x^\alpha = [\underline{x}(\alpha), \bar{x}(\alpha)]. \tag{2.1}$$

The distance between two element of E^1 is given by (see [1])

$$d(x,y) = \sup_{\alpha\in(0,1]} \max\{|\bar{x}(\alpha) - \bar{y}(\alpha)|, |\underline{x}(\alpha) - \underline{y}(\alpha)|\}. \tag{2.2}$$

And the following properties are valid:

1. $d(x+z, y+z) = d(x,y)$;
2. $d(\lambda x, \lambda y) = |\lambda| d(x,y)$;
3. $d(x+y, w+z) \leq d(x,w) + d(y,z)$.

The operations of addition and scalar multiplication of fuzzy numbers on $\mathbb{R}_{\mathcal{F}}$ have the form

$$[x \oplus y]^\alpha = [x]^\alpha + [y]^\alpha \quad \text{and} \quad [\lambda \odot x]^\alpha = \lambda [x]^\alpha, \quad \lambda \in \mathbb{R}, \tag{2.3}$$

where

$$[x]^\alpha + [y]^\alpha = \{a + b : a \in [x]^\alpha, b \in [y]^\alpha\} \tag{2.4}$$

is the Minkowski sum of $[x]^\alpha$ and $[y]^\alpha$ and

$$\lambda [x]^\alpha = \{\lambda a : a \in [x]^\alpha\}. \tag{2.5}$$

For $x, y \in \mathbb{R}_{\mathcal{F}}$, the gH difference [2] of x and y, denoted by $x \ominus_{gH} y$, is defined as the element $z \in \mathbb{R}_{\mathcal{F}}$ such that

$$x \ominus_{gH} y = z \Leftrightarrow \{\text{(i) } x = y + z \quad \text{or} \quad \text{(ii) } y = x + (-1)z\}. \tag{2.6}$$

In terms of α-levels we have

$$(x \ominus_{gH} y)^\alpha = [\min\{\underline{x}(\alpha) - \underline{y}(\alpha), \bar{x}(\alpha) - \bar{y}(\alpha)\}, \max\{\underline{x}(\alpha) - \underline{y}(\alpha), \bar{x}(\alpha) - \bar{y}(\alpha)\}].$$

And the conditions for the existence of $z = x \ominus_g y \in E^1$ are:

case (i):

$$\begin{cases} \underline{z}(\alpha) = \underline{x}(\alpha) - \underline{y}(\alpha) \text{ and } \overline{z}(\alpha) = \overline{x}(\alpha) - \overline{y}(\alpha), \\ \text{with } \underline{z}(\alpha) \text{ increasing, } \overline{z}(\alpha) \text{ decreasing, } \underline{z}(\alpha) \leq \overline{z}(\alpha); \end{cases} \tag{2.7}$$

case (ii):

$$\begin{cases} \underline{z}(\alpha) = \overline{x}(\alpha) - \overline{y}(\alpha) \text{ and } \overline{z}(\alpha) = \underline{x}(\alpha) - \underline{y}(\alpha), \\ \text{with } \underline{z}(\alpha) \text{ increasing, } \overline{z}(\alpha) \text{ decreasing, } \underline{z}(\alpha) \leq \overline{z}(\alpha), \end{cases} \tag{2.8}$$

for all $\alpha \in [0,1]$.

In general, with $x \in \mathbb{R}_{\mathcal{F}}$, there does not exist $y \in \mathbb{R}_{\mathcal{F}}$ such that $x \oplus y = 0$. Then, unfortunately, $\mathbb{R}_{\mathcal{F}}$ is not a linear space with addition and scalar multiplication. Consequently, $(\mathbb{R}_{\mathcal{F}}, \|\cdot\|)$ is not a Banach space, where $\|x\| = d_\infty(x, \hat{0})$, $x \in \mathbb{R}_{\mathcal{F}}$.

Denote $\mathcal{T}$ by the set of all triangular fuzzy numbers in $\mathbb{R}_{\mathcal{F}}$. $(\mathcal{T}, d_\infty)$ is a subspace of the metric space $(\mathbb{R}_{\mathcal{F}}, d_\infty)$. It is a complete metric space. Moreover, Bede [3] showed that if $x, y \in \mathcal{T}$, then the difference $x \ominus_{gH} y$ always exists in $\mathcal{T}$ and $x \ominus_{gH} y = (-1) \odot (y \ominus_{gH} x)$.

Let X be a subset of $\mathbb{R}_{\mathcal{F}}$, $J \subset \mathbb{R}$, and denote $\mathcal{C}(J, X)$ by the set of all continuous mappings $f : J \to X$.

2.2. **Hukuhara's derivative.** Let $f : [a,b] \subset \mathbb{R} \to E^1$ a fuzzy-valued function. The α-level of f is given by

$$f(t,\alpha) = \left[\underline{f}(t,\alpha), \overline{f}(t,\alpha)\right], \quad \text{for all } t \in [a,b], \alpha \in [0,1].$$

Definition 2.2 ([4])**.** Let $t_0 \in (a,b)$ and h be such that $t_0 + h \in (a,b)$, then the generalized Hukuhara derivative of a fuzzy value function $f : (a,b) \to E^1$ at t_0 is defined as

$$\lim_{h \to 0} \left\| \frac{f(t_0 + h) -_g f(t_0)}{h} -_g f'_{gH}(t_0) \right\|_1 = 0. \tag{2.9}$$

If $f_{gH}(t_0) \in E^1$ satisfying (3.4) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at t_0.

Definition 2.3 ([4])**.** Let $f : [a,b] \to E^1$ and $t_0 \in (a,b)$, with $\underline{f}(t,\alpha)$ and $\bar{f}(t,\alpha)$ both differentiable at t_0.

We say that

1. f is $[(i) - gH]$-differentiable at t_0 if

$$f'_{i,gH}(t_0) = \left[\underline{f}'(t,\alpha), \bar{f}'(t,\alpha)\right]; \tag{2.10}$$

2. f is $[(ii) - gH]$-differentiable at t_0 if

$$f'_{ii,gH}(t_0) = \left[\bar{f}'(t,\alpha), \underline{f}'(t,\alpha)\right]. \tag{2.11}$$

Theorem 2.1 ([6])**.** *Let $f : J \subset \mathbb{R} \to E^1$ and $\phi : J \to \mathbb{R}$ and $t \in J$. Suppose that $\phi(t)$ is differentiable function at t and the fuzzy-valued function $f(t)$ is gH-differentiable at t. So,*

$$(f\phi)'_g(t) = (f'\phi)_g(t) + (f\phi')_g(t). \tag{2.12}$$

Definition 2.4 ([5])**.** Let $f : [a,b] \to E^1$ and $f'_gH(t)$ be gH-differentiable at $t_0 \in (a,b)$, moreover there isn't any switching point on (a,b) and $\underline{f}(t,\alpha)$ and $\overline{f}(t,\alpha)$ both differentiable at t_0. We say that

- f' is $[(i) - gH]$-differentiable at t_0, if

$$f''_{i,gH}(t_0) = \left[\underline{f}''(t,\alpha), \overline{f}''(x,\alpha)\right];$$

- f' is $[(ii)-gH]$-differentiable at t_0, if

$$f''_{ii,gH}(t_0)=\Big[\overline{f}''(t,\alpha),\underline{f}''(t,\alpha)\Big].$$

2.3. **Fuzzy fractional derivative.** We present generalized fuzzy fractional derivative and their properties.

Definition 2.5 ([9])**.** Let $f\in L^{E^1}([a,b])$. The fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as following:

$$I^q_{RL}f(t)=\frac{1}{\Gamma(q)}\odot\int_a^t(t-s)^{q-1}\odot f(s)ds,\quad a<s<t,0<q<1. \tag{2.13}$$

Definition 2.6 ([6], Riemann-Liouville fractional derivative-RL)**.** Let us consider $f\in L^{E^1}([a,b])$ is a fuzzy number valued function,

$$D^q_{RL_{gH}}f(s)=\begin{cases}\frac{1}{\Gamma(n-q)}\odot\left(\frac{d}{ds}\right)^n\int_a^s(s-t)^{n-q-1}\odot f(t)dt, & n-1<q<n,\\ \left(\frac{d}{ds}\right)^{n-1}f(s), & q=n-1.\end{cases} \tag{2.14}$$

Definition 2.7 ([6])**.** In the definition of RL fractional derivative, suppose the integer order of the derivative is an operator inside of the integral and operating on operand function $f(t)\in E^1, t\in[a,b]$. We get the definition of Caputo gH derivative of $f(t)$

$${}^C_{gH}D^qf(s)=\begin{cases}\frac{1}{\Gamma(n-q)}\odot\int_a^s(s-t)^{n-q-1}\odot f^{(n)}_{gH}(t)dt, & n-1<q<n,\\ \left(\frac{d}{ds}\right)^{n-1}f(s), & q=n-1.\end{cases} \tag{2.15}$$

Also we say that f is $[(i)-gH]$-differentiable at t_0, if

$${}_{gH}D^q_tf(x,t;\alpha)=\Big[D^q\underline{f}(x,t;\alpha),D^q\bar{f}(x,t;\alpha)\Big],\quad\text{for all } q\in(0,1), \tag{2.16}$$

and f is $[(ii)-gH]$-differentiable at t_0, if

$${}_{gH}D^q_tf(x,t;\alpha)=\Big[D^q\bar{f}(x,t;\alpha),\underline{f}(x,t;\alpha)\Big],\quad\text{for all } q\in(0,1). \tag{2.17}$$

Definition 2.8 ([10])**.** Let $f:[0,+\infty)\to X\subset\mathbb{R}_{\mathcal{F}}$ be a continuous function such that $e^{-st}\odot f(t)$ is integrable. Then the fuzzy Laplace transform of f, denoted by $\mathcal{L}[f(t)]$, is

$$\mathcal{L}[f(t)]:=F(s)=\int_0^{+\infty}e^{-st}\odot f(t)dt,\quad s>0. \tag{2.18}$$

A fuzzy-valued function f is exponent bounded of order β if there exists $M>0$ similar that

$$(\exists t_0>0)\,d_{+\infty}(f(t),\hat{0})\le Me^{\beta t},\quad\text{for all } t\ge t_0.$$

Proposition 2.1. *If $x(t)$ is a fuzzy peace-wise continuous function on $[0,+\infty]$ and of exponential order a, then*

$$\boldsymbol{L}((x\star y)(t))=\boldsymbol{L}(x(t))\odot\boldsymbol{L}(y(t)), \tag{2.19}$$

where $y(t)$ is a peace-wise continuous real function on $[0,+\infty)$.

Proof. We have

$$\begin{aligned}\boldsymbol{L}(x(t)) \odot \boldsymbol{L}(y(t)) &= \left(\int_0^{+\infty} e^{-s\tau} \odot x(\tau)d\tau\right) \odot \left(\int_0^{+\infty} e^{-s\sigma} \odot y(\sigma)d\sigma\right)\\ &= \int_0^{+\infty} \left(\int_0^{+\infty} e^{-s(\tau+\sigma)} \odot x(\tau)d\tau\right) \odot y(\sigma)d\sigma.\end{aligned}$$

Let us to hold τ fixed in the interior integral, substituting $t = \tau + \sigma$ and $d\sigma = dt$, we obtain

$$\begin{aligned}\boldsymbol{L}(x(t)) \odot \boldsymbol{L}(y(t)) &= \int_0^{+\infty} \left(\int_\sigma^{+\infty} e^{-st} \odot x(\tau) \odot y(t-\tau)dt\right) d\tau\\ &= \int_0^{+\infty} \int_\sigma^{\infty} e^{-st} \odot x(\tau) \odot y(t-\tau)dtd\tau\\ &= \int_0^{+\infty} e^{-st} \odot \left(\int_0^{t} x(t-\sigma) \odot y(\sigma)d\tau\right) d\sigma\\ &= \boldsymbol{L}((x \star y)(t)).\end{aligned}$$

□

Definition 2.9 ([15]). 1. The Gamma function is given by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \quad \text{for all } x > 0. \tag{2.20}$$

2. The $\mathbb{B}$ function is defined by

$$\mathbb{B}(x,y) = \int_0^{1} t^{x-1}(1-t)^{y-1}, \quad \text{for all } x, y > 0. \tag{2.21}$$

Proposition 2.2 ([15]). 1. *For all* $x, y \in \mathbb{R}_+^*$, $\mathbb{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

2. *For all* $x > 0$, $\Gamma(x+1) = x\Gamma(x)$. *It is easy to show the following lemma.*

Proposition 2.3. *For all* $\alpha > 0$, *we get the following result*

$$\int_0^t E_{\alpha,1}\left(As^\alpha\right) ds = tE_{\alpha,2}\left(At^\alpha\right). \tag{2.22}$$

Proof.

$$\begin{aligned}\int_0^t E_{\alpha,1}\left(As^\alpha\right) ds &= \int_0^t \sum_{n=0}^{+\infty} \frac{s^{n\alpha}}{\Gamma(n\alpha+1)} A^n ds\\ &= \sum_{n=0}^{+\infty} \frac{\int_0^t s^{n\alpha}ds}{\Gamma(n\alpha+1)} A^n\\ &= \sum_{n=0}^{+\infty} \frac{t^{n\alpha+1}}{(n\alpha+1)\Gamma(n\alpha+1)} A^n\\ &= \sum_{n=0}^{+\infty} \frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)} A^n\\ &= tE_{\alpha,2}\left(At^\alpha\right).\end{aligned}$$

□

Lemma 2.1. *For all* $\alpha \in [1,2]$ *and* $s > 0$, *we have*

1. $s^{\alpha-1}\left(s^{\alpha}-A\right)^{-1} = \mathscr{L}\left(E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)$;
2. $s^{\alpha-2}\left(s^{\alpha}-A\right)^{-1} = \mathscr{L}\left(tE_{\alpha,2}\left(At^{\alpha}\right)\right)(s)$;
3. $\left(s^{\alpha}-A\right)^{-1} = \frac{1}{\Gamma(\alpha-1)}\mathscr{L}\left(\int_0^t (t-s)^{\alpha-2}E_{\alpha,1}\left(As^{\alpha}\right)ds\right)$.

Proof. 1. For $s > 0$,

$$\begin{aligned}\mathscr{L}\left(E_{\alpha,1}\left(At^{\alpha}\right)\right)(s) &= \mathscr{L}\left(\sum_{n=0}^{+\infty}\frac{t^{\alpha n}A^{n}}{\Gamma(\alpha n+1)}\right)\\ &= \sum_{n=0}^{+\infty}\mathscr{L}\left(t^{\alpha n}\right)\frac{A^{n}}{\Gamma(\alpha n+1)}\\ &= \sum_{n=0}^{+\infty}\frac{1}{s^{n\alpha+1}}A^{n}\\ &= s^{\alpha-1}\left(s^{\alpha}-A\right)^{-1}.\end{aligned}$$

2. For $s > 0$, $s^{\alpha-1}\left(s^{\alpha}-A\right)^{-1} = \mathscr{L}\left(E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)$, then

$$\begin{aligned}s^{\alpha-2}\left(s^{\alpha}-A\right)^{-1} &= s^{-1}s^{\alpha-1}\left(s^{\alpha}-A\right)^{-1}\\ &= \mathscr{L}(1)(s)\mathscr{L}\left(E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)\\ &= \mathscr{L}\left(1 * E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)\\ &= \mathscr{L}\left(\int_0^t E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)\\ &= \mathscr{L}\left(tE_{\alpha,2}\left(t^{\alpha}A\right)\right)(s).\end{aligned}$$

3. From (1), we get

$$\begin{aligned}\left(s^{\alpha}-A\right)^{-1} &= s^{1-\alpha}\mathscr{L}\left(E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)\\ &= \mathscr{L}\left(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\right)\mathscr{L}\left(E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)\\ &= \mathscr{L}\left(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} * E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)\\ &= \mathscr{L}\left(\int_0^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)}E_{\alpha,1}\left(A\delta^{\alpha}\right)d\delta\right)(s),\end{aligned}$$

hence the desired result. □

Lemma 2.2 ([10])**.** (1) *Let* $f, g : [0,+\infty) \to X$ *be continuous functions,* $c_1, c_2 \in \mathbb{R}^{+}$. *Then*

$$\mathcal{L}\left[c_1 \odot f(t) + c_2 \odot g(t)\right] = c_1 \odot \mathcal{L}[f(t)] + c_2 \odot \mathcal{L}[g(t)].$$

(2) *Let* $f : [0,+\infty) \to X$ *be a continuous function. Then*

$$\mathcal{L}\left[e^{at} \odot f(t)\right] = F(s-a), \quad s-a > 0.$$

(3) *Let* $f \in C^1([0,+\infty), X)$ *be exponent bounded of order* β. *Then*

(i) *if f is $[(i)-gH]$ differentiable, then* $\mathcal{L}\left[\mathcal{D}^{i}_{gH}f(t)\right] = s\odot\mathcal{L}[f(t)]\ominus f(0)$;

(ii) *if f is $[(ii)-gH]$ differentiable, then* $\mathcal{L}\left[\mathcal{D}^{ii}_{gH}f(t)\right] = (-1)\odot f(0)\ominus(-s)\odot\mathcal{L}[f(t)]$.

Now, we recall Schauder's fixed point theorem and the Ascoli-Arzela theorem as follows.

Theorem 2.2 (Schauder fixed point theorem)**.** *Let Y be a nonempty, closed, bounded and convex subset of a Banach space X, and suppose that $P : Y \to Y$ is a compact operator. Then P has at least one fixed point in Y.*

Theorem 2.3 (Ascoli-Arzela)**.** *Let $\phi_n(t)$ be a sequence of functions from $[a, b]$ to $\mathbb{R}$ which is uniformly bounded and equicontinuous. Then, $\phi_n(t)$ has a uniformly convergent subsequence.*

3. Fuzzy Fractional Integral Equation

In this section, we have performed the Cauchy problem for fuzzy fractional evolution equations involving Caputo gH derivatives

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{q}x(t) = Ax(t) + f\left(t, x(t), {}^{C}_{gH}\mathcal{D}^{q-1}x(t)\right), \quad t\in[0,a],\\ x(0) = x_0, \quad 1<q<2,\\ {}^{C}_{gH}\mathcal{D}^{q-i}x(0) = x_i, \quad i = 1,\dots,|q|, \end{cases} \tag{3.1}$$

where ${}^{C}_{gH}\mathcal{D}^{q}$ is the fuzzy Caputo fractional derivative of order $q\in(1,2)$, and A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{T}$.

Using Caputo's fuzzy fractional derivative definition, applying the Riemann-Liouville mixed fractional integral operator ${}^{RL}\mathcal{J}^{q}_{0+}$ member to member in (3.1) and using the Newton-Leibnitz formula for the gH derivative [3], we get

$$x(t)\ominus_{gH} x(0)\ominus_{gH} t\odot x'(0) = {}^{RL}\mathcal{J}^{q}_{0+}(Ax(t) + f\left(t, x(t), {}^{C}_{gH}\mathcal{D}^{q-1}x(t)\right). \tag{3.2}$$

From the definition of gH difference and (3.1), we get the following assertions.

(i) If x is Caputo $[(i)-\text{gH}]$ differentiable, then

$$x(t) = x(0) + t\odot x'(0) + {}^{RL}\mathcal{J}^{q}_{0+}\left(Ax(t) + f\left(t, x(t), {}^{C}_{gH}\mathcal{D}^{q-1}x(t)\right)\right). \tag{3.3}$$

(ii) If x is Caputo $[(ii)-gH]$ differentiable, then

$$x(t) = x(0) + t\odot x'(0)\ominus(-1)\odot {}^{RL}\mathcal{J}^{q}_{0+}\left(Ax(t) + f\left(t, x(t), {}^{C}_{gH}\mathcal{D}^{q-1}x(t)\right)\right). \tag{3.4}$$

By applying the fuzzy Laplace transform in [10], we obtain the precise integral formula of the Cauchy problem (3.1) as follows.

Lemma 3.1. (i) *If x is Caputo $[(i)-gH]$ differentiable satisfying the Cauchy problem* (3.1)*, then*

$$x(t) = E_{q,1}\left(At^{q}\right)\odot x_0 + t\odot E_{q,2}\left(At^{q}\right)\odot x_1 \tag{3.5}$$

$$+\int_0^t\int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^{q}\right)\odot f\left(s, x(s), {}^{C}_{gH}\mathcal{D}^{q-1}x(s)\right)d\delta ds. \tag{3.6}$$

(ii) If x is Caputo $[(ii)-gH]$ differentiable satisfying the Cauchy problem (3.1)*, then*

$$x(t) = E_{q,1}\left(At^q\right) \odot x_0 + t \odot E_{q,2}\left(At^q\right) \odot x_1 \tag{3.7}$$

$$\ominus (-1) \odot \int_0^t \int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta-s)^q\right) \odot f\left(s, x(s), {}_{gH}^{C}\mathfrak{D}^{q-1}x(s)\right) d\delta ds. \tag{3.8}$$

Here, $E_{q,1}\left(At^q\right)$ is the Mittag-Leffler function.

Proof. Set

$$X(s) = \mathfrak{L}[x(t)] = \int_0^\infty e^{-st} \odot x(t)dt \tag{3.9}$$

and

$$F(s) = \mathfrak{L}[f(t)] = \int_0^\infty e^{-st} \odot f\left(t, x(t), {}_{gH}^{C}\mathfrak{D}^{q-1}x(t)\right) dt. \tag{3.10}$$

Case 1. Assume that x is Caputo $[(i)-gH]$ differentiable on $[0,+\infty)$. Then from (3.3) and [10] we have

$$\begin{aligned} x(t) &= x(0) + t \odot x'(0) + {}^{RL}\mathfrak{I}_{0+}^{q}\left(Ax(t) + f\left(t, x(t), {}_{gH}^{C}\mathfrak{D}^{q-1}x(t)\right)\right), \\ X(s) &= \mathfrak{L}(x(0)) + \mathfrak{L}(t \odot x'(0)) + \mathfrak{L}\left({}^{RL}\mathfrak{I}_{0+}^{q}\left(Ax(t) + f\left(t, x(t), {}_{gH}^{C}\mathfrak{D}^{q-1}x(t)\right)\right)\right), \\ &= \frac{1}{s} \odot x_0 + \frac{1}{s^2} \odot x_1 + \frac{1}{s^q} \odot AX(s) + \frac{1}{s^q} \odot F(s), \end{aligned}$$

which implies

$$\begin{aligned} s^q \odot X(s) &= s^{q-1} \odot x_0 + s^{q-2} \odot x_1 + AX(s) + F(s), \\ (s^q \odot Id \ominus A) \odot X(s) &= s^{q-1} \odot x_0 + s^{q-2} \odot x_1 + F(s), \\ X(s) &= (s^q \odot Id \ominus A)^{-1} \odot s^{q-1} \odot x_0 + (s^q \odot Id \ominus A)^{-1} \odot s^{q-2} \odot x_1 \\ &\quad + (s^q \odot Id \ominus A)^{-1} \odot F(s) \end{aligned}$$

or

$$\begin{aligned} (s^q \odot Id \ominus A)^{-1} \odot s^{q-1} &= \mathfrak{L}\left(E_{q,1}\left(At^q\right)\right)(s), \\ (s^q \odot Id \ominus A)^{-1} \odot s^{q-2} &= \mathfrak{L}\left(t \odot E_{q,2}\left(At^q\right)\right)(s), \\ (s^q \odot Id \ominus A)^{-1} &= \frac{1}{\Gamma(q-1)} \odot \mathfrak{L}\left(\int_0^t (t-\tau)^{q-2} E_{q,1}\left(A\tau^q\right) d\tau\right), \end{aligned}$$

which give

$$X(s) = \mathfrak{L}\left(E_{q,1}\left(At^q\right)\right) \odot x_0 + \mathfrak{L}\left(t \odot E_{q,2}\left(At^q\right)\right) \odot x_1 + \mathfrak{L}\left(g * f\right), \tag{3.11}$$

with $g(t) = \int\limits_0^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A\tau^q\right) d\tau$.

Applying the inverse Laplace transformation, we get:

$$x(t) = E_{q,1}\left(At^q\right) \odot x_0 + t \odot E_{q,2}\left(At^q\right) \odot x_1 \tag{3.12}$$

$$+\int_0^t\int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta-s)^q\right) \odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) d\delta ds. \tag{3.13}$$

Case 2. Assume that x is Caputo $[(ii)-gH]$ differentiable on $[0,+\infty)$. Then from (3.4) we have in the same way

(3.14)

$$x(t) = E_{q,1}\left(At^q\right) \odot x_0 + t \odot E_{q,2}\left(At^q\right) \odot x_1$$

$$\ominus(-1)\int_0^t\int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta-s)^q\right) \odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) d\delta ds. \tag{3.15}$$

This completes the proof. □

4. Existence and Uniqueness of Mild Solutions

Definition 4.1. By a mild fuzzy solution in type 1 of the Cauchy problem (3.1), we mean a function $x \in \mathcal{C}([0,a],\mathcal{T})$ that satisfies (3.5). By a mild fuzzy solution in type 2 of the Cauchy problem (3.1), we mean a function $x \in \mathcal{C}([0,a],\mathcal{T})$ that satisfies (3.7).

The following hypotheses will be used in the next results.

(H1) For almost all $t \in [0,a]$, the function $f \in \mathcal{C}([0,a]\times\mathcal{T}\times\mathcal{T},\mathcal{T})$ is continuous and for each $z \in \mathcal{C}([0,a],\mathcal{T})$, the function $f(\cdot,z,{}_{gH}^{C}\mathcal{D}^{q-1}z) : [0,a] \to \mathcal{T}$ is strongly measurable.

(H2) There exist $q_2 \in [0,q)$, $B_r := \left\{x \in \mathcal{T} : d_\infty(x,\hat{0}) \le r\right\} \subset \mathcal{T}$, $r > 0$, and $\rho(\cdot) \in L^{\frac{1}{q_2}}([0,a],\mathbb{R}^+)$ such that for any $x,y \in \mathcal{C}([0,a],B_r)$ we have

(4.1)
$$d_\infty(f(t,x(t),{}_{gH}^{C}\mathcal{D}^{q-1}x(t)), f(t,y(t),{}_{gH}^{C}\mathcal{D}^{q-1}y(t))) \le \rho(t) d_\infty(x(t),y(t)), \quad t \in [0,a].$$

(H3) There exists a constant $q_1 \in [0,q)$ and $m \in L^{\frac{1}{q_1}}([0,a],\mathbb{R}^+)$ such that

$$d_\infty(f(t,z(t),{}_{gH}^{C}\mathcal{D}^{q-1}z(t)),\hat{0}) \le m(t), \tag{4.2}$$

for all $z \in \mathcal{C}([0,a],\mathcal{T})$ and for almost all $t \in [0,a]$.

(H4) $E_{q,n}\left(At^q\right)$ is a compact operator for every $t > 0$ and $n \in \mathbb{N}$.

Theorem 4.1. *Under hypotheses* (H_1)-(H_4) *the Cauchy problem* (3.1) *has a mild fuzzy solution in type* 1 *in space* $\mathcal{C}([0,a],\mathcal{T})$.

Proof. Let $x \in \mathcal{C}([0,a],\mathcal{T})$. Since x is continuous with respect to t and hypothesis (H_1), $f(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s))$ is a measurable function on $[0,a]$. Let

$$b = \frac{q-1}{1-q_1}, \quad M_1 = \|m\|_{L^{\frac{1}{q_1}}[0,a]}. \tag{4.3}$$

For $t \in [0,a]$, by applying Holder's inequality and (H_3), we have

$$d_\infty\left(\int_0^t (t-s)^{q-1} \odot f(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s))ds, \hat{0}\right)$$

$$\leq \int_0^t (t-s)^{q-1} \odot d_\infty(f(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)),\hat{0})ds$$
$$\leq \left(\int_0^t (t-s)^{\frac{q-1}{1-q_1}} ds\right)^{1-q_1} \|m\|_{L^{\frac{1}{q_1}}}[0,t]$$
$$\leq \frac{M_1 a^{(1+b)(1-q_1)}}{(1+b)^{1-q_1}}.$$

Therefore, we obtain

$$d_\infty\left(\int_0^t\int_s^t \left(\frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta-s)^q\right) \odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)\right.$$
$$\leq \int_0^t\int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot d_\infty\left(E_{q,1}\left(A(\delta-s)^q\right) \odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right),\hat{0}\right)d\delta ds$$
$$\leq \frac{M}{\Gamma(q)} \odot \int_0^t (t-s)^{q-1} d_\infty(f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right),\hat{0})ds$$
$$\leq \frac{M_1 M a^{(1+b)(1-q_1)}}{\Gamma(q)(1+b)^{1-q_1}}, \quad \text{for all } t\in[0,a].$$

Then

$$\int\limits_0^t\int\limits_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta-s)^q\right) \odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds$$

is bounded for all $t\in[0,a]$.

For $x\in\mathcal{C}([0,a],\mathcal{T})$, we define

$$(F_1x)(t) = E_{q,1}(At^q)\odot x_0 + t\odot E_{q,2}(At^q)\odot x_1, \quad t\in[0,a],$$
$$(F_2x)(t) = \int_0^t\int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta-s)^q\right) \odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds.$$

Set

$$k_0 = M\left(\|x_0\| + a\|x_1\|\right) + \frac{M_1 M a^{(1+b)(1-q_1)}}{\Gamma(q)(1+b)^{1-q_1}}$$

and $\mathcal{B}_{k_0} := \left\{x(\cdot)\in C([0,a],\mathcal{T}) : d_\infty(x(t),\hat{0}) \leq k_0 \text{ for all } t\in[0,a]\right\}$. We will prove that F_1x+F_2x has a fixed point on $\mathcal{B}_{k_0}$.

Step 1. We show for every $x\in\mathcal{B}_{k_0}$, $F_1x+F_2x\in\mathcal{B}_{k_0}$. Indeed, with $0\leq t_1\leq t_2\leq a$ we have

$$d_\infty\left((F_2x)(t_2),(F_2x)(t_1)\right)$$
$$=d_\infty\left(\int_0^{t_2}\int_s^{t_2} \frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta-s)^q\right) \odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\right.$$
$$\left.\int_0^{t_1}\int_s^{t_1} \frac{(t_1-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta-s)^q\right) \odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\right)$$

$$
\begin{aligned}
=&d_\infty\Bigg(\int_{t_1}^{t_2}\int_s^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\\
&+\int_0^{t_1}\int_s^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\\
&-\int_0^{t_1}\int_s^{t_1}\frac{(t_1-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\Bigg)\\
=&d_\infty\Bigg(\int_{t_1}^{t_2}\int_s^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\\
&+\int_0^{t_1}\int_s^{t_1}\frac{[(t_2-\delta)^{q-2}-(t_1-\delta)^{q-2}]}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\\
&\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\\
&+\int_0^{t_1}\int_{t_1}^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\Bigg)\\
\leq&d_\infty\left(\int_{t_1}^{t_2}\int_s^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)\\
&+d_\infty\Bigg(\int_0^{t_1}\int_s^{t_1}\frac{[(t_2-\delta)^{q-2}-(t_1-\delta)^{q-2}]}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\\
&\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\Bigg)\\
&+d_\infty\left(\int_0^{t_1}\int_{t_1}^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)\\
=&I_1+I_2+I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1=&d_\infty\left(\int_{t_1}^{t_2}\int_s^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)\\
I_2=&d_\infty\Bigg(\int_0^{t_1}\int_s^{t_1}\frac{[(t_2-\delta)^{q-2}-(t_1-\delta)^{q-2}]}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\\
&\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\Bigg)\\
I_3=&d_\infty\left(\int_0^{t_1}\int_{t_1}^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right).
\end{aligned}
$$

We have:

$$
I_1=d_\infty\left(\int_{t_1}^{t_2}\int_s^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)
$$

$$\leq \frac{MM_1}{\Gamma(q)}\left(\int_{t_1}^{t_2}(t_2-s)^{1-q_1}\right)$$
$$\leq \frac{MM_1}{(1+b)^{1-q_1}\Gamma(q)}(t_2-t_1)^{(b+1)(1-q_1)},$$

also

$$\begin{aligned}
I_2 =& d_\infty\Bigg(\int_0^{t_1}\int_s^{t_1}\frac{[(t_2-\delta)^{q-2}-(t_1-\delta)^{q-2}]}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\\
&\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\Bigg)\\
\leq&\frac{M}{\Gamma(q)}\Bigg(\int_0^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}-(t_2-t_1)^{q-1}\right]\\
&\odot d_\infty\left(f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)ds,\hat{0}\right)\Bigg)\\
\leq&\frac{M}{\Gamma(q)}\left(\int_0^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]\odot d_\infty\left(f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)ds,\hat{0}\right)\right.\\
&\left.-\int_0^{t_1}(t_2-t_1)^{q-1}\odot d_\infty\left(f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)ds,\hat{0}\right)\right)\\
\leq&\frac{M}{\Gamma(q)}\left(\int_0^{t_1}\left[(t_2-s)^{b}-(t_1-s)^{b}\right]^{1-q_1}\odot M_1-(t_2-t_1)^{q-1}\odot t_1^{1-q_1}M_1\right)\\
\leq&\frac{MM_1}{(b+1)^{1-q_1}\Gamma(q)}\left(-(t_2-t_1)^{b+1}+t_2^{b+1}-t_1^{b+1}-(t_2-t_1)^{q-1}(b+1)^{1-q_1}\right).
\end{aligned}$$

Likewise

$$\begin{aligned}
I_3 &= d_\infty\left(\int_0^{t_1}\int_{t_1}^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)\\
&\leq \frac{M}{\Gamma(q)}\left(\int_0^{t_1}(t_2-t_1)^{q-1}\odot d_\infty\left(f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)ds,\hat{0}\right)\right)\\
&\leq \frac{MM_1(t_2-t_1)^{q-1}}{\Gamma(q)}t_1^{1-q_1}.
\end{aligned}$$

Then it is straightforward that I_1, I_2 and I_3 tend to 0 as $t_2-t_1\to 0$. So, $(F_2x)(t)$ is continuous in $t\in[0,a]$. It is easy to see that $(F_1x)(t)$ is also continuous in $t\in[0,a]$. Now, for any $x\in\mathcal{B}_{k_0}$ and $t\in[0,a]$, we have

$$d_\infty\left((F_1x)(t)+(F_2x)(t),\hat{0}\right)\leq M\left(\|x_0\|+a\|x_1\|\right)+\frac{M_1Ma^{(1+b)(1-q_1)}}{\Gamma(q)(1+b)^{1-q_1}}\leq k_0. \tag{4.4}$$

Then F_1+F_2 is an operator from $\mathcal{B}_{k_0}$ into $\mathcal{B}_{k_0}$.

Step 2. We prove that F_2 is a fully continuous operator that can be decomposed into several small steps.

First, we show that F_2 is continuous in $\mathcal{B}_{k_0}$.

Let $\{x_n\} \subseteq \mathcal{B}_{k_0}$ with $x_n \to x$ on $\mathcal{B}_{k_0}$. Applying hypothesis (H_2), we get

$$f\left(s, x_n(s), {}_{gH}^{C}\mathcal{D}^{q-1}x_n(s)\right) \to f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right), \quad \text{as } n \to +\infty, \tag{4.5}$$

almost everywhere $t \in [0, a]$.

From the hypothesis (H_3),

$$d_\infty\left(f\left(s, x_n(s), {}_{gH}^{C}\mathcal{D}^{q-1}x_n(s)\right), f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)\right) \leq 2m(s).$$

Therefore, by the domination convergence theorem, we get

$$\begin{aligned}
&d_\infty\left(\left(F_2 x_n\right)(t), \left(F_2 x\right)(t)\right)\\
\leq &\int_0^t \int_s^t \frac{M(t-\delta)^{q-2}}{\Gamma(q-1)} \odot d_\infty\left(f\left(s, x_n(s), {}_{gH}^{C}\mathcal{D}^{q-1}x_n(s)\right), f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)\right) d\delta ds\\
\leq &\int_0^t (t-s)^{q-1} \frac{qM}{\Gamma(1+q)} \odot d_\infty\left(f\left(s, x_n(s), {}_{gH}^{C}\mathcal{D}^{q-1}x_n(s)\right), f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)\right) ds\\
\to &0,
\end{aligned}$$

when $n \to +\infty$, This means F_2 is continuous.

Next, we show that $F_2\left(\mathcal{B}_{k_0}\right)$ is relatively compact. This is the family of functions $\{F_2 x : x \in \mathcal{B}_{k_0}\}$ and $\{\left(F_2 x\right)(t)\}$ relative compactness: $x \in \mathcal{B}_{k_0}$, where $t \in [0, a]$.

We proved this for all $x \in \mathcal{B}_{k_0}$ and $0 \leq t_1 \leq t_2 \leq a$

$$d_\infty\left(\left(F_2 x\right)(t_2), \left(F_2 x\right)(t_1)\right) \leq I_1 + I_2 + I_3.$$

We now have

$$\begin{aligned}
I_1 &\leq \frac{MM_1}{(1+b)^{1-q_1}\Gamma(q)}(t_2 - t_1)^{(b+1)(1-q_1)},\\
I_2 &\leq \frac{MM_1}{(b+1)^{1-q_1}\Gamma(q)}\left(-(t_2-t_1)^{b+1} + t_2^{b+1} - t_1^{b+1} - (t_2-t_1)^{q-1}(b+1)^{1-q_1}\right),\\
I_3 &\leq \frac{MM_1(t_2-t_1)^{q-1}}{\Gamma(q)} t_1^{1-q_1}.
\end{aligned}$$

From **Step 1**, it is easy to see that $F_2\left(\mathcal{B}_{k_0}\right)$ is equicontinuous.

Proving this is enough for each $t \in [0, a]$, $V(t) = \{\left(F_2 x\right)(t) : x \in \mathcal{B}_{k_0}\}$ is relatively compact. For any fixed $0 < t \leq a$, for all $\epsilon \in (0, t)$ and for all $\delta > 0$, let the operator $F_{\epsilon,\delta}$ be define as

$$\begin{aligned}
&\left(F_{\epsilon,\delta} x\right)(t)\\
=&\int_0^{t-\epsilon} \int_{s+\eta}^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta - s)^q\right) \odot f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) d\delta ds\\
=&\int_0^{t-\epsilon} \int_{s+\eta}^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta - s)^q - A(\eta - \epsilon) + A(\eta - \epsilon)\right)\\
&\odot f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) d\delta ds
\end{aligned}$$

$$=E_{q,1}\left(A(\eta-\epsilon)\right)\int_0^{t-\epsilon}\int_{s+\eta}^{t}\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q-A(\eta-\epsilon)\right)$$
$$\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,$$

where $x\in\mathcal{B}_{k_0}$. From hypothesis $(H4)$, $E_{q,1}\left(A(\eta-\epsilon)\right)$ is the compact operator, then $V_{\epsilon,\delta}(t)=\{(F_{\epsilon,\delta}x)(t):x\in\mathcal{B}_{k_0}\}$ is relatively compact. Moreover, for all $x\in\mathcal{B}_{k_0}$, and we have

$$d_\infty\left((F_2x)(t),(F_{\epsilon,\delta}x)(t)\right)$$
$$=d_\infty\Bigg(\int_0^t\int_s^t\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds$$
$$-\int_0^{t-\epsilon}\int_{s+\eta}^t\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\Bigg)$$
$$=d_\infty\Bigg(\int_0^t\int_s^{s+\eta}\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds$$
$$+\int_0^t\int_{s+\eta}^t\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds$$
$$-\int_0^{t-\epsilon}\int_{s+\eta}^t\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\Bigg)$$
$$\le d_\infty\left(\int_0^t\int_s^{s+\eta}\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)$$
$$+d_\infty\left(\int_{t-\epsilon}^t\int_{s+\eta}^t\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)$$
$$\le\frac{M_1M}{(b+1)\Gamma(q)}\left[\left((-\eta)^{b+1}-(a-\eta)^{b+1}+a^{b+1}\right)^{1-q_1}\left(-(-\eta)^{b+1}+(-\epsilon-\eta)^{b+1}\right)^{1-q_1}\right]$$
$$\to 0,$$

when $\eta,\epsilon\to 0$. Then we have a relatively compact set arbitrarily close to $V(t)$, $t>0$, which means that $V(t)$, $t>0$, is also relatively compact.

Applying the Ascoli-Arzela theorem shows that $F_2\left(\mathcal{B}_{k_0}\right)$ is relatively compact. Since F_2 is continuous and $F_2\left(\mathcal{B}_{k_0}\right)$ is relatively compact, F_2 is a fully continuous operator.

According to Schauder's fixed point theorem, F_1+F_2 has a fixed point at $\mathcal{B}_{k_0}$. So the nonlocal Cauchy problem (3.1) has a mild fuzzy solution of type 1. □

Set

$$\hat{F}[x](t)=E_{q,1}\left(At^q\right)\odot x_0+tE_{q,2}\left(At^q\right)\odot x_1$$
$$\ominus(-1)\odot\int_0^t\int_s^t\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)$$
$$\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds$$

and

$$\hat{C}([0,a],\mathcal{T}) = \left\{x \in \mathcal{C}([0,a],\mathcal{T}) : \hat{F}[x](t) \text{ exists for all } t \in [0,a]\right\}. \tag{4.6}$$

The following results show that there exists a mild fuzzy solution for type 2 in the space $\mathcal{C}([0,a],\mathcal{T})$.

Theorem 4.2. *The hypothesises* $(H1)$*-*$(H4)$ *are true and*

(Q2) $\hat{C}([0,a],\mathcal{T}) \neq \emptyset$*;*

(Q3) *if* $x \in \hat{C}([0,a],\mathcal{T})$*, hence* $\hat{F}[x] \in \hat{C}([0,a],\mathcal{T})$*.*

In this case the Cauchy problem (3.1) *has a mild fuzzy solution of type* 2 *in space* $\mathcal{C}([0,a],\mathcal{T})$*.*

Proof. For $x \in \hat{C}([0,a],\mathcal{T})$, $\hat{F}[x](t) = (F_1x)(t) \ominus (-1) \odot (F_2x)(t)$.

Set

$$k_0 = M\left(\|x_0\| + a\|x_1\|\right) + \frac{M_1 M a^{(1+b)(1-q_1)}}{\Gamma(q)(1+b)^{1-q_1}}.$$

Using a similar method as before with the Caputo $[(i)-\text{gH}]$ derivative, we get: $F_1x \ominus (-1) \odot F_2y \in \mathcal{B}_{k_0}$ for any pair $x, y \in \mathcal{B}_{k_0} \subset \hat{C}([0,a],\mathcal{T})$, where $(F_1x)(t)$ and $(F_2x)(t)$ are continuous in $t \in [0,a]$.

Now for any $x, y \in \mathcal{B}_{k_0}$, we have

$$\begin{aligned} d_\infty\left((F_1x)(t) \ominus (-1) \odot (F_2x)(t), \hat{0}\right) &\leq d_\infty\left((F_1x)(t), \hat{0}\right) + d_\infty\left((F_2x)(t), \hat{0}\right) \\ &\leq M\left(\|x_0\| + a\|x_1\|\right) + \frac{M_1 M a^{(1+b)(1-q_1)}}{\Gamma(q)(1+b)^{1-q_1}} \\ &= k_0, \end{aligned}$$

which means that $F_1 \ominus (-1) \odot F_2$ is an operator from $\mathcal{B}_{k_0}$ into $\mathcal{B}_{k_0}$.

Since F_2 is a fully continuous operator, according to the Schauder fixed point theorem $F_1 \ominus (-1) \odot F_2$ has a fixed point on $\mathcal{B}_{k_0}$, this means that the Cauchy problem (3.1) has a mild fuzzy solution of type 2. □

5. An Example

Consider the following equations

(5.1)

$$\begin{cases} {}^C D_{0^+}^{\frac{3}{2}} u(t,x) = \dfrac{\partial}{\partial t} u(t,\ x) + \dfrac{e^{-t}}{9+e^t}\left(\dfrac{|u(t,\ x)|}{1+|u(t,\ x)|}\right), & (t,x) \in]0,1[\times]0,1[, \\ u(t,0) = u(t,1) = 0, \quad t \in]0,1[, \\ u(0,x) = \psi(x), \quad x \in]0,1[, \\ {}^C D_{0^+}^{\frac{1}{2}} u(0,x) = \phi(x), \quad x \in]0,1[. \end{cases}$$

We choose $\mathbb{X} = \mathrm{C}([0,\ 1] \times \mathcal{T}, \mathcal{T})$ and we do not forget the operator $\mathrm{A} : D(A) \subset \mathbb{X} \to \mathbb{X}$ defined by

$$D(A) = \left\{u \in \mathbb{X} : \frac{\partial}{\partial t} u \in \mathbb{X} \text{ and } u(0,0) = u(0,1) = 0\right\}, \quad Au = \frac{\partial}{\partial t} u.$$

Then, we get

$$\overline{D(A)} = \{u \in \mathbb{X} : u(t,0) = u(t,1) = 0\}. \tag{5.2}$$

This implies that A satisfies (H4).

As is well known that A generates a compact C_0-semigroup $E_{q,n}(At^q)$ on $\overline{D(A)}$. Let's pose $X(t) = u(t,\cdot)$, that is $X(t)(x) = u(t,x)$, for all $(t,x) \in]0,1[\times]0,1[$.

In this example, we have the function $f :]0,1[\times\mathcal{T} \longrightarrow \mathcal{T}$ is given by

$$f(t,X(t)) = \frac{e^{-t}}{9+e^t}\left(\frac{|X(t)|}{1+|X(t)|}\right).$$

It is clear that for all $X, Y \in \mathcal{C}([0,\ 1], B_r)$ we have

$$d_\infty\left(f(t,X(t)), f(t,Y(t))\right) \leq \rho(t) d_\infty(X(t),Y(t)), \quad \text{with } \rho(t) = \frac{e^{-t}}{9+e^t} \in L^1,$$

for all $t \in]0,1[$, and that

$$d_\infty\left(f(t,X(t)),\tilde{0}\right) \leq m(t), \quad \text{with } m(t) = \frac{1}{9+e^t} \in L^1, \text{ for all } t \in]0,1[.$$

Moreover, f is continuos, therefore it is strongly measurable. Hence, according to Theorem 4.1 and Definition 4.1, problem (5.1) admits two types of solutions expressed as follow

$$\begin{aligned} X(t) =& E_{q,1}\left(\frac{\partial}{\partial t}t^q\right) \odot x_0 + t \odot E_{q,2}\left(\frac{\partial}{\partial t}t^q\right) \odot x_1 \\ &+ \int_0^t \int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(\frac{\partial}{\partial s}(\delta-s)^q\right) \odot \frac{e^{-s}}{9+e^s}\left(\frac{|X(s)|}{1+|X(s)|}\right) d\delta ds \end{aligned}$$

and

$$\begin{aligned} X(t) =& E_{q,1}\left(\frac{\partial}{\partial t}t^q\right) \odot x_0 + t \odot E_{q,2}\left(\frac{\partial}{\partial t}t^q\right) \odot x_1 \\ &\ominus (-1) \int_0^t \int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(\frac{\partial}{\partial s}(\delta-s)^q\right) \odot \frac{e^{-s}}{9+e^s}\left(\frac{|X(s)|}{1+|X(s)|}\right) d\delta ds. \end{aligned}$$

6. Conclusion

In this work, the nonlocal Cauchy problem of fuzzy evolutionary equations in arbitrary Banach spaces of order $q \in (1,2)$ is discussed, and the existence and uniqueness criteria for mild fuzzy solutions are determined utilizing the Schauder fixed point theorem. An example illustrating the application of the main results is also provided.

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