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ON THE ENUMERATION OF THE SET OF ELEMENTARY NUMERICAL SEMIGROUPS WITH FIXED MULTIPLICITY, FROBENIUS NUMBER OR GENUS

J. C. ROSALES[1] AND M. B. BRANCO[2]

Abstract. In this paper we give algorithms that allow to compute the set of every elementary numerical semigroups with given genus, Frobenius number and multiplicity. As a consequence we obtain formulas for the cardinality of these sets.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. A numerical semigroup is a subset S of $\mathbb{N}$ which is closed under addition, $0 \in S$ and $\mathbb{N}\backslash S$ has finitely many elements. The cardinality of the set $\mathbb{N}\backslash S$ is called the genus of S and it is denoted by $\mathrm{g}(S)$.

Given a positive integer g, we denote by $\mathcal{S}(g)$ the set of all numerical semigroups with genus g. The problem of determining the cardinality of $\mathcal{S}(g)$ has been widely treated in the literature (see for example [2, 4–7] and [13]). Some of these works were motivated by Amorós's conjecture [5], which says that the sequence of cardinals of $\mathcal{S}(g)$ for $g = 1, 2, \ldots$ has a Fibonacci behavior. It is still not known in general if for a fixed positive integer g there are more numerical semigroups with genus $g+1$ than numerical semigroups with genus g.

An algorithm that allows us to compute the set of numerical semigroups with genus g is provided in [3], where elementary numerical semigroups play an important role. In fact, in [3] an equivalence binary relation R is defined over $\mathcal{S}(g)$ such that $\frac{\mathcal{S}(g)}{R} = \{[S] \mid S \text{ is a elementary numerical semigroup with genus } g\}$. Moreover, it is proved that if S and T are elementary numerical semigroups with genus g then $[S] = [T]$ if and only if $S = T$. The main idea of the algorithm in [3] is to compute

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every elementary numerical semigroups S with genus g and, then, to enumerate the elements in $[S]$ for each S.

For any numerical semigroup S, the smallest positive integer belonging to S (respectively, the greatest that does not belong to S) is called the multiplicity (respectively Frobenius number) of S and it is denoted by $\mathrm{m}(S)$ (respectively $\mathrm{F}(S)$) (see [9]).

We say that a numerical semigroup S is elementary if $\mathrm{F}(S) < 2\mathrm{m}(S)$. This type of numerical semigroups were also studied in [8] and [13]. We denote by $\mathcal{E}(m, F, g)$ the set of elementary numerical semigroups with multiplicity m, Frobenius number F and genus g (when one of the parameters to $\mathcal{E}(m, F, g)$ is replaced by the symbol $-$, it represents the set of elementary numerical semigroups in which no restrictions are placed on that parameter).

For any finite set A, $\#A$ denotes the cardinal of A. Given a rational number q we denote by $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid q \leq z\}$.

In Section 2, we review the results of Y. Zhao in [13] which give formulas for $\#\mathcal{E}(m, -, -)$, $\#\mathcal{E}(m, -, g)$ and $\#\mathcal{E}(-, -, g)$, and state that $\#\mathcal{E}(-, -, g+1) = \#\mathcal{E}(-, -, g) + \#\mathcal{E}(-, -, g-1)$. Therefore, we get that $\{\#\mathcal{E}(-, -, g)\}_{g \in \mathbb{N}}$ is a Fibonacci sequence.

In Section 3, we study the sets $\mathcal{E}(m, F, -)$ and $\mathcal{E}(-, F, -)$, find formulas for their cardinality, and describe the behavior of the sequence of cardinals of $\mathcal{E}(-, F, -)$.

In Section 4, we present algorithms for calculating $\mathcal{E}(-, F, g)$ and $\mathcal{E}(m, F, g)$. From these algorithms, we can derive the cardinality of these sets.

Finally, in Section 5 we show that the set of all elementary numerical semigroups $\mathcal{E}$ is a Frobenius variety. This fact, together with the results of [11], allows us to construct recursively the set $\mathcal{E}$.

2. Multiplicity and Genus

Our aim in this section is to see that $\{\#\mathcal{E}(-, -, g)\}_{g \in \mathbb{N}}$ is a Fibonacci sequence. The next result is easy to prove and appears in [13, Proposition 2.1].

Lemma 2.1. *Let m be an integer such that $m \geq 2$ and let A be a subset of $\{m+1, \ldots, 2m-1\}$. Then $\{0, m\} \cup A \cup \{2m, \rightarrow\}$ is an elementary numerical semigroup with multiplicity m. Moreover, every elementary numerical semigroup with multiplicity m is of this form.*

As consequence of the above lemma we have that $\#\mathcal{E}(m, -, -)$ is equal to the number of subsets of a set with $m-1$ elements.

Corollary 2.1. *If m is a positive integer, then $\#\mathcal{E}(m, -, -) = 2^{m-1}$.*

The following result is easy to prove and gives conditions imposed on two positive integers m and g so that there exists at least one elementary numerical semigroup with multiplicity m and genus g.

Proposition 2.1. *Let m and g be nonnegative integers with $m \neq 0$. Then $\mathcal{E}(m, -, g) \neq \emptyset$ if and only if $m - 1 \leq g \leq 2(m-1)$.*

From Lemma 2.1, we know that $S \in \mathcal{E}(m,-,g)$ if and only if $S = \{0,m\} \cup A \cup \{2m,\rightarrow\}$, where A is a subset of $\{m+1,\dots,2m-1\}$ and $\#A = 2(m-1)-g$. So we have the following result, which is also in [13, Corollary 2.2].

Corollary 2.2. *Let m and g be positive integers such that $m-1 \le g \le 2(m-1)$. Then $\#\mathcal{E}(m,-,g) = \binom{m-1}{g-(m-1)}$.*

From the results above we get

$$\mathcal{E}(-,-,g) = \bigcup_{m=\lceil \frac{g}{2} \rceil+1}^{g+1} \mathcal{E}(m,-,g).$$

Thus we have the following algorithm.

Algorithm 2.1. Input: g positive integer. Output: $\mathcal{E}(-,-,g)$.

1) For all $m \in \left\{\lceil \frac{g}{2} \rceil + 1,\dots,g+1\right\}$ compute the set $\mathcal{E}(m,-,g)$.
2) Return $\bigcup_{m=\lceil \frac{g}{2} \rceil+1}^{g+1} \mathcal{E}(m,-,g)$.

Clearly, we get

$$\#\mathcal{E}(-,-,g) = \sum_{m=\lceil \frac{g}{2} \rceil+1}^{g+1} \#\mathcal{E}(m,-,g).$$

By applying Corollary 2.2, we obtain the following result.

Corollary 2.3. *If g is a positive integer, then $\#\mathcal{E}(-,-,g) = \sum_{i=\lceil \frac{g}{2} \rceil}^{g} \binom{i}{g-i}$.*

The Fibonacci sequence is the sequence of positive integers defined by the linear recurrence equation $a_{n+1} = a_n + a_{n-1}$, with $a_0 = a_1 = 1$.

It is clear that $\mathcal{E}(-,-,0) = \{\mathbb{N}\}$ and $\mathcal{E}(-,-,1) = \{\{0,2,\rightarrow\}\}$ and so $\#\mathcal{E}(-,-,0) = \#\mathcal{E}(-,-,1) = 1$. By using Corollary 2.3, we can obtain [13, Proposition 2.3], which states that $\{\#\mathcal{E}(-,-,g)\}_{g\in\mathbb{N}}$ is a Fibonacci sequence.

Theorem 2.1. *If g is a positive integer, then $\#\mathcal{E}(-,-,g+1) = \#\mathcal{E}(-,-,g) + \#\mathcal{E}(-,-,g-1)$.*

3. Multiplicity and Frobenius Number

Our first goal in this section is to describe sufficient conditions for two positive integers m and F so that there exists at least one elementary numerical semigroups with multiplicity m and Frobenius number F.

Lemma 3.1. *If S is an elementary numerical semigroup such that $S \neq \mathbb{N}$, then $\frac{\mathrm{F}(S)+1}{2} \le \mathrm{m}(S) \le \mathrm{F}(S)+1$ and $\mathrm{m}(S) \neq \mathrm{F}(S)$.*

Proof. Since $S \neq \mathbb{N}$, then $\mathrm{m}(S) \ge 2$ and $\mathrm{m}(S)-1 \notin S$. Therefore, we have that $\mathrm{m}(S)-1 \le \mathrm{F}(S)$. In addition, as S is an elementary numerical semigroup then $\mathrm{F}(S) < 2(\mathrm{m}(S))$ and thus $\mathrm{F}(S)+1 \le 2(\mathrm{m}(S))$. $\square$

From the previous lemma we obtain the following result.

Proposition 3.1. *Let m and F be positive integers. Then $\mathcal{E}(m, F, -) \neq \emptyset$ if and only if $\frac{F+1}{2} \leq m \leq F+1$ and $m \neq F$.*

It is clear that $\mathcal{E}(F+1, F, -) = \{\{0, F+1, \rightarrow\}\}$ and $\mathcal{E}(F-1, F, -) = \{\{0, F-1, F+1, \rightarrow\}\}$. Hence, we can assume that $F = m + i$, where $i \in \{2, \ldots, m-1\}$. By applying Lemma 2.1, we deduce that $S \in \mathcal{E}(m, F, -)$ if and only if there exists $A \subseteq \{m+1, \ldots, m+i-1\}$ such that $S = \{0, m\} \cup A \cup \{F+1, \rightarrow\}$. As a consequence we have the following algorithm.

Algorithm 3.1. Input: m and F positive integers such that $\frac{F+1}{2} \leq m \leq F+1$ and $m \neq F$.

Output: $\mathcal{E}(m, F, -)$.

1) If $m = F+1$, then return $\{\{0, F+1, \rightarrow\}\}$.
2) If $m = F-1$, then return $\{\{0, F-1, F+1, \rightarrow\}\}$.
3) Compute the set $C = \{A \mid A \subseteq \{m+1, \ldots, F-1\}\}$.
4) Return $\{\{0, m\} \cup A \cup \{F+1, \rightarrow\} \mid A \in C\}$.

Gathering all this information, we obtain the following result which can also be deduced from equation (6) of [1].

Corollary 3.1. *Let m and F be positive integers such that $\frac{F+1}{2} \leq m \leq F+1$ and $m \neq F$. Then*

$$\#\mathcal{E}(m, F, -) = \begin{cases} 1, & \text{if } m = F+1, \\ 2^{F-m-1}, & \text{otherwise.} \end{cases}$$

Next we obtain an algorithm that allows us to compute every elementary numerical semigroup with a given Frobenius number. As a consequence of Proposition 3.1, we have

$$\mathcal{E}(-, F, -) = \bigcup_{m \in \left\{\lceil \frac{F+1}{2} \rceil, \ldots, F+1\right\} \setminus \{F\}} \mathcal{E}(m, F, -).$$

Algorithm 3.2. Input: F positive integer.

Output: $\mathcal{E}(-, F, -)$.

1) For all $m \in \left\{\lceil \frac{F+1}{2} \rceil, \ldots, F+1\right\} \setminus \{F\}$ compute (using Algorithm 3.1) the set $\mathcal{E}(m, F, -)$.
2) Return $\mathcal{E}(-, F, -) = \bigcup_{m \in \left\{\lceil \frac{F+1}{2} \rceil, \ldots, F+1\right\} \setminus \{F\}} \mathcal{E}(m, F, -)$.

Therefore, we have $\#\mathcal{E}(-, F, -) = \sum_{m \in \left\{\lceil \frac{F+1}{2} \rceil, \ldots, F+1\right\} \setminus \{F\}} \#\mathcal{E}(m, F, -)$. From Corollary 3.1 we obtain the following result.

Corollary 3.2. *If F is a positive integer, then $\#\mathcal{E}(-, F-) = 2^{F - \lceil \frac{F+1}{2} \rceil}$.*

We finish this section by describing the behavior of the sequence of cardinalities of $\mathcal{E}(-, F, -)$ for $F = 1, 2, \ldots$ Observe that $\#\mathcal{E}(-, 1, -) = \#\mathcal{E}(-, 2, -) = 1$.

Proposition 3.2. *Let F be an integer greater than or equal to 2.*

1) *If F is odd, then $\#\mathcal{E}(-, F+1, -) = \#\mathcal{E}(-, F, -)$.*
2) *If F is even, then $\#\mathcal{E}(-, F+1, -) = \#\mathcal{E}(-, F, -) + \#\mathcal{E}(-, F-1, -)$.*

Proof. 1) From Corollary 3.2 it is guaranteed that $\#\mathcal{E}(-, F, -) = 2^{F-\lceil \frac{F+1}{2} \rceil} = 2^{F-\frac{F+1}{2}} = 2^{\frac{F-1}{2}}$. By repeating this argument we obtain $\#\mathcal{E}(-, F+1, -) = 2^{\frac{F-1}{2}}$. Therefore, we have $\#\mathcal{E}(-, F+1, -) = \#\mathcal{E}(-, F, -)$.

2) Again, by Corollary 3.2, we know that $\#\mathcal{E}(-, F, -) + \#\mathcal{E}(-, F-1, -) = 2^{F-\lceil \frac{F+1}{2} \rceil} + 2^{F-1-\lceil \frac{F}{2} \rceil} = 2^{F-\frac{F+2}{2}} + 2^{F-1-\frac{F}{2}} = 2^{\frac{F}{2}}$. We obtain $\#\mathcal{E}(-, F+1, -) = 2^{F+1-\lceil \frac{F+2}{2} \rceil} = 2^{F+1-\frac{F+2}{2}} = 2^{\frac{F}{2}}$. Consequently, $\#\mathcal{E}(-, F+1, -) = \#\mathcal{E}(-, F, -) + \#\mathcal{E}(-, F-1, -)$ □

4. Multiplicity, Frobenius Number and Genus

In this section, we aim to find conditions for m, F and g positive integers so that there exists at least one elementary numerical semigroup with a given multiplicity m, Frobenius number F and genus g. The next results are a consequence of the results given in [3, Proposition 2 and Corollary 3].

Lemma 4.1. *Let F and g be two positive integers. Then $g \leq F \leq 2g-1$ if and only if $\mathcal{E}(-, F, g) \neq \emptyset$.*

Lemma 4.2. *Let F and g be two positive integers such that $g \leq F \leq 2g-1$, and let $\mathcal{A}_{F,g} = \left\{ A \mid A \subseteq \left\{ \left\lceil \frac{F+1}{2} \right\rceil, \ldots, F-1 \right\} \text{ and } \#A = F-g \right\}$. Then $\mathcal{E}(-, F, g) = \{\{0\} \cup A \cup \{F+1 \rightarrow\} \mid A \in \mathcal{A}_{F,g}\}$.*

As an immediate consequence of Lemmas 4.1 and 4.2 we have the following algorithm.

Algorithm 4.1. Input: F and g positive integers such that $g \leq F \leq 2g-1$.

Output: $\mathcal{E}(-, F, g)$.

1) Compute the set $C = \left\{ A \mid A \subseteq \left\{ \left\lceil \frac{F+1}{2} \right\rceil, \ldots, \ldots, F-1 \right\} \text{ and } \#A = F-g \right\}$.
2) Return $\{\{0\} \cup A \cup \{F+1, \rightarrow\} \mid A \in C\}$.

As a consequence of the previous algorithm we obtain the following result which also appears in [3, Corollary 4].

Corollary 4.1. *If F and g are positive integers such that $g \leq F \leq 2g-1$, then $\#\mathcal{E}(-, F, g) = \binom{\lceil \frac{F}{2} \rceil - 1}{F-g}$.*

Lemma 4.3. *If m, F and g are three positive integers such that $m \geq 2$ and $\mathcal{E}(m, F, g) \neq \emptyset$, then $m-1 \leq g \leq F < 2m$.*

Proof. Since $\mathcal{E}(m, F, g) \neq \emptyset$, then $\mathcal{E}(m, -, g) \neq \emptyset$ and we have that $m-1 \leq g$. From Lemma 4.1, we deduce that $g \leq F$. Finally, by Proposition 3.1, we conclude that $\frac{F+1}{2} \leq m$ and thus $F < 2m$. □

Finally, we present the main result of this section.

Proposition 4.1. *Let m, F and g be three positive integers such that $m \geq 2$. Then $\mathcal{E}(m, F, g) \neq \emptyset$ if and only if one of the following conditions holds:*

1) $(m, F, g) = (m, m-1, m-1)$;
2) $(m, F, g) = (m, F, m)$ *and* $m < F < 2m$;
3) $m < g < F < 2m$.

Proof. Necessity. If $\mathcal{E}(m, F, g) \neq \emptyset$ then by applying Lemma 4.3, we deduce that $m-1 \leq g \leq F < 2m$. Assume that $S \in \mathcal{E}(m, F, g)$. We distinguish the following four cases.

a) If $g = m-1$, then $S = \{0, m, \rightarrow\}$ and so $F = m-1$. Hence, $(m, F, g) = (m, m-1, m-1)$.
b) If $g = m$, then $m < F < 2m$ and $S = \{0, m, \rightarrow\} \setminus \{F\}$. Whence, $(m, F, g) = (m, F, m)$ and $m < F < 2m$.
c) If $g = F$, then $S = \{0, F+1, \rightarrow\}$ and thus $F + 1 = m$. Once again we have $(m, F, g) = (m, m-1, m-1)$.
d) If $g \notin \{m-1, m, F\}$, then as $m - 1 \leq g \leq F < 2m$ and we deduce that $m < g < F < 2m$.

Sufficiency. It is clear that $\{0, m, \rightarrow\} \in \mathcal{E}(m, m-1, m-1)$ and $\{0, m, \rightarrow\} \setminus \{F\} \in \mathcal{E}(m, F, m)$. Suppose that $m < g < F < 2m$. Let A be a subset of $\{m+1, \ldots, F-1\}$, with cardinality $F - g - 1$. Since $\mathrm{g}(S) = m - 1 + F - 1 - m - 1 + 1 - \#A + 1 = F - 1 - F + g + 1 = g$, then $S = \{0, m\} \cup A \cup \{F+1, \rightarrow\} \in \mathcal{E}(m, F, g)$. $\square$

Notice that, by the sufficiency condition of the proof above, we conclude that, if $m < g < F < 2m$, knowing an element in $\mathcal{E}(m, F, g)$ is the same as knowing a subset of $\{m+1, \ldots, F-1\}$ with cardinality $F - g - 1$. So we have the following algorithm.

Algorithm 4.2. Input: m, F and g integers such that $2 \leq m < g < F < 2m$.
Output: $\mathcal{E}(m, F, g)$.

1) Compute $C = \{A \mid A \subseteq \{m+1, \ldots, F-1\}$ and $\#A = F - g - 1\}$.
2) Return $\{\{0, m\} \cup A \cup \{F + 1 \rightarrow\}$ such that $A \in C\}$.

Clearly $\#\mathcal{E}(m, m-1, m-1) = \#\mathcal{E}(m, F, m) = 1$. For the remaining cases the following result gives us the cardinality of $\mathcal{E}(m, F, g)$.

Corollary 4.2. *Let m, F and g be positive integers such that $2 \leq m < g < F \leq 2m$. Then $\#\mathcal{E}(m, F, g) = \binom{F-m-1}{F-g-1}$.*

Proof. As a consequence of Algorithm 4.2 we have that $S \in \mathcal{E}(m, F, g)$ if and only if there exists $A \subseteq \{m+1, \ldots, F-1\}$, with cardinality $F - g - 1$ such that $S = \{0, m\} \cup A \cup \{F+1, \rightarrow\}$. $\square$

We conclude this section by giving an example that illustrates the previous results.

Example 4.1. Let us compute $\mathcal{E}(4,7,5)$. By Corollary 4.2 we have $\#\mathcal{E}(4,7,5) = \binom{7-4-1}{7-5-1} = \binom{2}{1} = 2$. Now by using Algorithm 4.2, with $m = 4$, $F = 7$ and $g = 5$ we can conclude that $C = \{\{5\},\{6\}\}$ and $\mathcal{E}(4,7,5) = \{\{0,4\} \cup \{5\} \cup \{8,\rightarrow\}, \{0,4\} \cup \{6\} \cup \{8,\rightarrow\}\}$.

5. Frobenius Variety

A Frobenius variety (see for example [11]) is a nonempty set V of numerical semigroups fulfilling the following conditions:

1) if S and T are in V, then $S \cap T \in V$;
2) if S is in V and $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\} \in V$.

Proposition 5.1. *$\mathcal{E} = \{S \mid S$ is an elementary numerical semigroup$\}$ is a Frobenius variety.*

Proof. If S and T belong to $\mathcal{E}$ it is clear that $S \cap T$ is a numerical semigroup,

$$\mathrm{F}(S \cap T) = \max\{\mathrm{F}(S), F(T)\}$$

and

$$\mathrm{m}(S \cap T) \geq \max\{\mathrm{m}(S), \mathrm{m}(T)\}.$$

Therefore, $\mathrm{F}(S \cap T) < 2\mathrm{m}(S \cap T)$ and thus $S \cap T \in \mathcal{E}$.

If S is an element in $\mathcal{E}$ and $S \neq \mathbb{N}$, then clearly $\overline{S} = S \cup \{\mathrm{F}(S)\}$ is a numerical semigroup such that $\mathrm{F}(\overline{S}) < \mathrm{F}(S)$ and $\mathrm{m}(\overline{S})$ is equal to $\mathrm{m}(S)$ or $\mathrm{F}(S)$. Therefore, $\mathrm{F}(\overline{S}) < 2\mathrm{m}(\overline{S})$ and thus $\overline{S} \in \mathcal{E}$. $\square$

We define a directed graph $G(\mathcal{E})$, with edges pointing from T to S, in the following way: the set of vertices is $\mathcal{E}$ and $(T,S) \in \mathcal{E} \times \mathcal{E}$ is an edge of $G(\mathcal{E})$ if and only if $S \cup \{\mathrm{F}(S)\} = T$.

The goal of this section is to see that $G(\mathcal{E})$ is a tree with root equal to $\mathbb{N}$ and to characterize the sons of a vertex. This fact allows us to recursively construct $G(\mathcal{E})$ and consequently $\mathcal{E}$. To this end we need to introduce some concepts and results.

Given a nonempty subset A of $\mathbb{N}$ we will denote by $\langle A \rangle$ the submonoid of $(\mathbb{N},+)$ generated by A, that is,

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}\backslash\{0\}, a_i \in A, \lambda_i \in \mathbb{N} \text{ for all } i \in \{1,\ldots,n\}\}.$$

It is well known (see for instance [12]) that every numerical semigroup S is finitely generated, and therefore there exists a finite subset A of $\mathbb{N}$ such that $S = \langle A \rangle$. Furthermore, we say that A is a minimal set of generators of S if no proper subset of A generates S. Every numerical semigroup admits an unique minimal set of generators of S and we denote this set by $\mathrm{msg}(S)$. It is well known (see for instance [12]) that $\mathrm{msg}(S) = (S\backslash\{0\}) \setminus (S\backslash\{0\} + S\backslash\{0\})$ and if $x \in S$ then $S\backslash\{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$.

As a consequence of [11, Proposition 24 and Theorem 27] we have the following result.

Theorem 5.1. *The graph $G(\mathcal{E})$ is a tree with root $\mathbb{N}$. Furthermore, the sons of a vertex S of $G(\mathcal{E})$ are in $\{S\backslash\{x\} \mid x \in \mathrm{msg}(S),\ x > \mathrm{F}(S) \text{ and } S\backslash\{x\} \in \mathcal{E}\}$.*

The following result is useful to compute the sons of a vertex of $G(\mathcal{E})$.

Proposition 5.2. *Let S be an elementary numerical semigroup and $x \in msg(S)$ such that $x > \mathrm{F}(S)$. Then $S\backslash\{x\}$ is an elementary numerical semigroup if and only if $x < 2\mathrm{m}(S)$.*

Proof. Suppose that $S = \{0, \mathrm{m}(S), \rightarrow\}$. Then

$$\mathrm{msg}(S) = \{\mathrm{m}(S), \mathrm{m}(S)+1, \ldots, 2\mathrm{m}(S)-1\}$$

and clearly the result is true. If $S \neq \{0, \mathrm{m}(S), \rightarrow\}$ then $\mathrm{m}(S\backslash\{x\}) = \mathrm{m}(S)$ and $\mathrm{F}(S\backslash\{x\}) = x$. Therefore, $S\backslash\{x\}$ is elementary numerical semigroup if and only if $x < 2\mathrm{m}(S)$. □

We illustrate the above results with the following example.

Example 5.1. Let us compute the sons of vertex $S = \{0, 5, 6, 9, \rightarrow\}$ of $G(\mathcal{E})$. We have $\mathrm{msg}(S) = \{5, 6, 9, 13\}$, $\mathrm{F}(S) = 8$ and $\mathrm{m}(S) = 5$. Whence $\{x \in \mathrm{msg}(S) \mid \mathrm{F}(S) < x < 2\mathrm{m}(S)\} = \{9\}$. Using Theorem 5.1 and Proposition 5.2 we conclude that S has an unique son $S\backslash\{9\} = \langle 5, 6, 13, 14\rangle$.

Now, we can recursively construct the tree $G(\mathcal{E})$ starting with $\mathbb{N}$ and connecting each vertex with their sons. First we construct $\mathrm{msg}(S\backslash\{x\})$ from $\mathrm{msg}(S)$, when x is a minimal generator of S greater than $\mathrm{F}(S)$. It is clear that if $\mathrm{msg}(S) = \{m, m+1, \ldots, 2m-1\}$ which is $S = \{0, m, \rightarrow\}$ then $\mathrm{msg}(S\backslash\{m\}) = \{m+1, m+2, \ldots, 2m+1\}$. For the remaining cases, we use the following result that appears in [10, Corollary 18].

Proposition 5.3. *Let S be a numerical semigroup with $\mathrm{msg}(S) = \{n_1, \ldots, n_p\}$. If $\mathrm{m}(S) = n_1 < n_p$ and $n_p > \mathrm{F}(S)$ then*

$$\mathrm{msg}(S\backslash\{n_p\}) = \begin{cases} \{n_1, \ldots, n_{p-1}\}, & \text{if exists } i \in \{2, \ldots, p-1\} \text{ such that} \\ & n_p + n_1 - n_i \in S, \\ \{n_1, \ldots, n_{p-1}, n_p + n_1\}, & \text{otherwise.} \end{cases}$$

Note that, in the previous proposition, the elements in $\mathrm{msg}(S)$ are not necessarily ordered.

Example 5.2. Let $S = \langle 5, 6, 9, 13\rangle$. Let us compute $\mathrm{msg}(S\backslash\{9\})$. By Proposition 5.3, as $9+5-6 \notin S$ and $9+5-13 \notin S$, we can conclude that $\{5, 6, 13, 14\}$ is the minimal system of generators of $S\backslash\{9\}$.

Using Theorem 5.1 and Proposition 5.2 and 5.3 we obtain the following:

. $\langle 1\rangle$ has only son $\langle 1\rangle\backslash\{1\} = \langle 2, 3\rangle$;
. $\langle 2, 3\rangle$ has two sons $\langle 2, 3\rangle\backslash\{2\} = \langle 3, 4, 5\rangle$ and $\langle 2, 3\rangle\backslash\{3\} = \langle 2, 5\rangle$;
. $\langle 2, 5\rangle$ has no sons;

$$\begin{tikzcd}
 & & & & & \langle 1\rangle = \mathbb{N} \arrow[d] & \\
 & & & & & \langle 2,3\rangle \arrow[dl] \arrow[dr] & \\
 & & & & \langle 3,4,5\rangle \arrow[dl] \arrow[d] \arrow[dr] & & \langle 2,5\rangle \\
 & & & \langle 4,5,6,7\rangle \arrow[dlll] \arrow[dll] \arrow[dl] \arrow[d] & \langle 3,5,7\rangle \arrow[d] & \langle 3,4\rangle & \\
\langle 5,6,7,8,9\rangle & \langle 4,6,7,9\rangle & \langle 4,5,7\rangle & \langle 4,5,6\rangle & \langle 3,7,8\rangle & \ldots &
\end{tikzcd}$$

. $\langle 3,4,5\rangle$ has three sons $\langle 3,4,5\rangle\backslash\{3\} = \langle 4,5,6,7\rangle$, $\langle 3,4,5\rangle\backslash\{4\} = \langle 3,5,7\rangle$ and $\langle 3,4,5\rangle\backslash\{5\} = \langle 3,4\rangle$;
. $\langle 3,4\rangle$ has no sons;
. $\langle 3,5,7\rangle$ has one son $\langle 3,5,7\rangle\backslash\{5\} = \langle 3,7,8\rangle$;
. $\langle 4,5,6,7\rangle$ has four sons $\langle 4,5,6,7\rangle\backslash\{4\} = \langle 5,6,7,8,9\rangle$, $\langle 4,5,6,7\rangle\backslash\{5\} = \langle 4,6,7,9\rangle$, $\langle 4,5,6,7\rangle\backslash\{6\} = \langle 4,5,7\rangle$ and $\langle 4,5,6,7\rangle\backslash\{7\} = \langle 4,5,6\rangle$;
. $\cdots$ $\cdots$ $\cdots$ $\cdots$

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[1]Departamento de Álgebra,
Universidad de Granada,
E-18071 Granada, Spain
Email address: `jrosales@ugr.es`

[2]Departamento de Matemática,
Universidade de Évora,
7000 Évora, Portugal
Email address: `mbb@uevora.pt`