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ON A NEW CLASS OF UNIFIED REDUCTION FORMULAS FOR SRIVASTAVA'S GENERAL TRIPLE HYPERGEOMETRIC FUNCTION $F^{(3)}[x,y,z]$

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Abstract. Very recently, by applying the so called Beta integral method to the well-known hypergeometric identities due to Bailey and Ramanujan, Choi et al. [*Reduction formula for Srivastava's triple hypergeometric series* $F^{(3)}[x,y,z]$, Kyungpook Math. J. **55** (2015), 439–447] have obtained three interesting reduction formulas for the Srivastava's triple hypergeometric series $F^{(3)}[x,y,z]$.

The aim of this paper is to provide three unified reduction formulas for the Srivastava's triple hypergeometric series $F^{(3)}[x,y,z]$ from which as many as reduction formulas desired (including those obtained by *Choi et al.*) can be deduced.

In the end, three unified relationships between Srivastava's triple hypergeometric series and Kampé de Fériet function have also been given.

1. Introduction

The generalized hypergeometric function ${}_pF_q$ with p numerator parameters $\alpha_1,\ldots,\alpha_p$ such that $\alpha_j\in\mathbb{C}$, $j=1,\ldots,p$, and q denominator parameters $\beta_1,\ldots,\beta_q$, $j=1,\ldots,q$, such that $\beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-$, $j=1,\ldots,q$, $\mathbb{Z}_0^-:=\mathbb{Z}\cup 0=\{0,-1,,-2,\cdots\}$, is defined by (see, for example [14, Chapter 4], see also [19, pp. 71–72]

$$\begin{aligned} {}_pF_q[\alpha_1,\,\ldots,\,\alpha_p;\,\beta_1,\,\ldots,\,\beta_q;\,z] &= {}_p\mathrm{F}_q\left[\begin{matrix}\alpha_1,\,\ldots,\,\alpha_p;\\ \beta_1,\,\ldots,\,\beta_q;\end{matrix}\;z\right]\\ &=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}, \end{aligned}\tag{1.1}$$

Key words and phrases. Generalized hypergeometric function ${}_pF_q$, gamma function, Pochhammer symbol, beta integral, Kampé de Fériet function, Srivastava's triple hypergeometric series $F^{(3)}[x,y,z]$.

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$p \leq q$ and $|z| < \infty$, $p = q+1$ and $|z| < 1$, $p = q+1$, $|z| = 1$ and $\mathrm{Re}(\omega) > 0$, where

$$\omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$

and $(\alpha)_n$ denotes the Pochhammer symbol defined in terms of Gamma functions by

$$(\alpha)_n := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} 1, & n = 0, \alpha \in \mathbb{C} \setminus \{0\}, \\ \alpha(\alpha+1)\cdots(\alpha+n-1), & n \in \mathbb{N}, \alpha \in \mathbb{C}. \end{cases} \tag{1.2}$$

It is to be noted here that whenever the hypergeometric function ${}_2F_1$ and the generalized hypergeometric function ${}_pF_q$ reduce to be expressed in terms of Gamma functions, the results are very important in view of applications as well as themselves. Thus the well known classical summation theorems [3] such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$, and those of Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$, and others have played important roles.

Moreover, it is well known that, if the product of two generalized hypergeometric series can be expressed as a generalized hypergeometric series with argument x, the coefficient of x^n in the product should be expressed in terms of gamma functions. Following this technique and using above mentioned classical summation theorems, in a well known, popular and very interesting paper [2], Bailey derived a large number of new as well as known results involving products of generalized hypergeometric series. Here, in our present investigation, we choose to recall some of those results in [2]:

$$e^{-x}\,{}_1\mathrm{F}_1(\alpha\,;\,\rho\,;x) = {}_1\mathrm{F}_1(\rho-\alpha\,;\,\rho\,;-x), \tag{1.3}$$

$${}_1\mathrm{F}_1(\alpha\,;\,2\alpha\,;x)\,{}_1\mathrm{F}_1(\beta\,;\,2\beta\,;-x) = {}_2\mathrm{F}_3\left[\begin{array}{lcc} \frac{1}{2}(\alpha+\beta),\, \frac{1}{2}(\alpha+\beta+1) & ; & \frac{x^2}{4} \\ \alpha+\frac{1}{2},\, \beta+\frac{1}{2},\, \alpha+\beta & ; & \end{array}\right], \tag{1.4}$$

$${}_1\mathrm{F}_1(\alpha\,;\,\rho\,;x)\,{}_1F_1(\alpha\,;\,\rho\,;-x) = {}_2\mathrm{F}_3\left[\begin{array}{lcc} \alpha,\, \rho-\alpha & ; & \frac{x^2}{4} \\ \rho,\, \frac{\rho}{2},\, \frac{\rho}{2}+\frac{1}{2} & ; & \end{array}\right], \tag{1.5}$$

$$\begin{aligned} &{}_1\mathrm{F}_1\left[\begin{array}{c} \alpha \\ \rho \end{array} ;x\right] {}_1F_1\left[\begin{array}{c} \alpha-\rho+1 \\ 2-\rho \end{array} ;-x\right] \\ &= {}_2\mathrm{F}_3\left[\begin{array}{ccc} \alpha-\frac{\rho}{2}+\frac{1}{2},\, \frac{\rho}{2}-\alpha+\frac{1}{2} & ; & \frac{x^2}{4} \\ \frac{1}{2},\, \frac{\rho}{2}+\frac{1}{2},\, \frac{3}{2}-\frac{\rho}{2} & ; & \end{array}\right] \\ &\quad + \frac{(2\alpha-\rho)\,(1-\rho)}{\rho\,(2-\rho)}\, x\; {}_2\mathrm{F}_3\left[\begin{array}{ccc} \alpha-\frac{\rho}{2}+1,\, \frac{\rho}{2}-\alpha+1 & ; & \frac{x^2}{4} \\ \frac{3}{2},\, \frac{\rho}{2}+1,\, 2-\frac{\rho}{2} & ; & \end{array}\right]. \end{aligned} \tag{1.6}$$

It is interesting to note here that if we use the result (1.3) in (1.4), (1.5) and (1.6), we get, respectively, the following alternative forms, that we will use in our present investigations:

$$e^{-x}\,{}_1\mathrm{F}_1(\alpha\,;\,2\alpha\,;x)\,{}_1F_1(\beta\,;\,2\beta\,;x) = {}_2\mathrm{F}_3\left[\begin{array}{ccc} \frac{1}{2}(\alpha+\beta),\, \frac{1}{2}(\alpha+\beta+1) & ; & \frac{x^2}{4} \\ \alpha+\frac{1}{2},\, \beta+\frac{1}{2},\, \alpha+\beta & ; & \end{array}\right], \tag{1.7}$$

$$e^{-x}\,{}_1\mathrm{F}_1(\alpha\,;\,\rho\,;x)\,{}_1F_1(\rho-\alpha\,;\,\rho\,;x) = {}_2\mathrm{F}_3\left[\begin{array}{c} \alpha,\, \rho-\alpha \\ \rho,\, \frac{\rho}{2},\, \frac{\rho}{2}+\frac{1}{2} \end{array} \begin{array}{c} ; \\ ; \end{array} \frac{x^2}{4}\right], \tag{1.8}$$

$$\begin{aligned} &e^{-x}\,{}_1\mathrm{F}_1\left[\begin{array}{c} \alpha \\ \rho \end{array} \begin{array}{c} ; \\ ; \end{array} x\right]{}_1\mathrm{F}_1\left[\begin{array}{c} 1-\alpha \\ 2-\rho \end{array} \begin{array}{c} ; \\ ; \end{array} x\right] \\ =&{}_2\mathrm{F}_3\left[\begin{array}{c} \alpha-\frac{\rho}{2}+\frac{1}{2},\, \frac{\rho}{2}-\alpha+\frac{1}{2} \\ \frac{1}{2},\, \frac{\rho}{2}+\frac{1}{2},\, \frac{3}{2}-\frac{\rho}{2} \end{array} \begin{array}{c} ; \\ ; \end{array} \frac{x^2}{4}\right] \\ &+\frac{(2\alpha-\rho)\,(1-\rho)}{\rho\,(2-\rho)}\,x\,{}_2\mathrm{F}_3\left[\begin{array}{c} \alpha-\frac{\rho}{2}+1,\, \frac{\rho}{2}-\alpha+1 \\ \frac{3}{2},\, \frac{\rho}{2}+1,\, 2-\frac{\rho}{2} \end{array} \begin{array}{c} ; \\ ; \end{array} \frac{x^2}{4}\right]. \end{aligned} \tag{1.9}$$

Also in 1987, Henrici [10] gave the following elegant result for a product of three generalized hypergeometric functions:

$$\begin{aligned} &{}_0\mathrm{F}_1\left[\begin{array}{c} - \\ 6c \end{array} \begin{array}{c} ; \\ ; \end{array} x\right]{}_0\mathrm{F}_1\left[\begin{array}{c} - \\ 6c \end{array} \begin{array}{c} ; \\ ; \end{array} \omega x\right]{}_0\mathrm{F}_1\left[\begin{array}{c} - \\ 6c \end{array} \begin{array}{c} ; \\ ; \end{array} \omega^2 x\right] \\ =&{}_2\mathrm{F}_7\left[\begin{array}{c} 3c-\frac{1}{4},\, 3c+\frac{1}{4} \\ 6c,\, 2c,\, 2c+\frac{1}{3},\, 2c+\frac{2}{3},\, 4c-\frac{1}{3},\, 4c,\, 4c+\frac{1}{3} \end{array} \begin{array}{c} ; \\ ; \end{array} \left(\frac{4x}{9}\right)^3\right], \end{aligned} \tag{1.10}$$

where $\omega = \exp(\frac{2\pi i}{3})$.

It is interesting to mention here that by making use of certain known transformations in the theory of generalized hypergeometric functions, in 1990, Karlsson and Srivastava [11] established a general triple series identity which readily yields the Henrici's identity (1.10).

On the other hand, just as the Gauss function ${}_2F_1$ was extended to ${}_pF_q$ by increasing the number of parameters in the numerator as well as in the denominator, the four Appell functions were introduced and generalized by Appell and Kampé de Fériet [1] who defined a general hypergeometric function in two variables. The notation defined and introduced by Kampé de Fériet for his double hypergeometric function of superior order was subsequently abbreviated by Burchnall and Chaundy [4, 5]. We, however, recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation given by Srivastava and Panda [22, pp. 423, (26)]. For this, let (H_h) denote the sequence of parameters $(H_1, H_2, \ldots, H_h)$ and for nonnegative integers define the Pochhammer symbols $((H_h))_n = (H_1)_n\,(H_2)_n \cdots (H_h)_n$, where, when $n = 0$, the product is understood to reduce to unity. Therefore, the convenient generalization of the Kampé de Fériet function is defined as follows:

$$\begin{aligned} &\mathrm{F}^{h:a;b}_{g:c;d}\left[\begin{array}{c} (H_h) : \\ (G_g) : \end{array} \begin{array}{c} (A_a)\,; \\ (C_c)\,; \end{array} \begin{array}{c} (B_b)\,; \\ (D_d)\,; \end{array} x,\, y\right] \\ =&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((H_h))_{m+n}\,((A_a))_m\,((B_b))_n}{((G_g))_{m+n}\,((C_c))_m\,((D_d))_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!}. \end{aligned} \tag{1.11}$$

The symbol (H) is a convenient contraction for the sequence of the parameters $H_1, H_2, \ldots, H_h$ and the Pochhammer symbol $(H)_n$ is defined by in (1.2). For more details about the convergence for this function, we refer to [21].

Later on, a unification of Lauricella's 14 triple hypergeometric series $F_1, \ldots, F_{14}$ [20] and the additional three triple hypergeometric series H_A, H_B and H_C was introduced by Srivastava [18] who defined the following general triple hypergeometric series $F^{(3)}[x,y,z]$ (see, e.g., [20, pp. 44, (14) and (15)]:

$$\begin{aligned}\mathrm{F}^{(3)}[x,\, y,\, z] \equiv F^{(3)}&\left[\begin{array}{llllllll}(a)::&(b);&(b');&(b''):&(c);&(c');&(c'');&\\(e)::&(g);&(g');&(g''):&(h);&(h');&(h'');&\end{array} x,\, y,\, z\right]\\ &= \sum_{m,\,n,\,p=0}^{\infty} \Lambda(m,\,n,\,p)\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!},\end{aligned} \tag{1.12}$$

where, for convenience,

$$\begin{aligned}\Lambda(m,\,n,\,p) =& \frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{B'}(b'_j)_{n+p}\prod_{j=1}^{B''}(b''_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{G'}(g'_j)_{n+p}\prod_{j=1}^{G''}(g''_j)_{p+m}}\\ &\times\frac{\prod_{j=1}^{C}(c_j)_m\prod_{j=1}^{C'}(c'_j)_n\prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{H'}(h'_j)_n\prod_{j=1}^{H''}(h''_j)_p},\end{aligned} \tag{1.13}$$

and (a) abbreviates the array of A parameters $a_1, \ldots, a_A$, with similar interpretations for (b), (b'), (b''), and so on.

Recently, Choi et al. [8,9] have obtained the following very interesting reduction formulas for the Srivastava's triple hypergeometric series $F^{(3)}[x,\, y,\, z]$ by applying the so-called Beta integral method (see [12], see also [7]) to the Henrici's triple product formula (1.10) and using (1.7) to (1.9):

$$\begin{aligned}&\mathrm{F}^{(3)}\left[\begin{array}{lllllll}e::&-;&-;&-:&-;&-;&-;\\d::&-;&-;&-:&6c;&6c;&6c;\end{array} 1,\,\omega,\,\omega^2\right]\\ &={}_5\mathrm{F}_{10}\left[\begin{array}{c}3c-\frac14,\ 3c+\frac14,\ \frac{e}{3},\ \frac{e}{3}+\frac13,\ \frac{e}{3}+\frac23\\ 6c,\,2c,\,2c+\frac13,\,2c+\frac23,\,4c-\frac13,\,4c,\,4c+\frac13,\,\frac{d}{3},\,\frac{d}{3}+\frac13,\,\frac{d}{3}+\frac23\end{array};\ \left(\frac49\right)^3\right],\end{aligned} \tag{1.14}$$

where $\omega = \exp(\frac{2\pi i}{3})$,

$$\begin{aligned}&\mathrm{F}^{(3)}\left[\begin{array}{lllllll}d::&-;&-;&-:&-;&\alpha;&\beta;\\e::&-;&-;&-:&-;&2\alpha;&2\beta;\end{array} -1,\,1,\,1\right]\\ &={}_4\mathrm{F}_5\left[\begin{array}{c}\frac12(\alpha+\beta),\ \frac12(\alpha+\beta+1),\ \frac{d}{2},\ \frac{d}{2}+\frac12;\\ \alpha+\frac12,\,\beta+\frac12,\,\alpha+\beta,\,\frac{e}{2},\,\frac{e}{2}+\frac12;\end{array}\ \frac14\right],\end{aligned} \tag{1.15}$$

$$\begin{aligned}&\mathrm{F}^{(3)}\left[\begin{array}{lllllll}d::&-;&-;&-:&-;&\alpha;&\rho-\alpha;\\e::&-;&-;&-:&-;&\rho;&\rho;\end{array} -1,\,1,\,1\right]\\ &={}_4\mathrm{F}_5\left[\begin{array}{c}\alpha,\ \rho-\alpha,\ \frac{d}{2},\ \frac{d}{2}+\frac12;\\ \rho,\,\frac{\rho}{2},\,\frac{\rho}{2}+\frac12,\,\frac{e}{2},\,\frac{e}{2}+\frac12;\end{array}\ \frac14\right],\end{aligned} \tag{1.16}$$

$$\mathrm{F}^{(3)}\left[\begin{array}{l} d::\ -;\ -;\ -:\ -;\ \alpha;\ 1-\alpha; \\ e::\ -;\ -;\ -:\ -;\ \rho;\ 2-\rho; \end{array} -1,1,1\right] \tag{1.17}$$

$$={}_4\mathrm{F}_5\left[\begin{array}{l} \alpha-\frac{\rho}{2}+\frac{1}{2},\ \frac{\rho}{2}-\alpha+\frac{1}{2},\ \frac{d}{2},\ \frac{d}{2}+\frac{1}{2}; \\ \frac{1}{2},\ \frac{\rho}{2}+\frac{1}{2},\ \frac{3}{2}-\frac{\rho}{2},\ \frac{e}{2},\ \frac{e}{2}+\frac{1}{2}; \end{array} \frac{1}{4}\right]$$

$$+\frac{d\,(2\alpha-\rho)(1-\rho)}{e\,\rho(2-\rho)}\,{}_4\mathrm{F}_5\left[\begin{array}{l} \alpha-\frac{\rho}{2}+1,\ \frac{\rho}{2}-\alpha+1,\ \frac{d}{2}+\frac{1}{2},\ \frac{d}{2}+1; \\ \frac{3}{2},\ \frac{\rho}{2}+1,\ 2-\frac{\rho}{2},\ \frac{e}{2}+\frac{1}{2},\ \frac{e}{2}+1; \end{array} \frac{1}{4}\right]. \tag{1.18}$$

In this sequel, motivated essentially by the results (1.14) to (1.17), we establish three unified reduction formulas for the function $F^{(3)}(x,\,y,\,z)$ by using the identities (1.7), (1.8) and (1.9) due to Bailey [2] and Ramanujan [15] which will be given in the next section.

2. Main Results

In this section, we shall establish three unified reduction formulas asserted by the following theorem.

Theorem 2.1. *For all finite x, each of the following reduction formulas holds true:*

$$\mathrm{F}^{(3)}\left[\begin{array}{l} d::\ -;\ -;\ -:\ -;\ \alpha;\ \beta; \\ e::\ -;\ -;\ -:\ -;\ 2\alpha;\ 2\beta; \end{array} -x,x,x\right] \tag{2.1}$$

$$={}_4\mathrm{F}_5\left[\begin{array}{l} \frac{1}{2}(\alpha+\beta),\ \frac{1}{2}(\alpha+\beta+1),\ \frac{d}{2},\ \frac{d}{2}+\frac{1}{2}; \\ \alpha+\frac{1}{2},\ \beta+\frac{1}{2},\ \alpha+\beta,\ \frac{e}{2},\ \frac{e}{2}+\frac{1}{2}; \end{array} \frac{x^2}{4}\right],$$

$$\mathrm{F}^{(3)}\left[\begin{array}{l} d::\ -;\ -;\ -:\ -;\ \alpha;\ \rho-\alpha; \\ e::\ -;\ -;\ -:\ -;\ \rho;\ \rho; \end{array} -x,x,x\right] \tag{2.2}$$

$$={}_4\mathrm{F}_5\left[\begin{array}{l} \alpha,\ \rho-\alpha,\ \frac{d}{2},\ \frac{d}{2}+\frac{1}{2}; \\ \rho,\ \frac{\rho}{2},\ \frac{\rho}{2}+\frac{1}{2},\ \frac{e}{2},\ \frac{e}{2}+\frac{1}{2}; \end{array} \frac{x^2}{4}\right],$$

$$\mathrm{F}^{(3)}\left[\begin{array}{l} d::\ -;\ -;\ -:\ -;\ \alpha;\ 1-\alpha; \\ e::\ -;\ -;\ -:\ -;\ \rho;\ 2-\rho; \end{array} -x,x,x\right] \tag{2.3}$$

$$={}_4\mathrm{F}_5\left[\begin{array}{l} \alpha-\frac{\rho}{2}+\frac{1}{2},\ \frac{\rho}{2}-\alpha+\frac{1}{2},\ \frac{d}{2},\ \frac{d}{2}+\frac{1}{2}; \\ \frac{1}{2},\ \frac{\rho}{2}+\frac{1}{2},\ \frac{3}{2}-\frac{\rho}{2},\ \frac{e}{2},\ \frac{e}{2}+\frac{1}{2}; \end{array} \frac{x^2}{4}\right]$$

$$+\,x\frac{d\,(2\alpha-\rho)(1-\rho)}{e\,\rho(2-\rho)}\,{}_4\mathrm{F}_5\left[\begin{array}{l} \alpha-\frac{\rho}{2}+1,\ \frac{\rho}{2}-\alpha+1,\ \frac{d}{2}+\frac{1}{2},\ \frac{d}{2}+1; \\ \frac{3}{2},\ \frac{\rho}{2}+1,\ 2-\frac{\rho}{2},\ \frac{e}{2}+\frac{1}{2},\ \frac{e}{2}+1; \end{array} \frac{x^2}{4}\right]. \tag{2.4}$$

3. Outline of Proof of Theorem 2.1

The proofs of the above results (2.1) to (2.3) are quite straightforward. In order to prove (2.1), we first replace e^{-x} by ${}_0F_0[-;-;\,-x]$ in (1.7), then multiply each side of the resulting identity by $x^{d-1}(1-x)^{e-d-1}$ and expand the involved generalized hypergeometric functions as series. Now integrate both sides of the present resulting identity with respect to x between 0 to 1 and then change the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the involved series. The integrations are easily evaluated to be expressed in terms of Gamma functions Γ by just recalling the well known relationship between the Beta function $B(\alpha, \beta)$ and the Gamma function (see, e.g., [19, pp. 8, (42)]. After some simplification, the left-hand side of the last resulting identity becomes

$$\frac{\Gamma(d)\,\Gamma(e-d)}{\Gamma(e)} \sum_{m,n,p=0}^{\infty} \frac{(d)_{m+n+p}}{(e)_{m+n+p}} \frac{(-1)^m}{m!} \frac{(\alpha)_n}{(2\alpha)_n\, n!} \frac{(\beta)_p}{(2\beta)_p\, p!} x^{m+n+p},$$

which, except for the Gamma fraction in front of the triple summations, in view of (1.12), is easily seen to correspond with the left-hand side of (2.1).

On the other hand, applying the Legendre's duplication formula for the Gamma function (see, e.g., [19, p. 6, (29)]):

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right), \quad z \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots, \tag{3.1}$$

to the right-hand side of the above last resulting identity, we obtain

$$\frac{\Gamma(d)\,\Gamma(e-d)}{\Gamma(e)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}(\alpha+\beta))_n\,(\frac{1}{2}(\alpha+\beta+1))_n\,(\frac{d}{2})_n\,(\frac{d}{2}+\frac{1}{2})_n}{(\alpha+\frac{1}{2})_n\,(\beta+\frac{1}{2})_n\,(\alpha+\beta)_n\,(\frac{e}{2})_n\,(\frac{e}{2}+\frac{1}{2})_n} \frac{1}{4^n\, n!} x^{2n},$$

which, except for the head-located Gamma fraction, is easily seen to correspond with the right-hand side of (2.1). This completes the proof of (2.1). A similar argument will establish the results (2.2) and (2.3).

We conclude this section by mentioning some special cases of our main results. The special case of (2.1) when $\beta = \alpha$ is equal to that of (2.2) when $\rho = 2\,\alpha$. The special case of (2.3) when $\rho = 2\alpha$ is also equal to that of (2.1) when $\beta = 1 - \alpha$.

4. Special Cases

In our unified reduction formulas (2.1), (2.2) and (2.3), if we take $x = 1$, we at once get the known results (1.5), (1.6) and (1.7) respectively due to Choi et al. [9]. Further since our results are valid for all finite x, so from our results, we can obtain as many reduction formulas as desired new and interesting results.

5. Further Observations

On the other hand, if we apply beta integral method to (1.4) to (1.6), we, after some simplification, obtain the following transformation formulas between Kampé de Fériet functions and generalized hypergeometric functions:

$$\begin{aligned} &\mathrm{F}^{1:1;1}_{1:1;1}\left[\begin{array}{ccc} d: & \alpha; & \beta; \\ e: & 2\alpha; & 2\beta; \end{array}\ x, -x\right] \\ &= {}_4\mathrm{F}_5\left[\begin{array}{c} \frac{1}{2}(\alpha+\beta),\ \frac{1}{2}(\alpha+\beta+1),\ \frac{1}{2}d,\ \frac{1}{2}d+\frac{1}{2} \\ \alpha+\frac{1}{2},\ \beta+\frac{1}{2},\ \alpha+\beta,\ \frac{1}{2}e,\ \frac{1}{2}e+\frac{1}{2} \end{array}\ ;\ \frac{x^2}{4}\right], \end{aligned} \tag{5.1}$$

$$\mathrm{F}^{1:1;1}_{1:1;1}\left[\begin{array}{ccc} d: & \alpha; & \alpha; \\ e: & \rho; & \rho; \end{array}\ x, -x\right] \tag{5.2}$$

$$={}_4\mathrm{F}_5\left[\begin{array}{c} \alpha,\, e-\alpha,\, \frac{1}{2}d,\, \frac{1}{2}d+\frac{1}{2} \\ \rho,\, \frac{1}{2}\rho,\, \frac{1}{2}\rho+\frac{1}{2},\, \frac{1}{2}e,\, \frac{1}{2}e+\frac{1}{2} \end{array} \begin{array}{c} ; \\ ; \end{array} \frac{x^2}{4}\right],$$

(5.3)
$$\mathrm{F}^{1:1;1}_{1:1;1}\left[\begin{array}{ccc} d: & \alpha\,; & \alpha-e+1\,; \\ e: & \rho\,; & 2-\rho\,; \end{array} x,\,-x\right]$$
$$={}_4\mathrm{F}_5\left[\begin{array}{c} \alpha-\frac{1}{2}\rho+\frac{1}{2},\, \frac{1}{2}\rho-\alpha+\frac{1}{2},\, \frac{1}{2}d,\, \frac{1}{2}d+\frac{1}{2} \\ \frac{1}{2},\, \frac{1}{2}\rho+\frac{1}{2},\, \frac{3}{2}-\frac{1}{2}\rho,\, \frac{1}{2}e,\, \frac{1}{2}e+\frac{1}{2} \end{array} ;\frac{x^2}{4}\right]$$
$$+\frac{d(2\alpha-\rho)(1-\rho)}{e\rho(2-\rho)}\, x\, {}_4\mathrm{F}_5\left[\begin{array}{c} \alpha-\frac{1}{2}\rho+1,\, \frac{1}{2}\rho-\alpha+1,\, \frac{1}{2}d+\frac{1}{2},\, \frac{1}{2}d+1 \\ \frac{3}{2},\, \frac{1}{2}\rho+1,\, 2-\frac{1}{2}\rho,\, \frac{1}{2}e+\frac{1}{2},\, \frac{1}{2}e+1 \end{array} ;\frac{x^2}{4}\right].$$

It is noted that the results (5.1) and (5.2) are seen to be special cases when $p=q=1$ of the more general results [21, pp. 31, (47) and (46)], while (5.3) is a special case of a more general result in [13, pp. 21, (2.6)].

Finally, comparing (1.15), (1.16) and (1.17) with (5.1), (5.2) and (5.3), respectively, we get the following transformation formulas between Srivastava's triple hypergeometric series $F^{(3)}(x,\,y,\,z)$ and Kampé de Fériet double series:

(5.4)
$$\mathrm{F}^{(3)}\left[\begin{array}{cccccc} d:: & \text{---}; & \text{---}; & \text{---}; & \alpha; & \beta; \\ e:: & \text{---}; & \text{---}; & \text{---}; & 2\alpha; & 2\beta; \end{array} -x,\,x,\,x\right]$$
$$=\mathrm{F}^{1:1;1}_{1:1;1}\left[\begin{array}{ccc} d: & \alpha\,; & \beta\,; \\ e: & 2\,\alpha\,; & 2\,\beta\,; \end{array} x,\,-x\right],$$

(5.5)
$$\mathrm{F}^{(3)}\left[\begin{array}{cccccc} d:: & \text{---}; & \text{---}; & \text{---}; & \alpha; & \rho-\alpha; \\ e:: & \text{---}; & \text{---}; & \text{---}; & \rho; & \rho; \end{array} -x,\,x,\,x\right]$$
$$=\mathrm{F}^{1:1;1}_{1:1;1}\left[\begin{array}{ccc} d: & \alpha\,; & \alpha\,; \\ e: & \rho\,; & \rho\,; \end{array} x,\,-x\right],$$

(5.6)
$$\mathrm{F}^{(3)}\left[\begin{array}{cccccc} d:: & \text{---}; & \text{---}; & \text{---}; & \alpha; & 1-\alpha; \\ e:: & \text{---}; & \text{---}; & \text{---}; & \rho; & 2-\rho; \end{array} -x,\,x,\,x\right]$$
$$=\mathrm{F}^{1:1;1}_{1:1;1}\left[\begin{array}{ccc} d: & \alpha\,; & \alpha-\rho+1\,; \\ e: & \rho\,; & 2-\rho\,; \end{array} x,\,-x\right].$$

Similarly other results can also be obtained.

Remark 5.1. For recent works in this area, we refer [6, 16, 17].

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