Kragujevac Journal of Mathematics
Volume 50(6) (2026), Pages 1017–1025.

SOME GENERALIZATIONS INVOLVING THE POLAR DERIVATIVE FOR AN INEQUALITY OF PAUL TURÁN

M. Y. MIR[1], S. L. WALI[1], AND W. M. SHAH[1]

Abstract. For a polynomial $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree n having all zeros in $|z| \leq 1$, It is known:

$$|P'(z)| \geq \frac{1}{2}\left(n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)|P(z)|.$$

In this paper, besides the generalization of the above inequality, we extend some well-known results to the polar derivative of a polynomial.

1. Introduction

For each positive integer n, let $\mathcal{P}_n$ denote the linear space of all polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n over the field $\mathbb{C}$ of complex numbers. If $P \in \mathcal{P}_n$ and P' be its derivative, then concerning the estimate of $|P'(z)|$ in terms of $|P(z)|$ on $|z| = 1$, Bernstein [2] proved the following:

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

Equality holds in (1.1) only if P has all its zeros at the origin. It stands natural to ask what happens to inequality (1.1), if we impose restrictions on the location of zeros of P. In this connection, the following inequalities are the earliest belonging to this domain of ideas, which have a clear impact on the subsequent work carried forward since then.

Key words and phrases. Polynomials, inequalities, polar derivative, s-fold zeros.
2010 *Mathematics Subject Classification.* Primary: 26D10. Secondary: 30A10, 30C15.
DOI
Received: January 28, 2024.
Accepted: August 18, 2024.

If $P \in \mathcal{P}_n$ has all zeros in $|z| \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{1.2}$$

and *if $P \in \mathcal{P}_n$ has all zeros in $|z| \leq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Inequality (1.2) was observed by Erdös and later verified by Lax [6], whereas inequality (1.3) was established by Turán [10].

By involving the coefficients of the polynomial $P(z)$, Dubinin [3] refined inequality (1.3). More precisely, he proved:

If all the zeros of a polynomial $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then

$$|P'(z)| \geq \frac{1}{2}\left(n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)|P(z)|.$$

Various Turán-type inequalities have appeared in the literature wherein the underlying polynomial is replaced by some more general class of functions and, the derivative of the function by some operators which, in particular generalize the ordinary derivative. Polar derivative of a polynomial with respect to a point α is one such operator for a polynomial $P(z)$ of degree n, denoted by $D_\alpha P(z)$ and is defined as (for references see [7]) $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$. It is to be observed that

$$\lim_{\alpha \to +\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Inequalities have been proved occasionally, which extend the ordinary derivative of a complex polynomial to the polar derivative. In this paper, we prove some results concerning the modulus of the polar derivative of a polynomial having all zeros in $|z| \leq k$, $k \geq 1$. Our results generalize as well as sharpen some already known inequalities of Turán type.

2. Lemmas

Lemma 2.1. *If $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s})$, $0 \leq s < n$, is a polynomial of degree $n \geq 1$ having all its zeros in $|z| \leq 1$, then for $P(z) \neq 0$, with $|z| = 1$, we have*

$$|P'(z)| \geq \frac{1}{2}\left\{n + s + \frac{\sqrt{|a_{n-s}|} - \sqrt{|a_0|}}{\sqrt{|a_{n-s}|}}\right\}|P(z)|. \tag{2.1}$$

The above lemma follows from a result due to Govil and Kumar [5].

Next lemma is due to Frappier et al. [4].

Lemma 2.2. *If $P(z)$ is a polynomial of degree $n \geq 1$, then for $R \geq 1$*

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| - (R^n - R^{n-2})|P(0)|, \quad \text{if } n > 1 \tag{2.2}$$

and

$$\max_{|z|=R}|P(z)| \leq R\max_{|z|=1}|P(z)| - (R-1)|P(0)|, \quad \text{if } n=1. \tag{2.3}$$

Lemma 2.3. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\geq 1$, then for $0\leq l<1$*

$$\max_{|z|=k}|P(z)| \geq \frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| - l\left(\frac{k^n-1}{k^n+1}\right)\min_{|z|=k}|P(z)| + \frac{2k^{n-1}}{1+k^n}|a_{n-1}|\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right), \quad \text{if } n>2, \tag{2.4}$$

and

$$\max_{|z|=k}|P(z)| \geq \frac{2k^2}{1+k^2}\max_{|z|=1}|P(z)| - l\left(\frac{k^2-1}{k^2+1}\right)\min_{|z|=k}|P(z)| + \frac{k(k-1)^2}{1+k^2}|a_1|, \quad \text{if } n=2. \tag{2.5}$$

The above lemma is due to Rather et al. [8].

We also need the following lemma which is a special case of a result due to Bernstein [1].

Lemma 2.4. *Let $P\in\mathcal{P}_n$ has all zeros in $|z|\leq 1$ and $Q(z)=z^n\overline{P\left(\frac{1}{\bar z}\right)}$. Then, for $|z|=1$*

$$|Q'(z)|\leq|P'(z)|.$$

3. Main Results.

In this paper, we first prove the following result which is a generalization of a result due to Rather et al. [9].

Theorem 3.1. *If $P\in\mathcal{P}_n$ be such that all the zeros of $P(z)=z^s(a_0+a_1z+\cdots+a_{n-s}z^{n-s})$, $0\leq s<n$, lie in $|z|\leq k$, $k\geq 1$, with a zero of multiplicity s, $0\leq s<n$, at origin, then for any complex number α, with $|\alpha|\geq k$*

$$\max_{|z|=1}|D_\alpha P(z)| \geq \frac{|\alpha|-k}{1+k^{n-s}}\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)\max_{|z|=1}|P(z)| + \frac{|a_{n-s-1}|(|\alpha|-k)}{k(1+k^{n-s})}\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)\phi(k) + \psi(k)|na_0+\alpha a_1|, \tag{3.1}$$

where $\phi(k)=\left(\frac{k^{n-s}-1}{n-s}-\frac{k^{n-s-2}-1}{n-s-2}\right)$ or $\frac{(k-1)^2}{2}$ and $\psi(k)=\left(1-\frac{1}{k^2}\right)$ or $\left(1-\frac{1}{k}\right)$ according as $n>2$ or $n=2$.

Proof. Since all zeros of $P(z)$ lie in $|z| \le k$, $k \ge 1$, with an s-fold zero at the origin, therefore $g(z) = P(kz)$ is a polynomial of degree $n \ge 1$, having all zeros in $|z| \le 1$ with an s-fold zero at the origin. Applying Lemma 2.1 to $g(z)$, we get

$$|g'(z)| \ge \frac{1}{2}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}|} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)|g(z)|. \tag{3.2}$$

Also,

$$|D_\alpha g(z)| = |ng(z) + (\alpha - z)g'(z)| \ge |\alpha| \cdot |g'(z)| - |ng(z) - zg'(z)|.$$

Since $g(z)$ has all its zeros in $|z| \le 1$, therefore by using Lemma 2.4, it can be easily shown that for $|z| = 1$

$$|g'(z)| \ge |ng(z) - zg'(z)|.$$

This gives $|D_\alpha g(z)| \ge (|\alpha| - 1)|g'(z)|$. Using (3.2), we get

$$|D_\alpha g(z)| \ge \frac{|\alpha| - 1}{2}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}|} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)|g(z)|. \tag{3.3}$$

Note that by hypothesis $|\alpha|/k \ge 1$, therefore on replacing $g(z)$ by $P(kz)$, we have

$$D_{\alpha/k}P(kz) = nP(kz) + \Big(\frac{\alpha}{k} - z\Big)kP'(kz).$$

Hence, from (3.3), we get

$$\max_{|z|=1}|D_{\alpha/k}P(kz)| \ge \frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}|} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)\max_{|z|=1}|P(kz)|. \tag{3.4}$$

Equivalently,

$$\max_{|z|=k}|D_\alpha P(z)| \ge \frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}|} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)\max_{|z|=k}|P(z)|. \tag{3.5}$$

Since $P(z) = z^s\phi(z)$, where $\phi(z)$ is a polynomial of degree $n - s$ and $D_\alpha P(z)$ has degree $n - 1$, therefore using (2.2) of Lemma 2.2 for $n > 2$, $R = k \ge 1$, we have

$$\max_{|z|=k}|D_\alpha P(z)| \le k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - (k^{n-1} - k^{n-3})|na_0 + \alpha a_1|. \tag{3.6}$$

Now using (3.6) in (3.5), we get

$$\begin{aligned} &k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - (k^{n-1} - k^{n-3})|na_0 + \alpha a_1| \\ \ge&\frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}|} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}}\right)\max_{|z|=k}|P(z)|. \end{aligned} \tag{3.7}$$

Now using (2.4) of Lemma 2.3 for $\phi(z)$ with $l = 0$, we have

$$\max_{|z|=k} |\phi(z)| \geq \left\{ \frac{2k^{n-s}}{1+k^{n-s}} \max_{|z|=1} |\phi(z)| + \frac{2k^{n-s-1}}{1+k^{n-s}} |a_{n-s-1}| \left(\frac{k^{n-s}-1}{n-s} - \frac{k^{n-s-2}-1}{n-s-2} \right) \right\}.$$

Since

$$\max_{|z|=k} |D_\alpha \phi(z)| = \frac{1}{k^s} \max_{|z|=1} |D_\alpha P(z)| \quad \text{and} \quad \max_{|z|=1} |D_\alpha \phi(z)| = \max_{|z|=1} |D_\alpha P(z)|.$$

We get for $n > 2$,

$$\max_{|z|=k} |P(z)| \geq \left\{ \frac{2k^{n}}{1+k^{n-s}} \max_{|z|=1} |P(z)| + \frac{2k^{n-1}}{1+k^{n-s}} |a_{n-s-1}| \left(\frac{k^{n-s}-1}{n-s} - \frac{k^{n-s-2}-1}{n-s-2} \right) \right\}. \tag{3.8}$$

Combining (3.8) and (3.7), we conclude that

$$\begin{aligned} \max_{|z|=1} |D_\alpha P(z)| \geq & \frac{|\alpha|-k}{1+k^{n-s}} \left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}} \right) \max_{|z|=1} |P(z)| \\ & + \frac{|a_{n-s-1}|(|\alpha|-k)}{k(1+k^{n-s})} \left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}} \right) \phi(k) \\ & + \psi(k)|na_0+\alpha a_1|, \end{aligned}$$

where $\phi(k) = \left(\frac{k^{n-s}-1}{n-s} - \frac{k^{n-s-2}-1}{n-s-2} \right)$ and $\psi(k) = \left(1 - \frac{1}{k^2} \right)$. This proves the result for the case $n > 2$. The result for the case $n = 2$ can be obtained by using inequalities (2.3) of Lemma 2.2 and (2.5) of Lemma 2.3. This completely proves Theorem 3.1. □

If we divide both sides of (3.1) by $|\alpha|$ and letting $\alpha \to +\infty$, we get the following.

Corollary 3.1. *If $P \in \mathcal{P}_n$ be such that all the zeros of $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s})$, $0 \leq s < n$, lie in $|z| \leq k$, $k \geq 1$, except a zero of multiplicity s, $0 \leq s < n$, at the origin, then*

$$\begin{aligned} \max_{|z|=1} |P'(z)| \geq & \frac{1}{1+k^{n-s}} \left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}} \right) \max_{|z|=1} |P(z)| \\ & + \frac{|a_{n-s-1}|}{k(1+k^{n-s})} \left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|}} \right) \phi(k) \\ & + \psi(k)|a_1|, \end{aligned} \tag{3.9}$$

where $\phi(k), \psi(k)$ are defined in Theorem 3.1.

Remark 3.1. For $s = 0$ Theorem 3.1 reduces to a result due to Rather et al. [9].

Remark 3.2. For $k = 1$, Corollary 3.1 reduces to inequality (2.1).

We next prove the following.

Theorem 3.2. *If $P \in \mathcal{P}_n$ be such that all the zeros of $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s})$, $0 \le s < n$, lie in $|z| \le k$, $k \ge 1$, with a zero of multiplicity s, $0 \le s < n$, at the origin, then for any complex number α with $|\alpha| \ge k$ and for $0 \le l < 1$, we have*

$$\begin{aligned}\max_{|z|=1} |D_\alpha P(z)| \ge &\frac{|\alpha| - k}{k^n(1 + k^{n-s})}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - lm} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - lm}}\right)\\ &\times (k^n \max_{|z|=1} |P(z)| - lm) + \frac{n+s}{1+k^{n-s}}\Bigg((|\alpha| - k)\max_{|z|=1}|P(z)|\\ &+ \left(\frac{|\alpha|}{k^s} + \frac{1}{k^{n-1}}\right)lm\Bigg) + \frac{|\alpha| - k}{k(1 + k^{n-s})}|a_{n-s-1}|\\ &\times \left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - lm} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - lm}}\right)\phi(k) + |na_0 + \alpha a_1|\psi(k),\end{aligned} \tag{3.10}$$

where $m = \min_{|z|=k}|P(z)|$ and $\phi(k), \psi(k)$ are defined in Theorem 3.1.

Proof. By given hypothesis all zeros of $P(z)$ lie in $|z| \le k, k \ge 1$ with an s-fold zero at the origin. Also, if at least one zero of $P(z)$ lie on $|z| = k$, then $m = \min_{|z|=k}|P(z)| = 0$ and the result follows from Theorem 3.1. So we assume all zeros of $P(z)$ lie in $|z| < k$ with s-fold zeros at the origin. Hence, in this case $m > 0$. Therefore, $f(z) = P(kz)$ has all zeros in $|z| < 1$ with $m = \min_{|z|=k}|P(z)| = \min_{|z|=1}|f(z)|$. Hence $|f(z)| > m$ for $|z| = 1$. Therefore, by Rouche's theorem for some complex δ with $|\delta| < 1$, $F(z) = f(z) - \delta m z^n$ has all zeros in $|z| < 1$ with s-fold zeros at the origin. Using value of $F(z)$ in (3.3), we get

$$|D_\alpha F(z)| \ge \frac{|\alpha| - 1}{2}\left(n + s + \frac{\sqrt{|k^{n-s}a_{n-s} - \delta m|} - \sqrt{|a_0|}}{\sqrt{|k^{n-s}a_{n-s} - \delta m|}}\right)|F(z)|.$$

Using the fact that $T(x) = \frac{x - |a|}{x + |a|}$ is non-decreasing function of x and

$$|k^{n-s}a_{n-s} + \delta m| \ge k^{n-s}|a_{n-s}| - |\delta|m \ge 0,$$

we get

$$|D_\alpha F(z)| \ge \frac{|\alpha| - 1}{2}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - |\delta|m} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - |\delta|m}}\right)|F(z)|.$$

Equivalently,

$$\begin{aligned}\left|D_{\alpha/k}f(z) - \frac{nm\alpha\delta}{k}z^{n-1}\right| \ge &\frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - |\delta|m} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - |\delta|m}}\right)\\ &\times |f(z) - \delta m z^n|.\end{aligned} \tag{3.11}$$

Now, by simple deduction of Laguerre theorem [6, p. 52] on the polar derivative of a polynomial that for any α with $|\alpha| \geq k$,

$$D_{\alpha/k}(f(z) - \delta m z^n) = D_{\alpha/k} f(z) - \frac{nm\alpha\delta}{k} z^{n-1}$$

has all its zeros in $|z| < 1$. This implies $|D_{\alpha/k}(f(z)| \geq \frac{nm|\alpha||\delta|}{k}|z|^{n-1}$. Hence, we can choose argument of δ suitably, so, that from (3.11) for $|z| = 1$

$$|D_{\alpha/k} f(z)| - \frac{nm|\alpha||\delta|}{k} \geq \frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - \delta m} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m}}\right) \times (|f(z)| - |\delta m|).$$

Equivalently,

$$\begin{aligned}|D_{\alpha/k} f(z)| \geq & \frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m}}\right)|f(z)| \\ & - \frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m}}\right)|\delta| m \\ & + \frac{nm|\alpha|\delta|}{k}.\end{aligned}$$

Now replacing $f(z)$ by $P(kz)$ and proceeding as in the proof of Theorem 3.1, we get

$$\begin{aligned}\max_{|z|=k} |D_\alpha P(z)| \geq & \frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m}}\right)\max_{|z|=k} |P(z)| \\ & - \frac{|\alpha| - k}{2k}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - |\delta| m} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - \delta m}}\right)|\delta| m + \frac{nm|\alpha||\delta|}{k}. \qquad (3.12)\end{aligned}$$

As in the proof of Theorem 3.1, $P(z) = z^s\phi(z)$ has all zeros in $|z| \leq k$, $k \geq 1$. Therefore, from (2.2) of Lemma 2.2 and (2.4) of Lemma 2.3, with

$$\max_{|z|=k} |D_\alpha \phi(z)| = \frac{1}{k^s} \max_{|z|=1} |D_\alpha P(z)|, \quad \min_{|z|=k} |D_\alpha \phi(z)| = \frac{1}{k^s} \min_{|z|=k} |D_\alpha P(z)|$$

and

$$\max_{|z|=1} |D_\alpha \phi(z)| = \max_{|z|=1} |D_\alpha P(z)|,$$

we get from (3.12) by taking $\delta = l$ with $n > 2$,

$$\begin{aligned}& k^n \max_{|z|=1} |D_\alpha P(z)| - (k^n - k^{n-2})|na_0 + \alpha a_1| \\ \geq & \frac{|\alpha| - k}{2}\left(n + s + \frac{\sqrt{k^{n-s}|a_{n-s}| - lm} - \sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}| - lm}}\right)\left\{\frac{2k^n}{1 + k^{n-s}} \max_{|z|=1} |P(z)|\right.\end{aligned}$$

$$+ l\left(\frac{k^{n-s}-1}{k^{n-s}+1}\right)\min_{|z|=k}|P(z)| + \frac{2k^{n-1}|a_{n-s-1}|}{k^{n-s}+1}\left(\frac{k^{n-s}-1}{n-s} - \frac{k^{n-s-2}-1}{n-s-2}\right)\Bigg\}$$
$$- \frac{|\alpha|-k}{2}\cdot\frac{\sqrt{k^{n-s}|a_{n-s}|-lm}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-lm}}lm + \frac{nml}{2k}(|\alpha|+k)$$
$$+ \frac{slm}{2k}(k-|\alpha|).$$

Thus on simplification, we get for $0 \leq l < 1$ and $n > 2$

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \geq & \frac{|\alpha|-k}{k^n(1+k^{n-s})}\left(\frac{\sqrt{k^{n-s}|a_{n-s}|-lm}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-lm}}\right)\\ & \times (k^n\max_{|z|=1}|P(z)|-lm)\\ & + \frac{n+s}{1+k^{n-s}}\left((|\alpha|-k)\max_{|z|=1}|P(z)| + \left(\frac{|\alpha|}{k^s}+\frac{1}{k^{n-1}}\right)lm\right)\\ & + \frac{|\alpha|-k}{k(1+k^{n-s})}|a_{n-s-1}|\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|-lm}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-lm}}\right)\\ & \times \phi(k) + |na_0+\alpha a_1|\psi(k) + \frac{slm}{2k}(k-|\alpha|),\end{aligned}$$

where $\phi(z)$ and $\psi(z)$ are defined in the statement of Theorem 3.1. This proves the theorem for case when $n > 2$. The result for $n = 2$ can be obtained by using (2.3) of Lemma 2.2 and (2.5) of Lemma 2.3. This completely proves Theorem 3.2. ☐

If we divide both sides of (3.10) by $|\alpha|$ and letting $\alpha \to +\infty$, we get the following.

Corollary 3.2. *If $P \in \mathcal{P}_n$ be such that all the zeros of $P(z) = z^s(a_0 + a_1z + \cdots + a_{n-s}z^{n-s})$, $0 \leq s < n$, lie in $|z| \leq k$, $k \geq 1$, with a zero of multiplicity s, $0 \leq s < n$, at the origin, then for $0 \leq l < 1$*

$$\begin{aligned}\max_{|z|=1}|P'(z)| \geq & \frac{1}{k^n(1+k^{n-s})}\left(\frac{\sqrt{k^{n-s}|a_{n-s}|-lm}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-lm}}\right)\\ & \times (k^n\max_{|z|=1}|P(z)|-lm) + \frac{n+s}{1+k^{n-s}}\left(\max_{|z|=1}|P(z)|+\frac{1}{k^s}lm\right) + \psi(k)|a_1|\\ & + k^{n-1}|a_{n-s-1}|\left(n+s+\frac{\sqrt{k^{n-s}|a_{n-s}|-lm}-\sqrt{|a_0|}}{\sqrt{k^{n-s}|a_{n-s}|-lm}}\right)\phi(k) - \frac{slm}{2k},\end{aligned} \tag{3.13}$$

where $m = \min_{|z|=k}|P(z)|$ and $\phi(k), \psi(k)$ are defined in Theorem 3.1.

Remark 3.3. For $s = 0$, (3.10) reduces to a result due to Rather et al. [9].

Acknowledgements. The authors are highly thankful to the referee for his/her valuable comments and suggestions.

References

[1] S. N. Bernstein, *Sur la limitation des dérivées des polynomes,* C. R. Acad. Sci. Paris. **190** (1930), 338–340.

[2] S. N. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions contimues par des polynomes de degré donné,* Mem. Acad. R. Belg. **4** (1912), 1-103.

[3] V. N. Dubinin, *Applications of the Schwarz Lemma to inequalities for entire functions with constraints on zeros*, J. Math. Sci. **143**(3) (2007), 3069–3076. https://doi.org/10.1007/s10958-007-0192-4

[4] C. Frappier, Q. I. Rahman and Rt. St. Ruscheweyh, *New inequalities for polynomials,* Trans. Amer. Math. Soc. **288** (1985), 69–99. https://doi.org/10.1090/S0002-9947-1985-0773048-1

[5] N. K. Govil and P. Kumar, *On sharpening of an inequality of Turán,* Appl. Anal. Discrete Math. **13** (2019), 711–720. https://doi.org/10.2298/aadm190326028g

[6] P. D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513. https://doi.org/10.1090/S0002-9904-1944-08177-9

[7] M. Marden, *Geometry of Polynomials*, American Math. Soc., USA 1966.

[8] N. A. Rather, A. Iqbal and I. Dar, *Inequalities for the derivative of polynomials with restricted zeros,* Korean J. Math. **28** (2020), 931–942. https://doi.org/10.11568/kjm.2020.28.4.931

[9] N. A. Rather, A. Iqbal and I. Dar, *Some extensions of a theorem of Paul-Turán concerning polynomials,* Kragujevac J. Math. **46** (2022), 969–979. https://doi.org/10.46793/kgjmat2206.969r

[10] P. Turán, *Über die ableitung von polynomen,* Compos. Math. **7** (1939), 89–95.

[1]Department of Mathematics,
Central University of Kashmir,
Ganderbal Jammu and Kashmir, India
Email address: myousf@cukashmir.ac.in
ORCID iD: https://orcid.org/0000-0001-6343-4736
Email address: shahlw@yahoo.co.in
ORCID iD: https://orcid.org/0000-0002-0258-042X
Email address: wali@cukashmir.ac.in