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CONSTRUCTING SYMMETRIC EQUALITY ALGEBRAS

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Abstract. In this paper, we introduce the notion of strong fuzzy filter on hyper equality algebras and investigate some equivalence definitions of it. Then by using this notion we constructed a symmetric equality algebra and define a special form of classes. By using these, we define the concept of a fuzzy hyper congruence relation on hyper equality algebra and we prove that the quotient is made by it is an equality algebra. Also, by using a fuzzy equivalence relation on hyper equality, we introduce a fuzzy hyper congruence relation and prove that this fuzzy hyper congruence is regular and finally we prove that the quotient structure that is made by it is a symmetric hyper equality algebra.

1. Introduction

The motivation for introducing equality algebras came from EQ-algebras which are defined by Novák in [18]. In EQ-algebras, compared to equality algebras, there is an additional operation $\otimes$, called product, which is very loosely related to the other operations. Therefore, there might not exist deep algebraic characterizations of EQ-algebras, and intention was to define a structure similar to EQ-algebras but without the product. This new logical algebra, the equality algebra, has two connectives, a meet operation and an equivalence, and a constant. Equality algebra is introduced by Jeni [9], and since then many mathematicians have studied this algebraic structure and it in various fields. For instance, Novák et al. in [18] introduced a closure operator in the equality algebra class, and investigated that under what condition an equality algebra is a BCK-algebra. Zebardast et al. in [23] investigated the relation among

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equality algebras and other logical algebra for instance, hoop, residuated lattice and etc. Also, Zebardast et al. in [23] studied commutative equality algebras and considered characterizations of commutative equality algebras. For more study we suggest [5, 7, 8, 16, 22, 23].

The hyper structure theory (called also multialgebra) was introduced in 1934 by F. Marty [14] at the 8th congress of Scandinavian Mathematicians. Nowadays, hyperstructures have a lot of applications in several domains of mathematics and computer science. In [15] Mittas et al. applied the hyperstructures to lattices and introduced the concepts of hyperlattice and superlattice. Many authors studied different aspects of semihypergroups, Borzooei et al. exerted hyper structuers to logical algebras and introduced some hyper logical algebras (see [1–4]). Hyper equality algebras are introduced and studied in [6, 12, 19] and authors provided many basic properties of this class of hyper algebras. Fuzzy type theory was developed by Novák in [17] as a fuzzy counterpart of the classical higher-order logic. Filters have momentous role in the perusing logical deductive systems and logical algebraic systems. The notion of filters on equality algebras is introduced by Jeni in [10]. Then some different kinds of filters on equality algebras are defined and studied, see [5, 22], for more details. Also, Zadeh [21], the idea of the fuzzy sets have been used to other algebraic structures by many mathematicians that we refer to [11, 13]. Fuzzy filters on equality algebras are defined recently in [20], where they have defined fuzzy congruences on equality algebras and have showed that there is one-to-one correspondence between fuzzy filters and fuzzy congruences.

In Section 2, we give some notions and statements of hyper equality algebras from [6] and we recall some facts about fuzzy set theories. In Section 3, we defined the notion of strong fuzzy filters on hyper equality algebras and investigate some properties of strong fuzzy filters on these algebras. Section 4, we introduce the concept of fuzzy hyper congruence on hyper equality algebras and we give a relation between strong fuzzy filters and fuzzy hyper congruence on hyper equality algebras.

2. Preliminaries

In this section, we present some of the main definitions and results of equality algebras used in this paper.

Let $\emptyset \neq \mathcal{L}$. Then a fuzzy subset of $\mathcal{L}$ is $\varsigma : \mathcal{L} \to [0,1]$, where for $\mathfrak{t} \in [0,1]$, the set $\varsigma_{\mathfrak{t}} = \{\mathfrak{x} \in \mathcal{L} \mid \varsigma(\mathfrak{x}) \succcurlyeq \mathfrak{t}\}$ is said to be a level subset of ς. We say ς satisfies the sup-property if for every $\emptyset \neq \mathcal{S} \subseteq \mathcal{L}$ there exists $\mathfrak{i} \in \mathcal{S}$, where $\varsigma(\mathfrak{i}) = \sup_{\mathfrak{x}\in\mathcal{S}} \varsigma(\mathfrak{x})$. The set of all fuzzy subsets of $\mathcal{L}$, is shown by $\mathcal{FS}(\mathcal{L})$. A function $\varrho : \mathcal{L} \times \mathcal{L} \to [0,1]$ is said to be a fuzzy relation on $\mathcal{L}$. Also, ϱ on $\mathcal{L}$ is said to be a fuzzy equivalence relation if for every $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$:

(i) $\varrho(\mathfrak{x}, \mathfrak{x}) = \bigvee_{(\mathfrak{y},\mathfrak{z})\in\mathcal{L}\times\mathcal{L}} \varrho(\mathfrak{y}, \mathfrak{z})$ (fuzzy reflexive);
(ii) $\varrho(\mathfrak{x}, \mathfrak{y}) = \varrho(\mathfrak{y}, \mathfrak{x})$ (fuzzy symmetric);
(iii) $\varrho(\mathfrak{x}, \mathfrak{y}) \succcurlyeq \bigvee_{\mathfrak{z}\in\mathcal{L}} (\varrho(\mathfrak{x}, \mathfrak{z}) \barwedge \varrho(\mathfrak{z}, \mathfrak{y}))$ (fuzzy transitive).

An equality algebra $\mathcal{E} = \langle \mathcal{E}, \backsim, \barwedge, 1\rangle$ is an algebra of type $(2, 2, 0)$ such that, for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{E}$, the following axioms are fulfilled:

($\mathcal{E}1$) $\langle \mathcal{E}, \barwedge, 1\rangle$ is a $\barwedge$-semilattice with top element 1;
($\mathcal{E}2$) $\mathfrak{x} \backsim \mathfrak{y} = \mathfrak{y} \backsim \mathfrak{x}$;
($\mathcal{E}3$) $\mathfrak{x} \backsim \mathfrak{x} = 1$;
($\mathcal{E}4$) $\mathfrak{x} \backsim 1 = \mathfrak{x}$;
($\mathcal{E}5$) $\mathfrak{x} \preccurlyeq \mathfrak{y} \preccurlyeq \mathfrak{z}$ implies $\mathfrak{x} \backsim \mathfrak{z} \preccurlyeq \mathfrak{y} \backsim \mathfrak{z}$ and $\mathfrak{x} \backsim \mathfrak{z} \preccurlyeq \mathfrak{x} \backsim \mathfrak{y}$;
($\mathcal{E}6$) $\mathfrak{x} \backsim \mathfrak{y} \preccurlyeq (\mathfrak{x} \barwedge \mathfrak{z}) \backsim (\mathfrak{y} \barwedge \mathfrak{z})$;
($\mathcal{E}7$) $\mathfrak{x} \backsim \mathfrak{y} \preccurlyeq (\mathfrak{x} \backsim \mathfrak{z}) \backsim (\mathfrak{y} \backsim \mathfrak{z})$.

From now, $\langle \mathcal{E}, \backsim, \barwedge, 1\rangle$ or $\mathcal{E}$ is an equality algebra.

Now, define two operations $\curvearrowright$ (implication) and $\leftrightarrows$ (equivalence operation) on $\mathcal{E}$ by $\mathfrak{x} \curvearrowright \mathfrak{y} = \mathfrak{x} \backsim (\mathfrak{x} \barwedge \mathfrak{y})$ and $\mathfrak{x} \leftrightarrows \mathfrak{y} = (\mathfrak{x} \curvearrowright \mathfrak{y}) \barwedge (\mathfrak{y} \curvearrowright \mathfrak{x})$ (see [9]).

Let $\mathcal{L}$ be a non-empty set. A function $\circ : \mathcal{L} \times \mathcal{L} \to P(\mathcal{L})^* = P(\mathcal{L}) \setminus \{\emptyset\}$ is a hyper operation on $\mathcal{L}$.

A hyper equality algebra $\mathcal{L} = \langle \mathcal{L}; \backsim, \barwedge, 1\rangle$ is a non-empty set $\mathcal{L}$ endowed with a binary operation $\barwedge$, a hyper operation $\backsim$ and a top element 1 where for each $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$:

($\mathcal{HE}1$) $\langle \mathcal{L}, \barwedge, 1\rangle$ is a meet-semilattice with top element 1;
($\mathcal{HE}2$) $\mathfrak{x} \backsim \mathfrak{y} \lll \mathfrak{y} \backsim \mathfrak{x}$;
($\mathcal{HE}3$) $1 \in \mathfrak{x} \backsim \mathfrak{x}$;
($\mathcal{HE}4$) $\mathfrak{x} \in 1 \backsim \mathfrak{x}$;
($\mathcal{HE}5$) $\mathfrak{x} \preccurlyeq \mathfrak{y} \preccurlyeq \mathfrak{z}$ implies $\mathfrak{x} \backsim \mathfrak{z} \lll \mathfrak{y} \backsim \mathfrak{z}$ and $\mathfrak{x} \backsim \mathfrak{z} \lll \mathfrak{x} \backsim \mathfrak{y}$;
($\mathcal{HE}6$) $\mathfrak{x} \backsim \mathfrak{y} \lll (\mathfrak{x} \barwedge \mathfrak{z}) \backsim (\mathfrak{y} \barwedge \mathfrak{z})$;
($\mathcal{HE}7$) $\mathfrak{x} \backsim \mathfrak{y} \lll (\mathfrak{x} \backsim \mathfrak{z}) \backsim (\mathfrak{y} \backsim \mathfrak{z})$, where $\mathfrak{x} \preccurlyeq \mathfrak{y}$ iff $\mathfrak{x} \barwedge \mathfrak{y} = \mathfrak{x}$ and $\mathcal{S} \lll \mathcal{R}$ is defined by, for all $\mathfrak{x} \in \mathcal{S}$, there is $\mathfrak{y} \in \mathcal{R}$ such that $\mathfrak{x} \preccurlyeq \mathfrak{y}$.

Notation. Throughout of this paper, we suppose $\mathcal{L} = \langle \mathcal{L}; \backsim, \barwedge, 1\rangle$ or $\mathcal{L}$ is a hyper equality algebra, unless otherwise stated (see [6]).

Define two operations, the implication and the equivalence on $\langle \mathcal{L}, \backsim, \barwedge, 1\rangle$ [6], such that for any $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, we have

$$\mathfrak{x} \curvearrowright \mathfrak{y} = \mathfrak{x} \backsim (\mathfrak{x} \barwedge \mathfrak{y}) \quad \text{and} \quad \mathfrak{x} \leftrightarrows \mathfrak{y} = (\mathfrak{x} \curvearrowright \mathfrak{y}) \barwedge (\mathfrak{y} \curvearrowright \mathfrak{x}).$$

Proposition 2.1 ([6]). *For all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, the next results are equivalent:*
($\mathcal{HE}5$) $\mathfrak{x} \preccurlyeq \mathfrak{y} \preccurlyeq \mathfrak{z}$ *implies* $\mathfrak{x} \backsim \mathfrak{z} \lll \mathfrak{y} \backsim \mathfrak{z}$ *and* $\mathfrak{x} \backsim \mathfrak{z} \lll \mathfrak{x} \backsim \mathfrak{y}$*;*
($\mathcal{HE}5a$) $\mathfrak{x} \backsim (\mathfrak{x} \barwedge \mathfrak{y} \barwedge \mathfrak{z}) \lll \mathfrak{x} \backsim (\mathfrak{x} \barwedge \mathfrak{y})$*;*
($\mathcal{HE}5b$) $\mathfrak{x} \curvearrowright (\mathfrak{y} \barwedge \mathfrak{z}) \lll \mathfrak{x} \curvearrowright \mathfrak{y}$.

Proposition 2.2 ([6]). *For all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$ and $\mathcal{S}, \mathcal{R}, \mathcal{T} \subseteq \mathcal{L}$, we have:*
(P1) $\mathfrak{x} \preccurlyeq \mathfrak{y}$ *and* $\mathfrak{y} \preccurlyeq \mathfrak{x}$ *imply* $\mathfrak{x} = \mathfrak{y}$*;*
(P2) $1 \in \mathfrak{x} \curvearrowright \mathfrak{x}$, $1 \in \mathfrak{x} \curvearrowright 1$, $\mathfrak{x} \lll \mathfrak{x} \backsim 1$, $\mathfrak{x} \in 1 \curvearrowright \mathfrak{x}$ *and* $1 \in \mathfrak{x} \leftrightarrows \mathfrak{x}$*;*
(P3) $\mathfrak{x} \backsim \mathfrak{y} \lll \mathfrak{x} \curvearrowright \mathfrak{y}$ *and* $\mathfrak{x} \backsim \mathfrak{y} \lll \mathfrak{y} \curvearrowright \mathfrak{x}$*;*
(P4) $\mathfrak{x} \preccurlyeq \mathfrak{y}$ *implies* $1 \in \mathfrak{x} \curvearrowright \mathfrak{y}$*;*
(P5) $\mathfrak{x} \preccurlyeq \mathfrak{y} \preccurlyeq \mathfrak{z}$ *implies* $\mathfrak{z} \backsim \mathfrak{x} \lll \mathfrak{z} \backsim \mathfrak{y}$ *and* $\mathfrak{z} \backsim \mathfrak{x} \lll \mathfrak{y} \backsim \mathfrak{x}$*;*
(P6) $\mathfrak{x} \lll \mathfrak{y} \curvearrowright \mathfrak{x}$ *and* $\mathcal{S} \lll \mathcal{R} \curvearrowright \mathcal{S}$*;*
(P7) $\mathfrak{x} \preccurlyeq \mathfrak{y}$ *implies* $\mathfrak{z} \curvearrowright \mathfrak{x} \lll \mathfrak{z} \curvearrowright \mathfrak{y}$ *and* $\mathfrak{y} \curvearrowright \mathfrak{z} \lll \mathfrak{x} \curvearrowright \mathfrak{z}$*;*

(P8) $\mathcal{S} \lll \mathcal{R}$ *implies* $\mathcal{T} \curvearrowright \mathcal{S} \lll \mathcal{T} \curvearrowright \mathcal{R}$ *and* $\mathcal{R} \curvearrowright \mathcal{T} \lll \mathcal{S} \curvearrowright \mathcal{T}$*;*
(P9) $\mathfrak{x} \preccurlyeq \mathfrak{y}$ *implies* $\mathfrak{x} \lll \mathfrak{y} \backsim \mathfrak{x}$*;*
(P10) $\mathfrak{y} \lll (\mathfrak{x} \curvearrowright \mathfrak{y}) \curvearrowright \mathfrak{y}$*;*
(P11) $\mathfrak{x} \curvearrowright \mathfrak{y} \lll (\mathfrak{y} \curvearrowright \mathfrak{z}) \curvearrowright (\mathfrak{x} \curvearrowright \mathfrak{z})$*;*
(P12) $\mathfrak{x} \curvearrowright (\mathfrak{y} \barwedge \mathfrak{z}) \lll (\mathfrak{x} \barwedge \mathfrak{z}) \curvearrowright \mathfrak{y}$.

Consider $\emptyset \neq \mathcal{G} \subseteq \mathcal{L}$ such that, for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, if $\mathfrak{x} \in \mathcal{G}$ and $\mathfrak{x} \preccurlyeq \mathfrak{y}$, then $\mathfrak{y} \in \mathcal{G}$. Thus, for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{G}$, $\mathcal{G}$ is called a

(WF) weak filter of $\mathcal{L}$ if $\mathfrak{x} \in \mathcal{G}$ and $\mathfrak{x} \backsim \mathfrak{y} \subseteq \mathcal{G}$ imply $\mathfrak{y} \in \mathcal{G}$;
(F) filter of $\mathcal{L}$ if $\mathfrak{x} \in \mathcal{G}$ and $\mathcal{G} \lll \mathfrak{x} \backsim \mathfrak{y}$ imply $\mathfrak{y} \in \mathcal{G}$;
(SF) strong filter of $\mathcal{L}$ if $\mathfrak{x} \in \mathcal{G}$ and $(\mathfrak{x} \backsim \mathfrak{y}) \cap \mathcal{G} \neq \emptyset$ imply $\mathfrak{y} \in \mathcal{G}$.

Clearly, if $\mathcal{G}$ is one of the above stated cases, then $1 \in \mathcal{G}$. Also, any strong filter of $\mathcal{L}$ is a (weak) filter but the converse is not true (see [6, Remark 1, Examples 10, 11]). $\mathcal{L}$ is a symmetric if $\mathfrak{x} \backsim \mathfrak{y} = \mathfrak{y} \backsim \mathfrak{x}$, where $=$ denotes the equality between subsets of $\mathcal{L}$. $\mathcal{L}$ is separated if for every $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, $1 \in \mathfrak{x} \backsim \mathfrak{y}$, then $\mathfrak{x} = \mathfrak{y}$. $\mathcal{L}$ is good if for each $\mathfrak{x} \in \mathcal{L}$, $\mathfrak{x} = 1 \backsim \mathfrak{x}$ (see [6]).

Assume θ is an equivalence relation on $\mathcal{L}$. For any $\mathcal{S}, \mathcal{R} \subseteq \mathcal{L}$, $\mathcal{S}\overline{\theta}\mathcal{R}$ means, for any $\mathfrak{i} \in \mathcal{S}$, there exists $\mathfrak{h} \in \mathcal{R}$ such that $\mathfrak{i}\theta\mathfrak{h}$ and for any $\mathfrak{h} \in \mathcal{R}$ there exists $\mathfrak{i} \in \mathcal{S}$ such that $\mathfrak{i}\theta\mathfrak{h}$, and $\mathcal{S}\overline{\overline{\theta}}\mathcal{R}$ means, for any $\mathfrak{i} \in \mathcal{S}$ and any $\mathfrak{h} \in \mathcal{R}$, $\mathfrak{i}\theta\mathfrak{h}$. Moreover, θ is called a congruence relation on $\mathcal{L}$ if, for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{u}, \mathfrak{v} \in \mathcal{L}$, $\mathfrak{x}\theta\mathfrak{y}$ and $\mathfrak{u}\theta\mathfrak{v}$ imply $(\mathfrak{x} \backsim \mathfrak{u})\ \overline{\theta}\ (\mathfrak{y} \backsim \mathfrak{v})$ and $(\mathfrak{x} \barwedge \mathfrak{u})\ \theta\ (\mathfrak{y} \barwedge \mathfrak{v})$. Also, θ is called a strong congruence relation if for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{u}, \mathfrak{v} \in \mathcal{L}$, $\mathfrak{x}\theta\mathfrak{y}$ and $\mathfrak{u}\theta\mathfrak{v}$ imply $(\mathfrak{x} \backsim \mathfrak{u})\ \overline{\overline{\theta}}\ (\mathfrak{y} \backsim \mathfrak{v})$ and $(\mathfrak{x} \barwedge \mathfrak{u})\ \theta\ (\mathfrak{y} \barwedge \mathfrak{v})$ (see [6, Definition 11]).

3. Strong Fuzzy Filters of Hyper Equality Algebras

In this section, we define the notion of strong fuzzy filter on hyper equality algebras and we characterize it. By using this notion we define a congruence relation on hyper equality algebra and prove that the quotient that is made by this is an equality algebra.

Definition 3.1. Let $\vartheta \in \mathcal{FS}(\mathcal{L})$. Then ϑ is a strong fuzzy filter of $\mathcal{L}$ if for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$
(FF1) $\vartheta(\mathfrak{x}) \barwedge \left(\sup_{\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}} \vartheta(\mathfrak{i})\right) \preccurlyeq \vartheta(\mathfrak{y})$;
(FF2) if $\mathfrak{x} \preccurlyeq \mathfrak{y}$, then $\vartheta(\mathfrak{x}) \preccurlyeq \vartheta(\mathfrak{y})$.
Note. For any $\mathcal{S}, \mathcal{R} \subseteq \mathcal{L}$, the above relations are equivalent with the following statements:
(FF1′) $(\sup_{\mathfrak{i} \in \mathcal{S}} \vartheta(\mathfrak{i})) \barwedge (\sup_{\mathfrak{o} \in \mathcal{S} \backsim \mathcal{R}} \vartheta(\mathfrak{o})) \preccurlyeq \sup_{\mathfrak{h} \in \mathcal{R}} \vartheta(\mathfrak{h})$;
(FF2′) If $\mathcal{S} \lll \mathcal{R}$, then $\sup_{\mathfrak{i} \in \mathcal{S}} \vartheta(\mathfrak{i}) \preccurlyeq \sup_{\mathfrak{h} \in \mathcal{R}} \vartheta(\mathfrak{h})$.

Example 3.1. (i) Assume $\mathcal{L} = [0, 1]$. For each $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, we define the operations $\backsim$ and $\barwedge$ on $\mathcal{L}$ by $\mathfrak{x} \barwedge \mathfrak{y} = \min\{\mathfrak{x}, \mathfrak{y}\}$ and $\mathfrak{x} \backsim \mathfrak{y} = \{0, 1 - |\mathfrak{x} - \mathfrak{y}|\}$. Then $\mathcal{L} = \langle \mathcal{L}; \backsim, \barwedge, 1\rangle$ is a hyper equality algebra. Define $\vartheta : \mathcal{L} \to [0, 1]$ by $\vartheta(1) = \beta$ and for any $\mathfrak{x} < 1$, $\vartheta(\mathfrak{x}) = \alpha$, where $0 < \alpha < \beta < 1$. Then ϑ is a strong fuzzy filter of $\mathcal{L}$.

(ii) If $\mathcal{L} = \{0, \mathfrak{i}, 1\}$ such that $0 \preccurlyeq \mathfrak{i} \preccurlyeq 1$, then, for any $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, define

$$\mathfrak{x} \barwedge \mathfrak{y} = \min\{\mathfrak{x}, \mathfrak{y}\} \quad \text{and} \quad \begin{array}{c|ccc} \backsim & 0 & \mathfrak{i} & 1 \\ \hline 0 & \{1\} & \{0,\mathfrak{i}\} & \{0,\mathfrak{i}\} \\ \mathfrak{i} & \{0,\mathfrak{i}\} & \{1\} & \{\mathfrak{i}\} \\ 1 & \{0,\mathfrak{i}\} & \{0,\mathfrak{i}\} & \{1\} \end{array}.$$

Then $\mathcal{L} = \langle \mathcal{L}; \backsim, \barwedge, 1\rangle$ is a hyper equality algebra. Suppose $\vartheta : \mathcal{L} \to [0,1]$ is defined by $\vartheta(1) = \beta$ and $\vartheta(0) = \vartheta(\mathfrak{i}) = \alpha$, where $0 < \alpha < \beta < 1$. Then ϑ is a strong fuzzy filter of $\mathcal{L}$.

Theorem 3.1. *Consider $\vartheta \in \mathcal{FS}(\mathcal{L})$. Then ϑ is a strong fuzzy filter of $\mathcal{L}$ which satisfies the sup-property if and only if for all $\mathfrak{t} \in [0,1]$, $\vartheta_{\mathfrak{t}} \neq \emptyset$ is a strong filter of $\mathcal{L}$.*

Proof. Suppose ϑ is a strong fuzzy filter of $\mathcal{L}$ and $\mathfrak{t} \in [0,1]$ such that $\vartheta_{\mathfrak{t}} \neq \emptyset$. Then $\mathfrak{x} \in \vartheta_{\mathfrak{t}}$. Assume $\mathfrak{y} \in \mathcal{L}$ such that $\mathfrak{x} \preccurlyeq \mathfrak{y}$. Since ϑ is strong, by (FF2) we have $\mathfrak{t} \preccurlyeq \vartheta(\mathfrak{x}) \preccurlyeq \vartheta(\mathfrak{y})$ and so $\mathfrak{y} \in \vartheta_{\mathfrak{t}}$. Now, suppose for any $\mathfrak{y} \in \mathcal{L}$, $(\mathfrak{x} \backsim \mathfrak{y}) \cap \vartheta_{\mathfrak{t}} \neq \emptyset$. Thus, there is $\mathfrak{z} \in (\mathfrak{x} \backsim \mathfrak{y}) \cap \vartheta_{\mathfrak{t}}$ where $\vartheta(\mathfrak{z}) \succcurlyeq \mathfrak{t}$. By (FF1), since $\vartheta(\mathfrak{z}) \preccurlyeq \sup_{\mathfrak{z} \in \mathfrak{x} \backsim \mathfrak{y}} \vartheta(\mathfrak{z})$, we have

$$\mathfrak{t} \preccurlyeq \vartheta(\mathfrak{x}) \barwedge \vartheta(\mathfrak{z}) \preccurlyeq \vartheta(\mathfrak{x}) \barwedge \Big(\sup_{\mathfrak{z} \in \mathfrak{x} \backsim \mathfrak{y}} \vartheta(\mathfrak{z})\Big) \preccurlyeq \vartheta(\mathfrak{y}).$$

Hence, $\mathfrak{y} \in \vartheta_{\mathfrak{t}}$. Therefore, $\vartheta_{\mathfrak{t}}$ is a strong filter of $\mathcal{L}$.

Conversely, suppose $\mathfrak{x} \in \mathcal{L}$. Clearly, $\mathfrak{x} \in \vartheta_{\vartheta(\mathfrak{x})}$ and so $\vartheta_{\vartheta(\mathfrak{x})} \neq \emptyset$. If for $\mathfrak{y} \in \mathcal{L}$, $\mathfrak{x} \preccurlyeq \mathfrak{y}$, from $\vartheta_{\vartheta(\mathfrak{x})}$ is strong, then $\mathfrak{y} \in \vartheta_{\vartheta(\mathfrak{x})}$. Thus, $\vartheta(\mathfrak{x}) \preccurlyeq \vartheta(\mathfrak{y})$ and so $(FF2)$ holds. Consider $\mathfrak{t}_1, \mathfrak{t}_2 \in [0,1]$ and $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$ such that $\vartheta(\mathfrak{x}) = \mathfrak{t}_1$ and $\mathfrak{t}_2 = \sup_{\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}} \vartheta(\mathfrak{i})$. Suppose $\mathfrak{s} = \min\{\mathfrak{t}_1, \mathfrak{t}_2\}$. Since $\mathcal{L}$ satisfies the sup-property, there exists $\mathfrak{z} \in \mathfrak{x} \backsim \mathfrak{y}$ such that $\vartheta(\mathfrak{z}) = \mathfrak{t}_2$ and so $\mathfrak{s} \preccurlyeq \vartheta(\mathfrak{z})$. Thus, $\mathfrak{z} \in (\mathfrak{x} \backsim \mathfrak{y}) \cap \vartheta_{\mathfrak{s}}$. Since $\mathfrak{x} \in \vartheta_{\mathfrak{s}}$, $(\mathfrak{x} \backsim \mathfrak{y}) \cap \vartheta_{\mathfrak{s}} \neq \emptyset$ and $\vartheta_{\mathfrak{s}}$ is a strong filter of $\mathcal{L}$, we get $\mathfrak{y} \in \vartheta_{\mathfrak{s}}$. Hence,

$$\vartheta(\mathfrak{x}) \barwedge \Big(\sup_{\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}} \vartheta(\mathfrak{i})\Big) = \vartheta(\mathfrak{x}) \barwedge \vartheta(\mathfrak{z}) = \min\{\mathfrak{t}_1, \mathfrak{t}_2\} = \mathfrak{s} \preccurlyeq \vartheta(\mathfrak{y}).$$

Therefore, ϑ is a strong fuzzy filter of $\mathcal{L}$. □

Example 3.2. Consider Example 3.1 (*ii*) and let $\alpha = 0.3$ and $\beta = 0.7$. Then ϑ is a strong fuzzy filter of $\mathcal{L}$ and $\vartheta_\alpha = \{0, \mathfrak{i}, 1\}$ and $\vartheta_\beta = \{1\}$. Obviously, ϑ_β and ϑ_α are strong filters of $\mathcal{L}$.

Theorem 3.2. *Assume $\vartheta \in \mathcal{FS}(\mathcal{L})$. Then, for each $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$ and $\mathcal{S}, \mathcal{R}, \mathcal{T} \subseteq \mathcal{L}$, the following statements are equivalent:*

(i) *ϑ is a strong fuzzy filter of $\mathcal{L}$;*

(ii) *$\vartheta(\mathfrak{x}) \preccurlyeq \vartheta(1)$ and $\vartheta(\mathfrak{x}) \barwedge \Big(\sup_{\mathfrak{z} \in \mathfrak{x} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{z})\Big) \preccurlyeq \vartheta(\mathfrak{y})$;*

(iii) *if $\mathcal{S} \lll \mathcal{R} \curvearrowright \mathcal{T}$, then for all $\mathfrak{x} \in \mathcal{S}$, there exists $\mathfrak{y} \in \mathcal{R}$ and $\mathfrak{z} \in \mathcal{T}$ such that*

$$\vartheta(\mathfrak{x}) \barwedge \vartheta(\mathfrak{y}) \preccurlyeq \vartheta(\mathfrak{z}), \quad (\sup_{\mathfrak{i} \in \mathcal{S}} \vartheta(\mathfrak{i})) \barwedge \Big(\sup_{\mathfrak{h} \in \mathcal{R}} \vartheta(\mathfrak{h})\Big) \preccurlyeq \sup_{\mathfrak{o} \in \mathcal{T}} \vartheta(\mathfrak{o}).$$

Proof. (i)⇒(ii) By (FF2) for any $\mathfrak{x} \in \mathcal{L}$ since $\mathfrak{x} \preccurlyeq 1$, we have $\vartheta(\mathfrak{x}) \preccurlyeq \vartheta(1)$ and for any $\mathfrak{x} \in \mathcal{S}$ and $\mathfrak{y} \in \mathcal{R}$

$$\vartheta(\mathfrak{x}) \barwedge \Big(\sup_{\mathfrak{z} \in \mathfrak{x} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{z})\Big) = \vartheta(\mathfrak{x}) \barwedge \Big(\sup_{\mathfrak{z} \in \mathfrak{x} \backsim (\mathfrak{x} \barwedge \mathfrak{y})} \vartheta(\mathfrak{z})\Big) \preccurlyeq \vartheta(\mathfrak{x} \barwedge \mathfrak{y}) \preccurlyeq \vartheta(\mathfrak{y}).$$

(ii)⇒(i) Suppose $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$ such that $\mathfrak{x} \preccurlyeq \mathfrak{y}$. Then by Proposition 2.2 (P4), $1 \in \mathfrak{x} \curvearrowright \mathfrak{y}$. Since, for any $\mathfrak{x} \in \mathcal{L}$, $\vartheta(\mathfrak{x}) \preccurlyeq \vartheta(1)$, we have $\sup_{\mathfrak{z} \in \mathfrak{x} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{z}) = \vartheta(1)$. Then by (ii) we have

$$\vartheta(\mathfrak{x}) = \vartheta(\mathfrak{x}) \barwedge \vartheta(1) = \vartheta(\mathfrak{x}) \barwedge \left(\sup_{\mathfrak{z} \in \mathfrak{x} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{z})\right) \preccurlyeq \vartheta(\mathfrak{y}).$$

By (ii) and Proposition 2.2 (P3), for any $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, we have

$$\vartheta(\mathfrak{x}) \barwedge \left(\sup_{\mathfrak{z} \in \mathfrak{x} \backsim \mathfrak{y}} \vartheta(\mathfrak{z})\right) \preccurlyeq \vartheta(\mathfrak{x}) \barwedge \left(\sup_{\mathfrak{z} \in \mathfrak{x} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{z})\right) \preccurlyeq \vartheta(\mathfrak{y}).$$

Hence, ϑ is a strong fuzzy filter of $\mathcal{L}$.

(ii)⇒(iii) Consider $\mathcal{S} \lll \mathcal{R} \curvearrowright \mathcal{T}$. Then, for any $\mathfrak{x} \in \mathcal{S}$, there exists $\mathfrak{v} \in \mathcal{R} \curvearrowright \mathcal{T}$ such that $\mathfrak{x} \preccurlyeq \mathfrak{v}$. Since $\mathfrak{v} \in \mathcal{R} \curvearrowright \mathcal{T}$, there are $\mathfrak{y} \in \mathcal{R}$ and $\mathfrak{z} \in \mathcal{T}$ such that $\mathfrak{v} \in \mathfrak{y} \curvearrowright \mathfrak{z}$ and so $\mathfrak{x} \lll \mathfrak{y} \curvearrowright \mathfrak{z}$. From $\mathfrak{x} \lll \mathfrak{y} \curvearrowright \mathfrak{z}$, there exists $\mathfrak{w} \in \mathfrak{y} \curvearrowright \mathfrak{z}$ such that $\mathfrak{x} \preccurlyeq \mathfrak{w}$. Then $\vartheta(\mathfrak{x}) \preccurlyeq \vartheta(\mathfrak{w})$, and so $\vartheta(\mathfrak{x}) \preccurlyeq \sup_{\mathfrak{w} \in \mathfrak{y} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{w})$. Hence,

$$\vartheta(\mathfrak{x}) \barwedge \vartheta(\mathfrak{y}) \preccurlyeq \left(\sup_{\mathfrak{w} \in \mathfrak{y} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{w})\right) \barwedge \vartheta(\mathfrak{y}) \preccurlyeq \vartheta(\mathfrak{z}).$$

Moreover,

$$\sup_{\mathfrak{i} \in \mathcal{S}} \vartheta(\mathfrak{i}) \preccurlyeq \sup_{\mathfrak{w} \in \mathfrak{y} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{w}) \preccurlyeq \sup_{\mathfrak{v} \in \mathcal{R} \curvearrowright \mathcal{T}} \vartheta(\mathfrak{v}).$$

Then, by (FF1′) and (FF2′), we get

$$\begin{aligned}(\sup_{\mathfrak{i} \in \mathcal{S}} \vartheta(\mathfrak{i})) \barwedge \left(\sup_{\mathfrak{h} \in \mathcal{R}} \vartheta(\mathfrak{h})\right) &\preccurlyeq \left(\sup_{\mathfrak{w} \in \mathfrak{y} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{w})\right) \barwedge \left(\sup_{\mathfrak{h} \in \mathcal{R}} \vartheta(\mathfrak{h})\right) \\ &\preccurlyeq (\sup_{\mathfrak{v} \in \mathcal{R} \curvearrowright \mathcal{T}} \vartheta(\mathfrak{v})) \barwedge \left(\sup_{\mathfrak{h} \in \mathcal{R}} \vartheta(\mathfrak{h})\right) \\ &= \left(\sup_{\mathfrak{v} \in \mathcal{R} \backsim (\mathcal{R} \barwedge \mathcal{T})} \vartheta(\mathfrak{v})\right) \barwedge \left(\sup_{\mathfrak{h} \in \mathcal{R}} \vartheta(\mathfrak{h})\right) \\ &\preccurlyeq \left(\sup_{\mathfrak{u} \in \mathcal{R} \barwedge \mathcal{T}} \vartheta(\mathfrak{u})\right) \preccurlyeq \sup_{\mathfrak{o} \in \mathcal{T}} \vartheta(\mathfrak{o}).\end{aligned} \tag{3.1}$$

(iii)⇒(ii) Since for any $\mathfrak{x} \in \mathcal{L}$, $\mathfrak{x} \preccurlyeq 1$ by Proposition 2.2 (P9), $\mathfrak{x} \lll 1 \backsim \mathfrak{x}$. Then by Proposition 2.2 (P3), $\mathfrak{x} \lll 1 \backsim \mathfrak{x} \lll \mathfrak{x} \curvearrowright 1$. Now, by (iii) for all $\mathfrak{x} \in \mathcal{L}$, $\vartheta(\mathfrak{x}) = \vartheta(\mathfrak{x}) \barwedge \vartheta(\mathfrak{x}) \preccurlyeq \vartheta(1)$. Also, since, for any $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, $\mathfrak{x} \curvearrowright \mathfrak{y} \lll \mathfrak{x} \curvearrowright \mathfrak{y}$, by (iii) (indeed by (3.1)), we have $\left(\sup_{\mathfrak{z} \in \mathfrak{x} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{z})\right) \barwedge \vartheta(\mathfrak{x}) \preccurlyeq \vartheta(\mathfrak{y})$. □

Example 3.3. Consider Example 3.1 (*ii*) and let $\alpha = 0.3$ and $\beta = 0.7$. Then ϑ is a strong fuzzy filter of $\mathcal{L}$. Clearly, Theorem 3.2 holds. For instance, $\vartheta(0) = \vartheta(\mathfrak{i}) \preccurlyeq \vartheta(1)$ and

$$\vartheta(0) \barwedge \left(\sup_{\mathfrak{z} \in 0 \curvearrowright \mathfrak{i}} \vartheta(\mathfrak{z})\right) = \vartheta(0) \barwedge \left(\sup_{\mathfrak{z} \in 0 \backsim (0 \barwedge \mathfrak{i}) = 0 \backsim 0 = \{1\}} \vartheta(\mathfrak{z})\right) = \vartheta(0) \barwedge \vartheta(1) \preccurlyeq \vartheta(\mathfrak{i}).$$

Assume $\vartheta \in \mathcal{FS}(\mathcal{L})$. For any $\mathcal{S} \subseteq \mathcal{L}$, we define a map $\varpi^{\mathcal{S}} : \mathcal{L} \to [0, 1]$ by $\varpi^{\mathcal{S}}(\mathfrak{x}) = \sup_{\mathfrak{z} \in \mathcal{S} \backsim \mathfrak{x}} \vartheta(\mathfrak{z})$, for any $\mathfrak{x} \in \mathcal{L}$. In particular, $\varpi^{\mathfrak{y}}(\mathfrak{x}) = \sup_{\mathfrak{z} \in \mathfrak{y} \backsim \mathfrak{x}} \vartheta(\mathfrak{z})$ and for any $\mathcal{S}, \mathcal{R} \subseteq \mathcal{L}$, $\varpi^{\mathcal{S}}(\mathcal{R}) = \sup_{\mathfrak{x} \in \mathcal{R}} \varpi^{\mathcal{S}}(\mathfrak{x}) = \sup_{\mathfrak{z} \in \mathcal{S} \backsim \mathcal{R}} \vartheta(\mathfrak{z})$. If for all $\mathfrak{z} \in \mathcal{L}$, $\varpi^{\mathfrak{x}}(\mathfrak{z}) \preccurlyeq \varpi^{\mathfrak{y}}(\mathfrak{z})$, then we denote it by $\varpi^{\mathfrak{x}} \preccurlyeq \varpi^{\mathfrak{y}}$.

Proposition 3.1. *Suppose ϑ is a strong fuzzy filter of $\mathcal{L}$. If for $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, $\varpi^{\mathfrak{x}} = \varpi^{\mathfrak{y}}$, then $\vartheta(\mathfrak{x}) = \vartheta(\mathfrak{y})$.*

Proof. Let $\varpi^{\mathfrak{x}} = \varpi^{\mathfrak{y}}$, for $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$. Then by Proposition 3.2 (i), $\sup_{\mathfrak{z}\in\mathfrak{x}\backsim\mathfrak{y}}\vartheta(\mathfrak{z}) = \varpi^{\mathfrak{x}}(\mathfrak{y}) = \vartheta(1)$. From ϑ is strong, $\vartheta(\mathfrak{x}), \vartheta(\mathfrak{y}) \preccurlyeq \vartheta(1)$ and $\vartheta(\mathfrak{x}) \barwedge \left(\sup_{\mathfrak{z}\in\mathfrak{x}\backsim\mathfrak{y}}\vartheta(\mathfrak{z})\right) \preccurlyeq \vartheta(\mathfrak{y})$. Thus, $\vartheta(\mathfrak{x}) = \vartheta(\mathfrak{x}) \barwedge \vartheta(1) \preccurlyeq \vartheta(\mathfrak{y})$. Similarly, $\vartheta(\mathfrak{y}) \preccurlyeq \vartheta(\mathfrak{x})$. Hence, $\vartheta(\mathfrak{x}) = \vartheta(\mathfrak{y})$. ☐

Example 3.4. Let $\mathcal{L} = \{0, \mathfrak{i}, \mathfrak{h}, 1\}$ be a set. Define the operation $\barwedge$ and $\backsim$ on $\mathcal{L}$ as follows:

$\backsim$	0	$\mathfrak{i}$	$\mathfrak{h}$	1
0	$\{1\}$	$\{1\}$	$\{1\}$	$\{0,\mathfrak{i}\}$
$\mathfrak{i}$	$\{1\}$	$\{1\}$	$\{\mathfrak{i},1\}$	$\{\mathfrak{i}\}$
$\mathfrak{h}$	$\{\mathfrak{h},1\}$	$\{\mathfrak{i},1\}$	$\{1\}$	$\{\mathfrak{h},1\}$
1	$\{0,\mathfrak{i}\}$	$\{\mathfrak{i}\}$	$\{\mathfrak{h}\}$	$\{1\}$

,

$\barwedge$	0	$\mathfrak{i}$	$\mathfrak{h}$	1
0	0	0	0	0
$\mathfrak{i}$	0	$\mathfrak{i}$	0	$\mathfrak{i}$
$\mathfrak{h}$	0	0	$\mathfrak{h}$	$\mathfrak{h}$
1	0	$\mathfrak{i}$	$\mathfrak{h}$	1

.

Define ϑ on $\mathcal{L}$ by $\vartheta(0) = \vartheta(\alpha) = \vartheta(\mathfrak{h}) = \alpha$ and $\vartheta(1) = \beta$, where $0 \prec \alpha \prec \beta \prec 1$. Clearly, $\varpi^{\mathfrak{i}} = \varpi^{0}$, then $\vartheta(\mathfrak{i}) = \vartheta(0)$. But the converse may not be true, since $\vartheta(\mathfrak{i}) = \vartheta(\mathfrak{h})$ but

$$\alpha = \vartheta(\mathfrak{i}) = \varpi^{\mathfrak{i}}(1) \neq \varpi^{\mathfrak{h}}(1) = \sup\{\vartheta(\mathfrak{h}), \vartheta(1)\} = \vartheta(1) = \beta.$$

Proposition 3.2. *Consider ϑ is a strong fuzzy filter of $\mathcal{L}$. Then for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{u}, \mathfrak{v} \in \mathcal{L}$ and $\mathcal{S}, \mathcal{R} \subseteq \mathcal{L}$, we have:*

(i) *$\varpi^{\mathfrak{x}} = \varpi^{\mathfrak{y}}$ if and only if $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \vartheta(1)$;*
(ii) *$\varpi^{\mathcal{S}} = \varpi^{\mathcal{R}}$ if and only if $\varpi^{\mathcal{S}}(\mathcal{R}) = \vartheta(1)$;*
(iii) *if $\varpi^{\mathfrak{x}} = \vartheta$, then $\vartheta(\mathfrak{x}) = \vartheta(1)$;*
(iv) *if $\mathfrak{x} \preccurlyeq \mathfrak{y}$, then $\vartheta(\mathfrak{x}) \preccurlyeq \varpi^{\mathfrak{y}}(\mathfrak{x})$;*
(v) *if $\varpi^{\mathfrak{x}} = \varpi^{\mathfrak{y}}$ and $\varpi^{\mathfrak{u}} = \varpi^{\mathfrak{v}}$, then $\varpi^{\mathfrak{x}\backsim\mathfrak{u}} = \varpi^{\mathfrak{y}\backsim\mathfrak{u}}$, $\varpi^{\mathfrak{u}\backsim\mathfrak{y}} = \varpi^{\mathfrak{v}\backsim\mathfrak{y}}$ and $\varpi^{\mathfrak{u}\barwedge\mathfrak{y}} = \varpi^{\mathfrak{y}\barwedge\mathfrak{v}}$;*
(vi) *if $\mathfrak{y} \backsim \mathfrak{u} = \mathfrak{u} \backsim \mathfrak{y}$ and $\mathfrak{y} \backsim \mathfrak{v} = \mathfrak{v} \backsim \mathfrak{y}$, then $\varpi^{\mathfrak{x}\backsim\mathfrak{u}} = \varpi^{\mathfrak{y}\backsim\mathfrak{v}}$.*

Proof. (i) Suppose $\varpi^{\mathfrak{x}} = \varpi^{\mathfrak{y}}$. Then for all $\mathfrak{z} \in \mathcal{L}$, $\varpi^{\mathfrak{x}}(\mathfrak{z}) = \varpi^{\mathfrak{y}}(\mathfrak{z})$. Consider $\mathfrak{y} = \mathfrak{z}$. Then $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \varpi^{\mathfrak{y}}(\mathfrak{y}) = \sup_{\mathfrak{z}\in\mathfrak{y}\backsim\mathfrak{y}} \vartheta(\mathfrak{z})$. By ($\mathcal{HE}3$), $1 \in \mathfrak{y} \backsim \mathfrak{y}$ and so $\sup_{\mathfrak{z}\in\mathfrak{y}\backsim\mathfrak{y}} \vartheta(\mathfrak{z}) = \vartheta(1)$. Hence, $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \vartheta(1)$.

Conversely, assume that for each $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \vartheta(1)$. By ($\mathcal{HE}2$) and ($\mathcal{HE}7$), for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, $\mathfrak{x} \backsim \mathfrak{y} \lll (\mathfrak{x} \backsim \mathfrak{z}) \backsim (\mathfrak{y} \backsim \mathfrak{z}) \lll (\mathfrak{y} \backsim \mathfrak{z}) \backsim (\mathfrak{x} \backsim \mathfrak{z})$. Then for any $\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}$, there exists $\mathfrak{h} \in (\mathfrak{y} \backsim \mathfrak{z}) \backsim (\mathfrak{x} \backsim \mathfrak{z})$ such that $\mathfrak{i} \preccurlyeq \mathfrak{h}$. Since ϑ is a strong fuzzy filter of $\mathcal{L}$, by (FF2), for any $\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}$, $\vartheta(\mathfrak{i}) \preccurlyeq \vartheta(\mathfrak{h})$ and so $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \sup_{\mathfrak{i}\in\mathfrak{x}\backsim\mathfrak{y}} \vartheta(\mathfrak{i}) \preccurlyeq \vartheta(\mathfrak{h}) \preccurlyeq \sup_{\mathfrak{h}\in(\mathfrak{y}\backsim\mathfrak{z})\backsim(\mathfrak{x}\backsim\mathfrak{z})} \vartheta(\mathfrak{h})$. Then

$$\begin{aligned}\varpi^{\mathfrak{y}}(\mathfrak{z}) \barwedge \varpi^{\mathfrak{x}}(\mathfrak{y}) &= \varpi^{\mathfrak{y}}(\mathfrak{z}) \barwedge \left(\sup_{\mathfrak{i}\in\mathfrak{x}\backsim\mathfrak{y}} \vartheta(\mathfrak{i})\right) \preccurlyeq \left(\sup_{\mathfrak{u}\in\mathfrak{y}\backsim\mathfrak{z}} \vartheta(\mathfrak{u})\right) \barwedge \left(\sup_{\mathfrak{h}\in(\mathfrak{y}\backsim\mathfrak{z})\backsim(\mathfrak{x}\backsim\mathfrak{z})} \vartheta(\mathfrak{h})\right)\\ &\preccurlyeq \sup_{\mathfrak{v}\in\mathfrak{x}\backsim\mathfrak{z}} \vartheta(\mathfrak{v}) = \varpi^{\mathfrak{x}}(\mathfrak{z}).\end{aligned}$$

Since for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \vartheta(1)$, by above relation, we have

$$\varpi^{\mathfrak{y}}(\mathfrak{z}) = \varpi^{\mathfrak{y}}(\mathfrak{z}) \barwedge \vartheta(1) = \varpi^{\mathfrak{y}}(\mathfrak{z}) \barwedge \varpi^{\mathfrak{x}}(\mathfrak{y}) \preccurlyeq \varpi^{\mathfrak{x}}(\mathfrak{z}).$$

This shows that $\varpi^{\mathfrak{y}} \preccurlyeq \varpi^{\mathfrak{x}}$. By the similar way, we have $\varpi^{\mathfrak{x}} \preccurlyeq \varpi^{\mathfrak{y}}$. Hence, $\varpi^{\mathfrak{x}} = \varpi^{\mathfrak{y}}$.

(ii) Similar to (i).

(iii) Suppose $\varpi^{\mathfrak{x}} = \vartheta$. Then for any $\mathfrak{y} \in \mathcal{L}$, $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \sup_{\mathfrak{z}\in\mathfrak{x}\backsim\mathfrak{y}} \vartheta(\mathfrak{z}) = \vartheta(\mathfrak{y})$. Let $\mathfrak{x} = \mathfrak{y}$. Since by ($\mathcal{HE}3$), $1 \in \mathfrak{x} \backsim \mathfrak{x}$, we get $\vartheta(\mathfrak{x}) = \varpi^{\mathfrak{x}}(\mathfrak{x}) = \sup_{\mathfrak{z}\in\mathfrak{x}\backsim\mathfrak{x}} \vartheta(\mathfrak{z}) = \vartheta(1)$.

(iv) Assume $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$ where $\mathfrak{x} \preccurlyeq \mathfrak{y}$. By Proposition 2.2 (P9), $\mathfrak{x} \lll \mathfrak{y} \backsim \mathfrak{x}$. Then there is $\mathfrak{i} \in \mathfrak{y} \backsim \mathfrak{x}$ such that $\mathfrak{x} \preccurlyeq \mathfrak{i}$. Since ϑ is strong, by (FF2) we have

$$\vartheta(\mathfrak{x}) \preccurlyeq \vartheta(\mathfrak{i}) \preccurlyeq \sup_{\mathfrak{i} \in \mathfrak{y} \backsim \mathfrak{x}} \vartheta(\mathfrak{i}) = \varpi^{\mathfrak{y}}(\mathfrak{x}).$$

(v) If $\varpi^{\mathfrak{x}} = \varpi^{\mathfrak{y}}$ and $\varpi^{\mathfrak{u}} = \varpi^{\mathfrak{v}}$, then by (i), $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \vartheta(1) = \varpi^{\mathfrak{u}}(\mathfrak{v})$. By $(\mathcal{HE}7)$,

$$\mathfrak{x} \backsim \mathfrak{y} \lll (\mathfrak{x} \backsim \mathfrak{u}) \backsim (\mathfrak{y} \backsim \mathfrak{u}), \quad \mathfrak{u} \backsim \mathfrak{v} \lll (\mathfrak{u} \backsim \mathfrak{y}) \backsim (\mathfrak{v} \backsim \mathfrak{y}).$$

Thus, for any $\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}$, there exists $\mathfrak{h} \in (\mathfrak{x} \backsim \mathfrak{u}) \backsim (\mathfrak{y} \backsim \mathfrak{u})$ such that $\mathfrak{i} \preccurlyeq \mathfrak{h}$. From ϑ is strong, by (FF2) we have $\vartheta(\mathfrak{i}) \preccurlyeq \vartheta(\mathfrak{h})$ and so

$$\vartheta(1) = \varpi^{\mathfrak{x}}(\mathfrak{y}) = \sup_{\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}} \vartheta(\mathfrak{i}) \preccurlyeq \vartheta(\mathfrak{h}) \preccurlyeq \sup_{\mathfrak{h} \in (\mathfrak{x} \backsim \mathfrak{u}) \backsim (\mathfrak{y} \backsim \mathfrak{u})} \vartheta(\mathfrak{h}).$$

Hence, $\varpi^{\mathfrak{x} \backsim \mathfrak{u}}(\mathfrak{y} \backsim \mathfrak{u}) = \vartheta(1)$. Now, by (i), we have $\varpi^{\mathfrak{x} \backsim \mathfrak{u}} = \varpi^{\mathfrak{y} \backsim \mathfrak{u}}$. Similarly, $\varpi^{\mathfrak{u} \backsim \mathfrak{y}} = \varpi^{\mathfrak{v} \backsim \mathfrak{y}}$. Moreover, by $(\mathcal{HE}6)$, $\mathfrak{x} \backsim \mathfrak{y} \lll (\mathfrak{x} \barwedge \mathfrak{u}) \backsim (\mathfrak{y} \barwedge \mathfrak{u})$ and $\mathfrak{u} \backsim \mathfrak{v} \lll (\mathfrak{u} \barwedge \mathfrak{y}) \backsim (\mathfrak{v} \barwedge \mathfrak{y})$. Then for any $\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}$ there is $\mathfrak{h} \in (\mathfrak{x} \barwedge \mathfrak{u}) \backsim (\mathfrak{y} \barwedge \mathfrak{u})$ such that $\mathfrak{i} \preccurlyeq \mathfrak{h}$. Since ϑ is strong, by (FF2), we have $\vartheta(\mathfrak{i}) \preccurlyeq \vartheta(\mathfrak{h})$ and so

$$\vartheta(1) = \varpi^{\mathfrak{x}}(\mathfrak{y}) = \sup_{\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}} \vartheta(\mathfrak{i}) \preccurlyeq \vartheta(\mathfrak{h}) \preccurlyeq \sup_{\mathfrak{h} \in (\mathfrak{x} \barwedge \mathfrak{u}) \backsim (\mathfrak{y} \barwedge \mathfrak{u})} \vartheta(\mathfrak{h}).$$

Hence, $\varpi^{\mathfrak{x} \barwedge \mathfrak{u}}(\mathfrak{y} \barwedge \mathfrak{u}) = \vartheta(1)$. Now, by (i), we have $\varpi^{\mathfrak{x} \barwedge \mathfrak{u}} = \varpi^{\mathfrak{y} \barwedge \mathfrak{u}}$.

(vi) Similar to (v). □

Corollary 3.1. *Let $\mathcal{L} = \langle \mathcal{L}; \backsim, \barwedge, 1 \rangle$ be symmetric and ϑ be a strong fuzzy filter of $\mathcal{L}$. If for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{u}, \mathfrak{v} \in \mathcal{L}$, $\varpi^{\mathfrak{x}} = \varpi^{\mathfrak{y}}$ and $\varpi^{\mathfrak{u}} = \varpi^{\mathfrak{v}}$, then $\varpi^{\mathfrak{x} \backsim \mathfrak{u}} = \varpi^{\mathfrak{y} \backsim \mathfrak{v}}$ and $\varpi^{\mathfrak{u} \barwedge \mathfrak{y}} = \varpi^{\mathfrak{y} \barwedge \mathfrak{v}}$.*

Note. Consider ϑ is a strong fuzzy filter of $\mathcal{L}$ and

$$\mathcal{L}/\varpi = \{\varpi^{S} \mid \mathcal{S} \subseteq \mathcal{L}\}.$$

For any $\varpi^{\mathcal{S}}, \varpi^{\mathcal{R}} \in \mathcal{L}/\varpi$, we consider the operations $\overline{\backsim}$ and $\overline{\overline{\barwedge}}$ on $\mathcal{L}/\varpi$ as follow:

$$\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{R}} = \varpi^{\mathcal{S} \backsim \mathcal{R}} \quad \text{and} \quad \varpi^{\mathcal{S}} \overline{\overline{\barwedge}} \varpi^{\mathcal{R}} = \varpi^{\mathcal{S} \barwedge \mathcal{R}},$$

where $\mathcal{S} \backsim \mathcal{R} = \bigcup_{\mathfrak{i} \in \mathcal{S}, \mathfrak{h} \in \mathcal{R}} \mathfrak{i} \backsim \mathfrak{h}$ and $\mathcal{S} \barwedge \mathcal{R} = \{\mathfrak{i} \barwedge \mathfrak{h} \mid \mathfrak{i} \in \mathcal{S}, \mathfrak{h} \in \mathcal{R}\}$. Also, we consider $\varpi^1 = \varpi^{\mathcal{L}}$. Now, we prove that these operations are well-defined. Assume $\varpi^{\mathcal{S}}, \varpi^{\mathcal{R}}, \varpi^{\mathcal{T}} \in \mathcal{L}/\varpi$ such that $\varpi^{\mathcal{S}} = \varpi^{\mathcal{R}}$. Then by Proposition 3.2 (ii), $\varpi^{\mathcal{S}}(\mathcal{R}) = \vartheta(1)$, and so $\sup_{\alpha \in \mathcal{S} \backsim \mathcal{R}} \vartheta(\alpha) = \vartheta(1)$. At first we prove $\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{T}} = \varpi^{\mathcal{R}} \overline{\backsim} \varpi^{\mathcal{T}}$. For this, by Proposition 3.2 (ii), we show $\varpi^{\mathcal{S} \backsim \mathcal{T}}(\mathcal{R} \backsim \mathcal{T}) = \omega(1)$. By definition, for any $\alpha \in \mathcal{S} \backsim \mathcal{R}$, there exists $\mathfrak{i} \in \mathcal{S}$ and $\mathfrak{h} \in \mathcal{R}$ such that $\alpha \in \mathfrak{i} \backsim \mathfrak{h}$. By $(\mathcal{HE}7)$, for any $\mathfrak{o} \in \mathcal{T}$, we get $\mathfrak{i} \backsim \mathfrak{h} \lll (\mathfrak{i} \backsim \mathfrak{o}) \backsim (\mathfrak{h} \backsim \mathfrak{o})$. So, for any $\alpha \in \mathfrak{i} \backsim \mathfrak{h}$, there exists $\beta \in (\mathfrak{i} \backsim \mathfrak{o}) \backsim (\mathfrak{h} \backsim \mathfrak{o})$ such that $\alpha \preccurlyeq \beta$. Since ϑ is strong, we have $\vartheta(\alpha) \preccurlyeq \vartheta(\beta)$, and so $\sup_{\alpha \in \mathcal{S} \backsim \mathcal{R}} \vartheta(\alpha) \preccurlyeq \vartheta(\beta)$. Then

$$\varpi^{\mathcal{S} \backsim \mathcal{T}}(\mathcal{R} \backsim \mathcal{T}) = \sup_{\mathfrak{z} \in (\mathcal{S} \backsim \mathcal{T}) \backsim (\mathcal{R} \backsim \mathcal{T})} \vartheta(\mathfrak{z}) \succcurlyeq \vartheta(\beta) \succcurlyeq \sup_{\alpha \in \mathcal{S} \backsim \mathcal{R}} \vartheta(\alpha) \succcurlyeq \vartheta(1).$$

Hence, $\varpi^{\mathcal{S} \backsim \mathcal{T}}(\mathcal{R} \backsim \mathcal{T}) = \vartheta(1)$. By Proposition 3.2 (ii), $\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{T}} = \varpi^{\mathcal{R}} \overline{\backsim} \varpi^{\mathcal{T}}$.

Now, we prove that $\varpi^{\mathcal{S}} \overline{\overline{\barwedge}} \varpi^{\mathcal{T}} = \varpi^{\mathcal{R}} \overline{\overline{\barwedge}} \varpi^{\mathcal{T}}$. For this, by Proposition 3.2 (ii), we show $\varpi^{\mathcal{S} \barwedge \mathcal{T}}(\mathcal{R} \barwedge \mathcal{T}) = \omega(1)$. By definition, for any $\alpha \in \mathcal{S} \backsim \mathcal{R}$, there exists $\mathfrak{i} \in \mathcal{S}$ and $\mathfrak{h} \in \mathcal{R}$ such that $\alpha \in \mathfrak{i} \backsim \mathfrak{h}$. By $(\mathcal{HE}6)$, for any $\mathfrak{o} \in \mathcal{T}$, we have $\mathfrak{i} \backsim \mathfrak{h} \lll (\mathfrak{i} \barwedge \mathfrak{o}) \backsim (\mathfrak{h} \barwedge \mathfrak{o})$.

So for any $\alpha \in \mathfrak{i} \backsim \mathfrak{h}$, there exists $\beta \in (\mathfrak{i} \barwedge \mathfrak{o}) \backsim (\mathfrak{h} \barwedge \mathfrak{o})$ such that $\alpha \preccurlyeq \beta$. Since ϑ is strong, we obtain $\vartheta(\alpha) \preccurlyeq \vartheta(\beta)$, and so $\sup_{\alpha\in\mathcal{S}\backsim\mathcal{R}} \vartheta(\alpha) \preccurlyeq \vartheta(\beta)$. Then

$$\varpi^{\mathcal{S}\barwedge\mathcal{T}}(\mathcal{R} \barwedge \mathcal{T}) = \sup_{\mathfrak{z}\in(\mathcal{S}\barwedge\mathcal{T})\backsim(\mathcal{R}\barwedge\mathcal{T})} \vartheta(\mathfrak{z}) \succcurlyeq \vartheta(\beta) \succcurlyeq \sup_{\alpha\in\mathcal{S}\backsim\mathcal{R}} \vartheta(\alpha) \succcurlyeq \vartheta(1).$$

Hence, $\varpi^{\mathcal{S}\barwedge\mathcal{T}}(\mathcal{R} \barwedge \mathcal{T}) = \vartheta(1)$. By Proposition 3.2 (ii), $\varpi^{\mathcal{S}}\overline{\barwedge}\varpi^{\mathcal{T}} = \varpi^{\mathcal{R}}\overline{\barwedge}\varpi^{\mathcal{T}}$. Therefore, these operations are well-defined.

Now, suppose $\mathcal{S}, \mathcal{R} \subseteq \mathcal{L}$. Then the relation $\preccurlyeq_\varpi$ on $\mathcal{L}/\varpi$ by $\varpi^{\mathcal{S}} \preccurlyeq_\varpi \varpi^{\mathcal{R}}$ if and only if for any $\mathcal{T} \subseteq \mathcal{L}$, $\varpi^{\mathcal{S}}(\mathcal{T}) \preccurlyeq \varpi^{\mathcal{R}}(\mathcal{T})$, is an order on $\mathcal{L}/\varpi$. By routine calculation, it is easy to see that $\varpi^{\mathcal{S}} \preccurlyeq_\varpi \varpi^{\mathcal{R}}$ if and only if $\varpi^{\mathcal{S}}\overline{\barwedge}\varpi^{\mathcal{R}} = \varpi^{\mathcal{S}}$ if and only if $\varpi^{\mathcal{S}} = \varpi^{\mathcal{S}\barwedge\mathcal{R}}$ if and only if $\varpi^{\mathcal{S}}(\mathcal{S} \barwedge \mathcal{R}) = \vartheta(1)$ (by Proposition 3.2 (ii)) if and only if $\sup_{\mathfrak{h}\in\mathcal{S}\backsim(\mathcal{S}\barwedge\mathcal{R})} \vartheta(\mathfrak{h}) = \vartheta(1)$ if and only if $\sup_{\mathfrak{h}\in\mathcal{S}\frown\mathcal{R}} \vartheta(\mathfrak{h}) = \vartheta(1)$.

Theorem 3.3. *Let $\mathcal{L} = \langle \mathcal{L}; \backsim, \barwedge, 1\rangle$ be symmetric and ϑ be a strong fuzzy filter of $\mathcal{L}$. Then $\mathcal{L}/\varpi = \langle \mathcal{L}/\varpi; \overline{\backsim}, \overline{\barwedge}, \varpi^1\rangle$ is a symmetric equality algebra.*

Proof. We prove that $\overline{\backsim}$ and $\overline{\barwedge}$ are well-defined. Clearly, $(\mathcal{L}/\varpi, \preccurlyeq_\varpi)$ is a poset. Now, we show that $\mathcal{L}/\varpi = \langle \mathcal{L}/\varpi; \overline{\backsim}, \overline{\barwedge}, \varpi^1\rangle$ is an equality algebra.

We have to prove that for any $\mathcal{S} \subseteq \mathcal{L}$, $\varpi^{\mathcal{S}} \preccurlyeq_\varpi \varpi^{\mathcal{L}}$. For this, suppose $\mathcal{R} \subseteq \mathcal{L}$. Then $\varpi^{\mathcal{L}}(\mathcal{R}) = \sup_{\mathfrak{z}\in\mathcal{L}\backsim\mathcal{R}} \vartheta(\mathfrak{z})$. Since $\mathcal{L} \backsim \mathcal{R} = \bigcup_{\mathfrak{g}\in\mathcal{L},\mathfrak{h}\in\mathcal{R}} \mathfrak{g} \backsim \mathfrak{h}$ and $\mathcal{R} \subseteq \mathcal{L}$, we get

$$1 \in \mathfrak{h} \backsim \mathfrak{h} \in \bigcup\nolimits_{\mathfrak{g}\in\mathcal{L},\mathfrak{h}\in\mathcal{R}} \mathfrak{g} \backsim \mathfrak{h} = \mathcal{L} \backsim \mathcal{R}.$$

Then $\varpi^{\mathcal{L}}(\mathcal{R}) = \vartheta(1)$. Moreover, since for any $\mathfrak{z} \in \mathcal{S} \backsim \mathcal{R}$, $\vartheta(\mathfrak{z}) \preccurlyeq \vartheta(1)$, we obtain $\varpi^{\mathcal{S}}(\mathcal{R}) = \sup_{\mathfrak{z}\in\mathcal{S}\backsim\mathcal{R}} \vartheta(\mathfrak{z}) \preccurlyeq \vartheta(1) = \varpi^{\mathcal{L}}(\mathcal{R})$. Hence for any $\mathcal{R} \subseteq \mathcal{L}$, $\varpi^{\mathcal{S}}(\mathcal{R}) \preccurlyeq \varpi^{\mathcal{L}}(\mathcal{R})$, and so $\varpi^{\mathcal{S}} \preccurlyeq_\varpi \varpi^{\mathcal{L}}$.

($\mathcal{E}2$) Since $\mathcal{L}$ is symmetric, for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, $\mathfrak{x} \backsim \mathfrak{y} = \mathfrak{y} \backsim \mathfrak{x}$. Consider $\mathcal{S}, \mathcal{R} \subseteq \mathcal{L}$. Then

$$\mathcal{S} \backsim \mathcal{R} = \bigcup\nolimits_{\mathfrak{i}\in\mathcal{S},\mathfrak{h}\in\mathcal{R}} \mathfrak{i} \backsim \mathfrak{h} = \bigcup\nolimits_{\mathfrak{h}\in\mathcal{R},\mathfrak{i}\in\mathcal{S}} \mathfrak{h} \backsim \mathfrak{i} = \mathcal{R} \backsim \mathcal{S}.$$

Then $\varpi^{\mathcal{S}}\overline{\backsim}\varpi^{\mathcal{R}} = \varpi^{\mathcal{S}\backsim\mathcal{R}} = \varpi^{\mathcal{R}\backsim\mathcal{S}} = \varpi^{\mathcal{R}}\overline{\backsim}\varpi^{\mathcal{S}}$.

($\mathcal{E}3$) Assume $\mathcal{S} \subseteq \mathcal{L}$. Then for any $\mathfrak{i} \in \mathcal{S}$ we have

$$1 \in 1 \backsim \mathfrak{i} \in (\mathfrak{i} \backsim \mathfrak{i}) \backsim \mathfrak{i} \in \bigcup\nolimits_{\mathfrak{g}\in\mathcal{L}\ \ \mathfrak{i},\mathfrak{h}\in\mathcal{S}} (\mathfrak{i} \backsim \mathfrak{h}) \backsim \mathfrak{g} = (\mathcal{S} \backsim \mathcal{S}) \backsim \mathcal{L}.$$

Then $\varpi^{\mathcal{S}\backsim\mathcal{S}}(\mathcal{L}) = \sup_{\mathfrak{i}\in(\mathcal{S}\backsim\mathcal{S})\backsim\mathcal{L}} \vartheta(\mathfrak{i}) = \vartheta(1)$. Hence, by Proposition 3.2 (ii), $\varpi^{\mathcal{S}\backsim\mathcal{S}} = \varpi^{\mathcal{L}}$. Therefore, $\varpi^{\mathcal{S}}\overline{\backsim}\varpi^{\mathcal{S}} = \varpi^{\mathcal{L}}$.

($\mathcal{E}4$) Similar to ($\mathcal{E}3$), suppose $\mathcal{S} \subseteq \mathcal{L}$. Then for any $\mathfrak{i} \in \mathcal{S}$ we have

$$1 \in 1 \backsim \mathfrak{i} \in (\mathfrak{i} \backsim \mathfrak{i}) \backsim \mathfrak{i} \in \bigcup\nolimits_{\mathfrak{g}\in\mathcal{L}\ \ \mathfrak{i},\mathfrak{h}\in\mathcal{S}} (\mathfrak{g} \backsim \mathfrak{i}) \backsim \mathfrak{h} = (\mathcal{L} \backsim \mathcal{S}) \backsim \mathcal{S}.$$

Then $\varpi^{\mathcal{L}\backsim\mathcal{S}}(\mathcal{S}) = \sup_{\mathfrak{i}\in(\mathcal{L}\backsim\mathcal{S})\backsim\mathcal{S}} \vartheta(\mathfrak{i}) = \vartheta(1)$. Hence, by Proposition 3.2 (ii), $\varpi^{\mathcal{L}\backsim\mathcal{S}} = \varpi^{\mathcal{S}}$. Therefore, $\varpi^{\mathcal{L}}\overline{\backsim}\varpi^{\mathcal{S}} = \varpi^{\mathcal{S}}$.

($\mathcal{E}5$) Let $\mathcal{S}, \mathcal{R}, \mathcal{T} \subseteq \mathcal{L}$ such that $\varpi^{\mathcal{S}} \preccurlyeq_\varpi \varpi^{\mathcal{R}} \preccurlyeq_\varpi \varpi^{\mathcal{T}}$. Then

$$\varpi^{\mathcal{S}} = \varpi^{\mathcal{S}}\overline{\barwedge}\varpi^{\mathcal{R}} = \varpi^{\mathcal{S}\barwedge\mathcal{R}}, \quad \varpi^{\mathcal{R}\barwedge\mathcal{T}} = \varpi^{\mathcal{R}}\overline{\barwedge}\varpi^{\mathcal{T}} = \varpi^{\mathcal{R}}.$$

Suppose $\mathfrak{x} \in \mathcal{S} \backsim \mathcal{R}$. Then there exist $\mathfrak{i} \in \mathcal{S}$ and $\mathfrak{h} \in \mathcal{R}$ where $\mathfrak{x} \in \mathfrak{i} \backsim \mathfrak{h}$. By Proposition 2.2 (P3), $\mathfrak{i} \backsim \mathfrak{h} \lll \mathfrak{i} \curvearrowright \mathfrak{h}$, then there exists $\mathfrak{y} \in \mathfrak{i} \curvearrowright \mathfrak{h}$ such that $\mathfrak{x} \preccurlyeq \mathfrak{y}$. Since

$$\mathfrak{i} \curvearrowright \mathfrak{h} \in \bigcup_{\mathfrak{i}\in\mathcal{S},\mathfrak{h}\in\mathcal{R}} \mathfrak{i} \curvearrowright \mathfrak{h} = \mathcal{S} \curvearrowright \mathcal{R},$$

we get for any $\mathfrak{x} \in \mathcal{S} \backsim \mathcal{R}$, there exists $\mathfrak{y} \in \mathcal{S} \curvearrowright \mathcal{R}$ such that $\mathfrak{x} \preccurlyeq \mathfrak{y}$. Hence, $\mathcal{S} \backsim \mathcal{R} \lll \mathcal{S} \curvearrowright \mathcal{R}$. By using this method, Proposition 3.2 (iv) and Proposition 2.2 (P12), we can prove that

$$\mathcal{T} \curvearrowright (\mathcal{S} \barwedge \mathcal{R}) \lll (\mathcal{T} \barwedge \mathcal{R}) \curvearrowright \mathcal{R} = (\mathcal{T} \barwedge \mathcal{R}) \backsim (\mathcal{S} \barwedge \mathcal{R} \barwedge \mathcal{T}).$$

Then

$$\begin{aligned}
\varpi^{\mathcal{S}\backsim\mathcal{T}} &= \varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{T}} = \varpi^{(\mathcal{S}\barwedge\mathcal{R})} \overline{\backsim} \varpi^{\mathcal{T}} = \varpi^{(\mathcal{S}\barwedge\mathcal{R})\backsim\mathcal{T}} = \varpi^{\mathcal{T}\backsim(\mathcal{S}\barwedge\mathcal{R})} \\
&\lll_{\varpi} \varpi^{\mathcal{T}\curvearrowright(\mathcal{S}\barwedge\mathcal{R})} \lll_{\varpi} \varpi^{(\mathcal{T}\barwedge\mathcal{R})\curvearrowright\mathcal{S}} = \varpi^{(\mathcal{T}\barwedge\mathcal{R})\backsim(\mathcal{S}\barwedge\mathcal{R}\barwedge\mathcal{T})} \\
&= \varpi^{(\mathcal{T}\barwedge\mathcal{R})} \overline{\backsim} \varpi^{(\mathcal{S}\barwedge\mathcal{R}\barwedge\mathcal{T})} = \varpi^{\mathcal{R}} \overline{\backsim} \varpi^{\mathcal{S}} = \varpi^{\mathcal{R}\backsim\mathcal{S}} = \varpi^{\mathcal{S}\backsim\mathcal{R}}.
\end{aligned}$$

Hence, $\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{T}} \preccurlyeq_{\varpi} \varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{R}}$. Again, since $\varpi^{\mathcal{S}\barwedge\mathcal{R}} = \varpi^{\mathcal{S}}$, similar to the above proof, by Propositions 3.2 (iv), 2.2 (P3) and 2.1 (HE5b), we have

$$\begin{aligned}
\varpi^{\mathcal{S}\backsim\mathcal{T}} &= \varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{T}} = \varpi^{(\mathcal{S}\barwedge\mathcal{R})} \overline{\backsim} \varpi^{\mathcal{T}} = \varpi^{(\mathcal{S}\barwedge\mathcal{R})\backsim\mathcal{T}} = \varpi^{\mathcal{T}\backsim(\mathcal{R}\barwedge\mathcal{S})} \\
&\lll_{\varpi} \varpi^{\mathcal{T}\curvearrowright(\mathcal{R}\barwedge\mathcal{S})} \lll_{\varpi} \varpi^{\mathcal{T}\curvearrowright\mathcal{R}} \\
&= \varpi^{\mathcal{T}\backsim(\mathcal{T}\barwedge\mathcal{R})} = \varpi^{\mathcal{T}} \overline{\backsim} \varpi^{(\mathcal{R}\barwedge\mathcal{T})} = \varpi^{\mathcal{T}} \overline{\backsim} \varpi^{\mathcal{R}} = \varpi^{\mathcal{R}\backsim\mathcal{T}}.
\end{aligned}$$

Hence, $\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{T}} \preccurlyeq_{\varpi} \varpi^{\mathcal{R}} \overline{\backsim} \varpi^{\mathcal{T}}$.

($\mathcal{E}6$) For all $\mathcal{S}, \mathcal{R}, \mathcal{T} \subseteq \mathcal{L}$, we have $\mathcal{S} \backsim \mathcal{R} \lll (\mathcal{S} \barwedge \mathcal{T}) \backsim (\mathcal{R} \barwedge \mathcal{T})$. Because if $\mathfrak{x} \in \mathcal{S} \backsim \mathcal{R}$, then there exist $\mathfrak{i} \in \mathcal{S}$ and $\mathfrak{h} \in \mathcal{R}$ such that $\mathfrak{x} \in \mathfrak{i} \backsim \mathfrak{h}$. Thus for any $\mathfrak{o} \in \mathcal{T}$, by ($\mathcal{HE}6$), $\mathfrak{y} \in (\mathfrak{i} \barwedge \mathfrak{o}) \backsim (\mathfrak{h} \barwedge \mathfrak{o})$ such that $\mathfrak{x} \preccurlyeq \mathfrak{y}$. Hence for any $\mathfrak{x} \in \mathcal{S} \backsim \mathcal{R}$, there exists $\mathfrak{y} \in (\mathcal{S} \barwedge \mathcal{T}) \backsim (\mathcal{R} \barwedge \mathcal{T})$ such that $\mathfrak{x} \preccurlyeq \mathfrak{y}$. Then, for any $\mathcal{S}, \mathcal{R}, \mathcal{T} \subseteq \mathcal{L}$, we have

$$\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{R}} = \varpi^{\mathcal{S}\backsim\mathcal{R}} \preccurlyeq_{\varpi} \varpi^{(\mathcal{S}\barwedge\mathcal{T})\backsim(\mathcal{R}\barwedge\mathcal{T})} = \varpi^{\mathcal{S}\barwedge\mathcal{T}} \overline{\backsim} \varpi^{\mathcal{R}\barwedge\mathcal{T}} = \left(\varpi^{\mathcal{S}} \overline{\barwedge} \varpi^{\mathcal{T}}\right) \overline{\backsim} \left(\varpi^{\mathcal{R}} \overline{\barwedge} \varpi^{\mathcal{T}}\right).$$

($\mathcal{E}7$) Similar to the proof of ($\mathcal{E}6$), for any $\mathcal{S}, \mathcal{R}, \mathcal{T} \subseteq \mathcal{L}$, by ($\mathcal{HE}7$),

$$\mathcal{S} \backsim \mathcal{R} \lll (\mathcal{S} \backsim \mathcal{T}) \backsim (\mathcal{R} \backsim \mathcal{T}).$$

Then

$$\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{R}} = \varpi^{\mathcal{S}\backsim\mathcal{R}} \preccurlyeq_{\varpi} \varpi^{(\mathcal{S}\backsim\mathcal{T})\backsim(\mathcal{R}\backsim\mathcal{T})} = \varpi^{(\mathcal{S}\backsim\mathcal{T})} \overline{\backsim} \varpi^{(\mathcal{R}\backsim\mathcal{T})} = \left(\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{T}}\right) \overline{\backsim} \left(\varpi^{\mathcal{R}} \overline{\backsim} \varpi^{\mathcal{T}}\right).$$

Therefore, $\mathcal{L}/\varpi$ is an equality algebra. $\square$

In Theorem 3.3, the condition symmetric is essential, because we need it for ($\mathcal{E}5$) and in the absence of this assumption the axiom ($\mathcal{E}5$) does not hold.

Example 3.5. Let $\mathcal{L} = \{0, \mathfrak{o}, \mathfrak{i}, \mathfrak{h}, 1\}$ be a set with the following operations:

$\backsim$	0	$\mathfrak{o}$	$\mathfrak{i}$	$\mathfrak{h}$	1
0	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\mathfrak{o}$	$\{0\}$	$\{0,1\}$	$\{0,\mathfrak{h}\}$	$\{0,\mathfrak{i}\}$	$\{\mathfrak{o}\}$
$\mathfrak{i}$	$\{0\}$	$\{0,\mathfrak{h}\}$	$\{\mathfrak{i},1\}$	$\{0,\mathfrak{o}\}$	$\{\mathfrak{i}\}$
$\mathfrak{h}$	$\{0\}$	$\{0,\mathfrak{i}\}$	$\{0,\mathfrak{o}\}$	$\{1\}$	$\{\mathfrak{h}\}$
1	$\{0\}$	$\{\mathfrak{o}\}$	$\{\mathfrak{i}\}$	$\{\mathfrak{h}\}$	$\{1\}$

,

$\bar{\wedge}$	0	$\mathfrak{o}$	$\mathfrak{i}$	$\mathfrak{h}$	1
0	0	0	0	0	0
$\mathfrak{o}$	0	$\mathfrak{o}$	$\mathfrak{o}$	$\mathfrak{o}$	$\mathfrak{o}$
$\mathfrak{i}$	0	$\mathfrak{o}$	$\mathfrak{i}$	$\mathfrak{o}$	$\mathfrak{i}$
$\mathfrak{h}$	0	$\mathfrak{o}$	$\mathfrak{o}$	$\mathfrak{h}$	$\mathfrak{h}$
1	0	$\mathfrak{o}$	$\mathfrak{i}$	$\mathfrak{h}$	1

.

Define ϑ on $\mathcal{L}$ by $\vartheta(0)=\vartheta(\mathfrak{o})\preccurlyeq\vartheta(\mathfrak{i}),\vartheta(\mathfrak{h})\preccurlyeq\vartheta(1)$. By routine calculation, we have $\mathcal{L}/\varpi=\{\varpi^0,\varpi^{\mathfrak{o}},\varpi^{\mathfrak{i}},\varpi^{\mathfrak{h}},\varpi^1\}$ which is a symmetric equality algebra.

Theorem 3.4. *Let $\mathcal{L}=\langle\mathcal{L};\backsim,\bar{\wedge},1\rangle$ be symmetric such that for any $\mathfrak{x},\mathfrak{y},\mathfrak{z}\in\mathcal{L}$, $(\mathfrak{x}\backsim\mathfrak{y})\backsim\mathfrak{z}=(\mathfrak{x}\backsim\mathfrak{z})\backsim(\mathfrak{y}\backsim\mathfrak{z})$ and ϑ be a strong fuzzy filter of $\mathcal{L}$. Then there exists a strong fuzzy filter ϵ on $\mathcal{L}/\varpi$ such that $\epsilon\circ\pi\succcurlyeq\vartheta$, where π is the canonical epimorphism.*

Proof. Define $\epsilon:\mathcal{L}/\varpi\to[0,1]$, for any $\mathcal{S}\subseteq\mathcal{L}$, by $\epsilon(\varpi^{\mathcal{S}})=\sup_{\mathfrak{z}\in\mathcal{L}}\varpi^{\mathcal{S}}(\mathfrak{z})$. First we prove that ϵ is well defined. Assume $\mathcal{S},\mathcal{R}\subseteq\mathcal{L}$ such that $\varpi^{\mathcal{S}}=\varpi^{\mathcal{R}}$. Then for any $\mathfrak{z}\in\mathcal{L}$, $\varpi^{\mathcal{S}}(\mathfrak{z})=\varpi^{\mathcal{R}}(\mathfrak{z})$. Thus,

$$\epsilon(\varpi^{\mathcal{S}})=\sup_{\mathfrak{z}\in\mathcal{L}}\varpi^{\mathcal{S}}(\mathfrak{z})=\sup_{\mathfrak{z}\in\mathcal{L}}\varpi^{\mathcal{R}}(\mathfrak{z})=\epsilon(\varpi^{\mathcal{R}}).$$

Since for any $\mathfrak{i}\in\mathcal{L}$, $1\in\mathfrak{i}\backsim\mathfrak{i}$, we have $\varpi^{\mathcal{L}}(\mathfrak{i})=\sup_{\mathfrak{z}\in\mathcal{L}\backsim\mathfrak{i}}\vartheta(\mathfrak{z})=\vartheta(1)$. Suppose $\mathcal{S}\subseteq\mathcal{L}$. Then

$$\epsilon(\varpi^{\mathcal{L}})=\sup_{\mathfrak{i}\in\mathcal{L}}\varpi^{\mathcal{L}}(\mathfrak{i})=\sup_{\mathfrak{i}\in\mathcal{L}}\sup_{\mathfrak{z}\in\mathcal{L}\backsim\mathfrak{i}}\vartheta(\mathfrak{z})=\sup_{\mathfrak{i}\in\mathcal{L}}\vartheta(1)=\vartheta(1)\tag{3.2}$$

$$\succcurlyeq\sup_{\mathfrak{y}\in\mathcal{L}}\sup_{\mathfrak{h}\in\mathcal{S}\backsim\mathfrak{y}}\vartheta(\mathfrak{h})=\sup_{\mathfrak{y}\in\mathcal{L}}\varpi^{\mathcal{S}}(\mathfrak{y})=\epsilon(\varpi^{\mathcal{S}}).\tag{3.3}$$

Since for any $\mathfrak{x},\mathfrak{y},\mathfrak{z}\in\mathcal{L}$, $(\mathfrak{x}\backsim\mathfrak{y})\backsim\mathfrak{z}=(\mathfrak{x}\backsim\mathfrak{z})\backsim(\mathfrak{y}\backsim\mathfrak{z})$, obviously, for any $\mathcal{S},\mathcal{R},\mathcal{T}\subseteq\mathcal{L}$, we have $(\mathcal{S}\backsim\mathcal{R})\backsim\mathcal{T}=(\mathcal{S}\backsim\mathcal{T})\backsim(\mathcal{R}\backsim\mathcal{T})$. Moreover, from ϑ is a strong fuzzy filter of $\mathcal{L}$, by $(FF1')$ we have

$$\begin{aligned}
\epsilon(\varpi^{\mathcal{R}})&=\sup_{\mathfrak{v}\in\mathcal{L}}\varpi^{\mathcal{R}}(\mathfrak{v})=\sup_{\mathfrak{v}\in\mathcal{L}}\Big(\sup_{\mathfrak{h}\in\mathcal{R}\backsim\mathfrak{v}}\vartheta(\mathfrak{h})\Big)\\
&\succcurlyeq\sup_{\mathfrak{v}\in\mathcal{L}}\Big((\sup_{\mathfrak{i}\in\mathcal{S}\backsim\mathfrak{v}}\vartheta(\mathfrak{i}))\,\bar{\wedge}\,\Big(\sup_{\mathfrak{h}\in(\mathcal{S}\backsim\mathfrak{v})\backsim(\mathcal{R}\backsim\mathfrak{v})}\vartheta(\mathfrak{h})\Big)\Big)\\
&=\sup_{\mathfrak{v}\in\mathcal{L}}\Big((\sup_{\mathfrak{i}\in\mathcal{S}\backsim\mathfrak{v}}\vartheta(\mathfrak{i}))\,\bar{\wedge}\,\Big(\sup_{\mathfrak{h}\in(\mathcal{S}\backsim\mathcal{R})\backsim\mathfrak{v}}\vartheta(\mathfrak{h})\Big)\Big)\\
&\succcurlyeq\sup_{\mathfrak{v}\in\mathcal{L}}\Big(\varpi^{\mathcal{S}}(\mathfrak{v})\,\bar{\wedge}\,\varpi^{\mathcal{S}\backsim\mathcal{R}}(\mathfrak{v})\Big)\\
&=\Big(\sup_{\mathfrak{v}\in\mathcal{L}}\varpi^{\mathcal{S}}(\mathfrak{v})\Big)\,\bar{\wedge}\,\Big(\sup_{\mathfrak{v}\in\mathcal{L}}\varpi^{\mathcal{S}\backsim\mathcal{R}}(\mathfrak{v})\Big)=\epsilon(\varpi^{\mathcal{S}})\,\bar{\wedge}\,\epsilon(\varpi^{\mathcal{S}\backsim\mathcal{R}})\\
&=\epsilon(\varpi^{\mathcal{S}})\,\bar{\wedge}\,\epsilon(\varpi^{\mathcal{S}}\,\bar{\backsim}\,\varpi^{\mathcal{R}}).
\end{aligned}$$

Hence, for any $\mathcal{S},\mathcal{R}\subseteq\mathcal{L}$,

$$\epsilon(\varpi^{\mathcal{R}})\succcurlyeq\epsilon(\varpi^{\mathcal{S}})\,\bar{\wedge}\,\epsilon(\varpi^{\mathcal{S}}\,\bar{\backsim}\,\varpi^{\mathcal{R}}).\tag{3.4}$$

Thus, by (3.2), (3.4) and Theorem 3.2, we have ϵ is a strong fuzzy filter of $\mathcal{L}/\varpi$. Consider $\mathcal{S}\subseteq\mathcal{L}$. Define $\vartheta(\mathcal{S})=\sup_{\mathfrak{z}\in\mathcal{S}}\vartheta(\mathfrak{z})$. Since $\mathcal{L}$ is symmetric, by Proposition 2.2

(P6), we get

$$\begin{aligned}\epsilon \circ \pi(\mathcal{S}) =& \epsilon(\varpi^{\mathcal{S}}) = \sup_{\mathfrak{z}\in\mathcal{L}} \varpi^{\mathcal{S}}(\mathfrak{z}) = \sup_{\mathfrak{z}\in\mathcal{L}} \sup_{\mathfrak{y}\in\mathcal{S}\backsim\mathfrak{z}} \vartheta(\mathfrak{y}) \\ =& \sup_{\mathfrak{z}\in\mathcal{L}} \sup_{\mathfrak{y}\in\mathfrak{z}\backsim\mathcal{S}} \vartheta(\mathfrak{y}) \succcurlyeq \sup_{\mathfrak{i}\in\mathcal{S}} \vartheta(\mathfrak{i}) = \vartheta(\mathcal{S}).\end{aligned}$$

□

4. Fuzzy Hyper Congruence Relation

In this section, we introduce the notion of a fuzzy regular relation on hyper equality algebras and then we give some results related to quotient hyper equality algebras.

Definition 4.1. Let Ω be an equivalence relation on $\mathcal{L}$. Then Ω is called a regular relation on $\mathcal{L}$, if $(\mathfrak{x} \backsim \mathfrak{y})\Omega 1$ and $(\mathfrak{y} \backsim \mathfrak{x})\Omega 1$ imply $\mathfrak{x}\Omega\mathfrak{y}$.

Example 4.1. According to Example 3.1 (*ii*), define

$$\Omega = \{(0,0),(\mathfrak{i},\mathfrak{i}),(\mathfrak{i},0),(0,\mathfrak{i}),(1,1)\}.$$

Then Ω is a regular relation on $\mathcal{L}$.

Definition 4.2. Consider ϱ is a fuzzy equivalence relation on $\mathcal{L}$. Then we say ϱ is a fuzzy hyper congruence relation on $\mathcal{L}$ if for all $\mathfrak{x},\mathfrak{y},\mathfrak{z} \in \mathcal{L}$

$$\varrho(\mathfrak{x},\mathfrak{y}) \preccurlyeq \bigvee\nolimits_{\mathfrak{z}\in\mathcal{L}} \varrho(\mathfrak{x} \backsim \mathfrak{z}, \mathfrak{y} \backsim \mathfrak{z}) \quad \text{and} \quad \varrho(\mathfrak{x},\mathfrak{y}) \preccurlyeq \bigvee\nolimits_{\mathfrak{z}\in\mathcal{L}} \varrho(\mathfrak{x} \barwedge \mathfrak{z}, \mathfrak{y} \barwedge \mathfrak{z}),$$

where

$$\varrho(\mathfrak{x} \backsim \mathfrak{z}, \mathfrak{y} \backsim \mathfrak{z}) = \sup_{\mathfrak{i}\in\mathfrak{x}\backsim\mathfrak{z},\mathfrak{h}\in\mathfrak{y}\backsim\mathfrak{z}} \varrho(\mathfrak{i},\mathfrak{h}). \tag{4.1}$$

The fuzzy hyper congruence relation ϱ on $\mathcal{L}$ is called a fuzzy regular relation on $\mathcal{L}$, if for any $\mathfrak{x},\mathfrak{y} \in \mathcal{L}$

$$\varrho(\mathfrak{x},\mathfrak{y}) \succcurlyeq \min\left\{\bigvee \varrho(\mathfrak{x} \backsim \mathfrak{y}, 1), \bigvee \varrho(\mathfrak{y} \backsim \mathfrak{x}, 1)\right\}. \tag{4.2}$$

Proposition 4.1. *Assume ϱ is a fuzzy regular relation on $\mathcal{L}$. Then for any $\mathfrak{t} \in [0,1]$, $\varrho_{\mathfrak{t}} \neq \emptyset$ is a regular relation on $\mathcal{L}$.*

Proof. First, we prove that for any $\mathfrak{t} \in [0,1]$, $\varrho_{\mathfrak{t}}$ is an equivalence relation on $\mathcal{L}$. Since $\varrho_{\mathfrak{t}}$ is a non-empty set, there exist $\mathfrak{y},\mathfrak{z} \in \mathcal{L}$ such that $\varrho(\mathfrak{y},\mathfrak{z}) \succcurlyeq \mathfrak{t}$. Then for all $\mathfrak{x} \in \mathcal{L}$

$$\varrho(\mathfrak{x},\mathfrak{x}) = \bigvee\nolimits_{(\mathfrak{y},\mathfrak{z})\in\mathcal{L}\times\mathcal{L}} \varrho(\mathfrak{y},\mathfrak{z}) \succcurlyeq \varrho(\mathfrak{y},\mathfrak{z}) \succcurlyeq \mathfrak{t}.$$

Hence, $\varrho_{\mathfrak{t}}$ is a fuzzy reflexive on $\mathcal{L}$. Let $\mathfrak{x}\varrho_{\mathfrak{t}}\mathfrak{y}$. Then $\varrho(\mathfrak{x},\mathfrak{y}) \succcurlyeq \mathfrak{t}$. From ϱ is regular, we have $\varrho(\mathfrak{y},\mathfrak{x}) = \varrho(\mathfrak{x},\mathfrak{y}) \succcurlyeq \mathfrak{t}$, and so $\mathfrak{y}\varrho_{\mathfrak{t}}\mathfrak{x}$. Hence, $\varrho_{\mathfrak{t}}$ is symmetric. Now, suppose $\mathfrak{x}\varrho_{\mathfrak{t}}\mathfrak{y}$ and $\mathfrak{y}\varrho_{\mathfrak{t}}\mathfrak{z}$. Then $\varrho(\mathfrak{x},\mathfrak{y}) \succcurlyeq \mathfrak{t}$ and $\varrho(\mathfrak{y},\mathfrak{z}) \succcurlyeq \mathfrak{t}$. Since ϱ is a fuzzy regular relation on $\mathcal{L}$, we have

$$\varrho(\mathfrak{x},\mathfrak{z}) \succcurlyeq \bigvee\nolimits_{\mathfrak{y}\in\mathcal{L}} (\varrho(\mathfrak{x},\mathfrak{y}) \barwedge \varrho(\mathfrak{y},\mathfrak{z})) \succcurlyeq \varrho(\mathfrak{x},\mathfrak{y}) \barwedge \varrho(\mathfrak{y},\mathfrak{z}) \succcurlyeq \mathfrak{t}.$$

Hence, $\varrho(\mathfrak{x},\mathfrak{z}) \succcurlyeq \mathfrak{t}$ and so $\mathfrak{x}\varrho_{\mathfrak{t}}\mathfrak{z}$. Thus, $\varrho_{\mathfrak{t}}$ is transitive. Therefore, $\varrho_{\mathfrak{t}}$ is an equivalence relation on $\mathcal{L}$.

Now, we prove that $\varrho_{\mathfrak{t}}$ is regular. For this, suppose $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$ where $(\mathfrak{x} \backsim \mathfrak{y})\varrho_{\mathfrak{t}}1$ and $(\mathfrak{y} \backsim \mathfrak{x})\varrho_{\mathfrak{t}}1$. Then there exists $\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}$ and $\mathfrak{h} \in \mathfrak{y} \backsim \mathfrak{x}$ such that $\mathfrak{i}\varrho_{\mathfrak{t}}1$ and $\mathfrak{h}\varrho_{\mathfrak{t}}1$, respectively. Thus, $\varrho(\mathfrak{i}, 1) \succcurlyeq \mathfrak{t}$ and $\varrho(\mathfrak{h}, 1) \succcurlyeq \mathfrak{t}$. Hence,

$$\varrho(\mathfrak{x}, \mathfrak{y}) \succcurlyeq \min\left\{\bigvee \varrho(\mathfrak{x} \backsim \mathfrak{y}, 1), \bigvee \varrho(\mathfrak{y} \backsim \mathfrak{x}, 1)\right\} \succcurlyeq \min\left\{\varrho(\mathfrak{i}, 1), \varrho(\mathfrak{h}, 1)\right\} \succcurlyeq \mathfrak{t}.$$

Therefore, $\varrho_{\mathfrak{t}}$ is a regular relation on $\mathcal{L}$. ☐

Proposition 4.2. *Consider ϱ is a fuzzy relation on $\mathcal{L}$ which satisfies the sup-property. If for each $\mathfrak{t} \in [0, 1]$, $\varrho_{\mathfrak{t}} \neq \emptyset$ is a regular relation on $\mathcal{L}$, then ϱ is a fuzzy regular relation on $\mathcal{L}$.*

Proof. Suppose $\mathfrak{t} = \bigvee_{(\mathfrak{u},\mathfrak{v})\in\mathcal{L}\times\mathcal{L}} \varrho(\mathfrak{u}, \mathfrak{v})$. By assumption ϱ is a fuzzy relation on $\mathcal{L}$ which satisfies the sup-property, then there exists $(\mathfrak{u}, \mathfrak{v}) \in \mathcal{L} \times \mathcal{L}$ such that $\mathfrak{t} = \varrho(\mathfrak{u}, \mathfrak{v})$. Since $\mathfrak{t} \in [0, 1]$, we get $\mathfrak{u}\varrho_{\mathfrak{t}}\mathfrak{v}$, and so $\varrho_{\mathfrak{t}} \neq \emptyset$. Reflexivity of $\varrho_{\mathfrak{t}}$ implies that for all $\mathfrak{x} \in \mathcal{L}$, $(\mathfrak{x}, \mathfrak{x}) \in \varrho_{\mathfrak{t}}$. Thus for all $\mathfrak{x} \in \mathcal{L}$, $\varrho(\mathfrak{x}, \mathfrak{x}) \succcurlyeq \mathfrak{t}$. Then, for any $\mathfrak{x} \in \mathcal{L}$,

$$\varrho(\mathfrak{x}, \mathfrak{x}) \preccurlyeq \bigvee_{(\mathfrak{y},\mathfrak{z})\in\mathcal{L}\times\mathcal{L}} \varrho(\mathfrak{y}, \mathfrak{z}) = \mathfrak{t} \preccurlyeq \varrho(\mathfrak{x}, \mathfrak{x}).$$

Hence, ϱ is a fuzzy reflexive relation on $\mathcal{L}$. Now, since $\varrho_{\mathfrak{t}} \neq \emptyset$, suppose $(\mathfrak{x}, \mathfrak{y}) \in \varrho_{\mathfrak{t}}$ and by symmetry property, we have $\varrho_{\mathfrak{t}}(\mathfrak{x}, \mathfrak{y}) = \varrho_{\mathfrak{t}}(\mathfrak{y}, \mathfrak{x})$ and so $\varrho(\mathfrak{x}, \mathfrak{y}) \succcurlyeq \mathfrak{t}$ and $\varrho(\mathfrak{y}, \mathfrak{x}) \succcurlyeq \mathfrak{t}$. Then for any $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, we get that

$$\varrho(\mathfrak{x}, \mathfrak{y}) \preccurlyeq \bigvee_{(\mathfrak{u},\mathfrak{v})\in\mathcal{L}\times\mathcal{L}} \varrho(\mathfrak{u}, \mathfrak{v}) = \mathfrak{t} \preccurlyeq \varrho(\mathfrak{y}, \mathfrak{x}).$$

Similarly, $\varrho(\mathfrak{y}, \mathfrak{x}) \preccurlyeq \varrho(\mathfrak{x}, \mathfrak{y})$. Then ϱ is a fuzzy symmetric relation on $\mathcal{L}$. By a similar argument, it is easy to see that ϱ is a fuzzy transitive relation on $\mathcal{L}$. Thus ϱ is a fuzzy equivalent relation on $\mathcal{L}$. Now, we show that ϱ is a fuzzy regular relation on $\mathcal{L}$. Let $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$ and

$$\mathfrak{t} = \min\left\{\bigvee \varrho(\mathfrak{x} \backsim \mathfrak{y}, 1), \bigvee \varrho(\mathfrak{y} \backsim \mathfrak{x}, 1)\right\}.$$

Since ϱ satisfies the sup-property, there exist $\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}$ and $\mathfrak{h} \in \mathfrak{y} \backsim \mathfrak{x}$ such that $\mathfrak{t} \preccurlyeq \bigvee \varrho(\mathfrak{x} \backsim \mathfrak{y}, 1) = \varrho(\mathfrak{i}, 1)$ and $\mathfrak{t} \preccurlyeq \bigvee \varrho(\mathfrak{y} \backsim \mathfrak{x}, 1) = \varrho(\mathfrak{h}, 1)$. Thus, $(\mathfrak{i}, 1), (\mathfrak{h}, 1) \in \varrho_{\mathfrak{t}}$. Moreover, since ϱ satisfies the sup-property, by (4.1) we have

$$\varrho(\mathfrak{x} \backsim \mathfrak{y}, 1) = \sup_{m\in\mathfrak{x}\backsim\mathfrak{y}} \varrho(\mathfrak{m}, 1) = \varrho(\mathfrak{i}, 1) \succcurlyeq \mathfrak{t} \text{ and}$$
$$\varrho(\mathfrak{y} \backsim \mathfrak{x}, 1) = \sup_{\neg\in\mathfrak{y}\backsim\mathfrak{x}} \varrho(\neg, 1) = \varrho(\mathfrak{h}, 1) \succcurlyeq \mathfrak{t}.$$

Hence, $(\mathfrak{x} \backsim \mathfrak{y}, 1), (\mathfrak{y} \backsim \mathfrak{x}, 1) \in \varrho_{\mathfrak{t}}$. By our assumption, $\varrho_{\mathfrak{t}}$ is a regular relation on $\mathcal{L}$, then $(\mathfrak{x}, \mathfrak{y}) \in \varrho_{\mathfrak{t}}$. Thus for any $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, we get that

$$\varrho(\mathfrak{x}, \mathfrak{y}) \succcurlyeq \mathfrak{t} = \min\left\{\bigvee \varrho(\mathfrak{x} \backsim \mathfrak{y}, 1), \bigvee \varrho(\mathfrak{y} \backsim \mathfrak{x}, 1)\right\}.$$

Therefore, ϱ is a fuzzy regular relation on $\mathcal{L}$. ☐

Corollary 4.1. *Consider ϱ is a fuzzy relation on $\mathcal{L}$ such that satisfies the sup-property. Then ϱ is a fuzzy regular relation on $\mathcal{L}$ iff, for all $\mathfrak{t} \in [0, 1]$, $\varrho_{\mathfrak{t}} \neq \emptyset$ is a regular relation on $\mathcal{L}$.*

Proof. By Propositions 4.1 and 4.2, the proof is clear. □

By the following result, we show a relation between strong fuzzy filters on hyper equality algebras and fuzzy hyper congruence relation that is made by them.

Theorem 4.1. *Assume ϑ is a strong fuzzy filter on $\mathfrak{L}$. Then for any $\mathfrak{x}, \mathfrak{y} \in \mathfrak{L}$, relation $\varrho : \mathfrak{L} \times \mathfrak{L} \to [0, 1]$ which is defined by*

$$\varrho(\mathfrak{x}, \mathfrak{y}) = \left(\sup_{\mathfrak{u} \in \mathfrak{x} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{u})\right) \barwedge \left(\sup_{\mathfrak{v} \in \mathfrak{y} \curvearrowright \mathfrak{x}} \vartheta(\mathfrak{v})\right)$$

is a fuzzy hyper congruence relation on $\mathfrak{L}$.

Proof. Clearly, ϱ is reflexive and symmetry. We show that ϱ is a fuzzy transitive relation on $\mathfrak{L}$. For this by Proposition 2.2 (P11), for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathfrak{L}$, $\mathfrak{x} \curvearrowright \mathfrak{y} \lll (\mathfrak{y} \curvearrowright \mathfrak{z}) \curvearrowright (\mathfrak{x} \curvearrowright \mathfrak{z})$. Since ϑ is a strong fuzzy filter of $\mathfrak{L}$, by Theorem 3.2 (iii), we have

$$\left(\sup_{\mathfrak{u} \in \mathfrak{x} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{u})\right) \barwedge \left(\sup_{\mathfrak{v} \in \mathfrak{y} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{v})\right) \preccurlyeq \sup_{\mathfrak{w} \in \mathfrak{x} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{w}). \tag{4.3}$$

Then for any $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathfrak{L}$, we get

$$\begin{aligned}
&\bigvee_{\mathfrak{z} \in \mathfrak{L}} (\varrho(\mathfrak{x}, \mathfrak{z}) \barwedge \varrho(\mathfrak{z}, \mathfrak{y})) \\
&= \bigvee_{\mathfrak{z} \in \mathfrak{L}} \left[\left(\sup_{\mathfrak{i} \in \mathfrak{x} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{i})\right) \barwedge \left(\sup_{\mathfrak{h} \in \mathfrak{z} \curvearrowright \mathfrak{x}} \vartheta(\mathfrak{h})\right) \barwedge \left(\sup_{\mathfrak{u} \in \mathfrak{z} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{u})\right) \barwedge \left(\sup_{\mathfrak{v} \in \mathfrak{y} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{v})\right)\right] \\
&= \bigvee_{\mathfrak{z} \in \mathfrak{L}} \left[\left(\left(\sup_{\mathfrak{i} \in \mathfrak{x} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{i})\right) \barwedge \left(\sup_{\mathfrak{u} \in \mathfrak{z} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{u})\right)\right) \barwedge \left(\left(\sup_{\mathfrak{v} \in \mathfrak{y} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{v})\right) \barwedge \left(\sup_{\mathfrak{h} \in \mathfrak{z} \curvearrowright \mathfrak{x}} \vartheta(\mathfrak{h})\right)\right)\right] \\
&\preccurlyeq \left(\sup_{\mathfrak{o} \in \mathfrak{x} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{o})\right) \barwedge \left(\sup_{\mathfrak{w} \in \mathfrak{y} \curvearrowright \mathfrak{x}} \vartheta(\mathfrak{w})\right) \quad \text{(by (4.3))}, \\
&= \varrho(\mathfrak{x}, \mathfrak{y}).
\end{aligned}$$

Thus ϱ is a fuzzy transitive relation on $\mathfrak{L}$. Hence, ϱ is a fuzzy equivalence relation on $\mathfrak{L}$. Now, we investigate the condition of Definition 3.1. By Proposition 2.2 (P9), for any $\mathfrak{x}, \mathfrak{y} \in \mathfrak{L}$

$$\mathfrak{x} \barwedge \mathfrak{y} \preccurlyeq \mathfrak{y} \lll \mathfrak{y} \backsim (\mathfrak{x} \barwedge \mathfrak{y}) = \mathfrak{y} \curvearrowright \mathfrak{x}.$$

Then there exists $\mathfrak{z} \in \mathfrak{y} \curvearrowright \mathfrak{x}$ such that $\mathfrak{x} \barwedge \mathfrak{y} \preccurlyeq \mathfrak{y} \preccurlyeq \mathfrak{z}$. By $(\mathcal{HE}5)$ we have

$$(\mathfrak{x} \barwedge \mathfrak{y}) \backsim \mathfrak{z} \lll \mathfrak{y} \backsim \mathfrak{z}. \tag{4.4}$$

Thus,

$$\begin{aligned}
\mathfrak{x} \curvearrowright \mathfrak{y} &= \mathfrak{x} \backsim (\mathfrak{x} \barwedge \mathfrak{y}) \quad \text{(by } (\mathcal{HE}7)) \\
&\lll (\mathfrak{x} \backsim \mathfrak{z}) \backsim ((\mathfrak{x} \barwedge \mathfrak{y}) \backsim \mathfrak{z}) \quad \text{(by Proposition 2.2(P3))} \\
&\lll (\mathfrak{x} \backsim \mathfrak{z}) \curvearrowright ((\mathfrak{x} \barwedge \mathfrak{y}) \backsim \mathfrak{z}) \quad \text{(by (4.4) and Proposition 2.2(P8))} \\
&\lll (\mathfrak{x} \backsim \mathfrak{z}) \curvearrowright (\mathfrak{y} \backsim \mathfrak{z}).
\end{aligned}$$

Hence, by (FF2) for any $\mathfrak{z} \in \mathfrak{y} \curvearrowright \mathfrak{x}$, we have

$$\sup_{\mathfrak{u} \in \mathfrak{x} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{u}) \preccurlyeq \sup_{\mathfrak{v} \in (\mathfrak{x} \backsim \mathfrak{z}) \curvearrowright (\mathfrak{y} \backsim \mathfrak{z})} \vartheta(\mathfrak{v}). \tag{4.5}$$

By the similar way, for any $\mathfrak{z} \in \mathfrak{x} \curvearrowright \mathfrak{y}$, we get

$$\sup_{\mathfrak{i}\in\mathfrak{y}\curvearrowright\mathfrak{x}} \vartheta(\mathfrak{i}) \preccurlyeq \sup_{\mathfrak{h}\in(\mathfrak{y}\backsim\mathfrak{z})\curvearrowright(\mathfrak{x}\backsim\mathfrak{z})} \vartheta(\mathfrak{h}). \tag{4.6}$$

Then by (4.5) and (4.6), for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, we have

$$\begin{aligned}
\varrho(\mathfrak{x}, \mathfrak{y}) &= \left(\sup_{\mathfrak{u}\in\mathfrak{x}\curvearrowright\mathfrak{y}} \vartheta(\mathfrak{u})\right) \barwedge \left(\sup_{\mathfrak{i}\in\mathfrak{y}\curvearrowright\mathfrak{x}} \vartheta(\mathfrak{i})\right) \\
&\preccurlyeq \left(\sup_{\mathfrak{v}\in(\mathfrak{x}\backsim\mathfrak{z})\curvearrowright(\mathfrak{y}\backsim\mathfrak{z})} \vartheta(\mathfrak{v})\right) \barwedge \left(\sup_{\mathfrak{h}\in(\mathfrak{y}\backsim\mathfrak{z})\curvearrowright(\mathfrak{x}\backsim\mathfrak{z})} \vartheta(\mathfrak{h})\right) \\
&\preccurlyeq \bigvee_{\mathfrak{z}\in\mathcal{L}} \left[\left(\sup_{\mathfrak{v}\in(\mathfrak{x}\backsim\mathfrak{z})\curvearrowright(\mathfrak{y}\backsim\mathfrak{z})} \vartheta(\mathfrak{v})\right) \barwedge \left(\sup_{\mathfrak{h}\in(\mathfrak{y}\backsim\mathfrak{z})\curvearrowright(\mathfrak{x}\backsim\mathfrak{z})} \vartheta(\mathfrak{h})\right)\right] \\
&= \bigvee_{\mathfrak{z}\in\mathcal{L}} \varrho(\mathfrak{x} \backsim \mathfrak{z}, \mathfrak{y} \backsim \mathfrak{z}).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\mathfrak{x} \curvearrowright \mathfrak{y} &= \mathfrak{x} \backsim (\mathfrak{x} \barwedge \mathfrak{y}) \quad \text{(by } (\mathcal{HE}6)) \\
&\lll (\mathfrak{x} \barwedge \mathfrak{z}) \backsim (\mathfrak{x} \barwedge \mathfrak{y} \barwedge \mathfrak{z}) \quad \text{(by Proposition 2.2 (P3))} \\
&\lll (\mathfrak{x} \barwedge \mathfrak{z}) \curvearrowright (\mathfrak{x} \barwedge \mathfrak{y} \barwedge \mathfrak{z}) \quad \text{(by Proposition 2.2 (P7))} \\
&\lll (\mathfrak{x} \barwedge \mathfrak{z}) \curvearrowright (\mathfrak{y} \barwedge \mathfrak{z}).
\end{aligned} \tag{4.7}$$

Similarly, for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, we have

$$\mathfrak{y} \curvearrowright \mathfrak{x} \lll (\mathfrak{y} \barwedge \mathfrak{z}) \curvearrowright (\mathfrak{x} \barwedge \mathfrak{z}). \tag{4.8}$$

Then by (4.7), (4.8) and (FF2), for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, we get that

$$\begin{aligned}
\varrho(\mathfrak{x}, \mathfrak{y}) &= \left(\sup_{\mathfrak{u}\in\mathfrak{x}\curvearrowright\mathfrak{y}} \vartheta(\mathfrak{u})\right) \barwedge \left(\sup_{\mathfrak{i}\in\mathfrak{y}\curvearrowright\mathfrak{x}} \vartheta(\mathfrak{i})\right) \\
&\preccurlyeq \left(\sup_{\mathfrak{v}\in(\mathfrak{x}\barwedge\mathfrak{z})\curvearrowright(\mathfrak{y}\barwedge\mathfrak{z})} \vartheta(\mathfrak{v})\right) \barwedge \left(\sup_{\mathfrak{h}\in(\mathfrak{y}\barwedge\mathfrak{z})\curvearrowright(\mathfrak{x}\barwedge\mathfrak{z})} \vartheta(\mathfrak{h})\right) \\
&= \varrho(\mathfrak{x} \barwedge \mathfrak{z}, \mathfrak{y} \barwedge \mathfrak{z}).
\end{aligned}$$

Hence, ϱ is a fuzzy hyper congruence relation on $\mathcal{L}$. □

Let ϱ be a fuzzy hyper congruence relation on $\mathcal{L}$, we define the fuzzy subset $\vartheta^{\varrho}_{\mathfrak{x}} : \mathcal{L} \to [0, 1]$ by $\vartheta^{\varrho}_{\mathfrak{x}}(\mathfrak{y}) = \varrho(\mathfrak{y}, \mathfrak{x})$ for all $\mathfrak{y} \in \mathcal{L}$.

Lemma 4.1. *Consider ϱ is a fuzzy hyper congruence relation on $\mathcal{L}$. Then for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, $\vartheta^{\varrho}_{\mathfrak{x}} = \vartheta^{\varrho}_{\mathfrak{y}}$ iff $\varrho(\mathfrak{x}, \mathfrak{y}) = \bigvee_{\mathfrak{s},\mathfrak{t}\in\mathcal{L}} \varrho(\mathfrak{s}, \mathfrak{t})$.*

Proof. Suppose $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$ such that $\vartheta^{\varrho}_{\mathfrak{x}} = \vartheta^{\varrho}_{\mathfrak{y}}$. Since ϱ is a fuzzy reflexive relation, we have

$$\vartheta^{\varrho}_{\mathfrak{y}}(\mathfrak{x}) = \vartheta^{\varrho}_{\mathfrak{x}}(\mathfrak{x}) = \varrho(\mathfrak{x}, \mathfrak{x}) = \bigvee_{\mathfrak{s},\mathfrak{t}\in\mathcal{L}} \varrho(\mathfrak{s}, \mathfrak{t}).$$

Conversely, assume $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$ such that $\varrho(\mathfrak{x}, \mathfrak{y}) = \bigvee_{\mathfrak{s},\mathfrak{t}\in\mathcal{L}} \varrho(\mathfrak{s}, \mathfrak{t})$. Then by fuzzy symmetric and fuzzy transitive relations defined on ϱ, for all $\mathfrak{z} \in \mathcal{L}$, we get

$$\begin{aligned}\vartheta^{\varrho}_{\mathfrak{x}}(\mathfrak{z}) &= \varrho(\mathfrak{z}, \mathfrak{x}) = \varrho(\mathfrak{x}, \mathfrak{z})\\ &\succcurlyeq \bigvee\nolimits_{\mathfrak{y}\in\mathcal{L}} (\varrho(\mathfrak{x}, \mathfrak{y}) \,\overline{\curlywedge}\, \varrho(\mathfrak{y}, \mathfrak{z})) \succcurlyeq \varrho(\mathfrak{x}, \mathfrak{y}) \,\overline{\curlywedge}\, \varrho(\mathfrak{y}, \mathfrak{z})\\ &= \Big(\bigvee\nolimits_{\mathfrak{s},\mathfrak{t}\in\mathcal{L}} \varrho(\mathfrak{s}, \mathfrak{t})\Big) \,\overline{\curlywedge}\, \varrho(\mathfrak{y}, \mathfrak{z}) = \varrho(\mathfrak{y}, \mathfrak{z}) = \varrho(\mathfrak{z}, \mathfrak{y})\\ &= \vartheta^{\varrho}_{\mathfrak{y}}(\mathfrak{z}).\end{aligned}$$

By replacing $\mathfrak{x}$ by $\mathfrak{y}$ throughout the above statements, we get $\vartheta^{\varrho}_{\mathfrak{x}}(\mathfrak{z}) \preccurlyeq \vartheta^{\varrho}_{\mathfrak{y}}(\mathfrak{z})$. Hence, $\vartheta^{\varrho}_{\mathfrak{x}} = \vartheta^{\varrho}_{\mathfrak{y}}$. □

Theorem 4.2. *Consider* $\mathcal{L} = \langle \mathcal{L}; \backsim, \overline{\curlywedge}, 1\rangle$ *is symmetric and* ϱ *be a fuzzy hyper congruence relation on* $\mathcal{L}$*, satisfies the sup-property. Define* $\frac{\mathcal{L}}{\varrho} = \{\vartheta^{\varrho}_{\mathfrak{x}} \mid \mathfrak{x} \in \mathcal{L}\}$*. Then* $\frac{\mathcal{L}}{\varrho} = \left\langle \frac{\mathcal{L}}{\varrho}; \backsim_{\varrho}, \overline{\curlywedge}_{\varrho}, \vartheta^{\varrho}_{1}\right\rangle$ *is symmetric, where the operations* $\backsim_{\varrho}$ *and* $\overline{\curlywedge}_{\varrho}$ *are defined on* $\frac{\mathcal{L}}{\varrho}$ *as follows:*

$$\vartheta^{\varrho}_{\mathfrak{x}} \backsim_{\varrho} \vartheta^{\varrho}_{\mathfrak{y}} = \vartheta^{\varrho}_{\mathfrak{x}\backsim\mathfrak{y}} = \{\vartheta^{\varrho}_{\mathfrak{z}} \mid \mathfrak{z} \in \mathfrak{x} \backsim \mathfrak{y}\}, \quad \vartheta^{\varrho}_{\mathfrak{x}} \,\overline{\curlywedge}_{\varrho}\, \vartheta^{\varrho}_{\mathfrak{y}} = \vartheta^{\varrho}_{\mathfrak{x}\overline{\curlywedge}\mathfrak{y}},$$
$$\vartheta^{\varrho}_{\mathfrak{x}} \preccurlyeq_{\varrho} \vartheta^{\varrho}_{\mathfrak{y}} \Leftrightarrow \vartheta^{\varrho}_{\mathfrak{x}} \,\overline{\curlywedge}_{\varrho}\, \vartheta^{\varrho}_{\mathfrak{y}} = \vartheta^{\varrho}_{\mathfrak{x}},$$

and for any $\mathcal{S}, \mathcal{R} \subseteq \frac{\mathcal{L}}{\varrho}$, $\mathcal{S} \lll_{\varrho} \mathcal{R}$ *if and only if for all* $\mathfrak{i} \in \mathcal{S}$*, there exists* $\mathfrak{h} \in \mathcal{R}$ *such that* $\vartheta^{\varrho}_{\mathfrak{i}} \preccurlyeq_{\varrho} \vartheta^{\varrho}_{\mathfrak{h}}$.

Proof. Let $\mathfrak{x}, \mathfrak{y}, \mathfrak{s}, \mathfrak{t} \in \mathcal{L}$, $\vartheta^{\varrho}_{\mathfrak{x}} = \vartheta^{\varrho}_{\mathfrak{y}}$ and $\vartheta^{\varrho}_{\mathfrak{s}} = \vartheta^{\varrho}_{\mathfrak{t}}$. Then by Lemma 4.1, $\varrho(\mathfrak{x}, \mathfrak{y}) = \varrho(\mathfrak{s}, \mathfrak{t}) = \bigvee_{\mathfrak{u},\mathfrak{v}\in\mathcal{L}} \varrho(\mathfrak{u}, \mathfrak{v})$. Set $\bigvee_{\mathfrak{u},\mathfrak{v}\in\mathcal{L}} \varrho(\mathfrak{u}, \mathfrak{v}) = \mathfrak{m}$. Then by Proposition 4.1, $\varrho_{\mathfrak{m}}$ is a regular relation on $\mathcal{L}$ and since $\varrho(\mathfrak{x}, \mathfrak{y}) = \varrho(\mathfrak{s}, \mathfrak{t}) = \mathfrak{m}$, we have $\mathfrak{x}\varrho_{\mathfrak{m}}\mathfrak{y}$ and $\mathfrak{s}\varrho_{\mathfrak{m}}\mathfrak{t}$. Moreover, since ϱ is a fuzzy regular relation on $\mathcal{L}$ such that satisfies the sup-property and for any $\mathfrak{z} \in \mathcal{L}$, $\mathfrak{z} \preccurlyeq 1$, we get $\varrho(\mathfrak{x}, \mathfrak{z}) \preccurlyeq \varrho(\mathfrak{x}, 1)$ and so

$$\varrho(\mathfrak{x}, \mathfrak{y}) = \varrho(\mathfrak{x}, 1) \,\overline{\curlywedge}\, \varrho(1, \mathfrak{y}) \succcurlyeq \bigvee\nolimits_{\mathfrak{z}\in\mathcal{L}} (\varrho(\mathfrak{x}, \mathfrak{z}) \,\overline{\curlywedge}\, \varrho(\mathfrak{z}, \mathfrak{y})).$$

Moreover, since ϱ is a fuzzy relation on $\mathcal{L}$, we have $\varrho(\mathfrak{t}, \mathfrak{t}) = \bigvee_{\mathfrak{u},\mathfrak{v}\in\mathcal{L}} \varrho(\mathfrak{u}, \mathfrak{v}) = \mathfrak{m}$. Also, from $\mathcal{L}$ is symmetric, by (4.2), Proposition 2.2(P9) and ($\mathcal{HE}7$), we have

$$\begin{aligned}\varrho(\mathfrak{x} \backsim \mathfrak{t}, \mathfrak{y} \backsim \mathfrak{t}) &\succcurlyeq \min\left\{\bigvee \varrho\left((\mathfrak{x} \backsim \mathfrak{t}) \backsim (\mathfrak{y} \backsim \mathfrak{t}), 1\right), \bigvee \varrho\left((\mathfrak{y} \backsim \mathfrak{t}) \backsim (\mathfrak{x} \backsim \mathfrak{t}), 1\right)\right\}\\ &\succcurlyeq \min\left\{\bigvee \varrho\left(\mathfrak{x} \backsim \mathfrak{y}, 1\right), \bigvee \varrho\left(\mathfrak{y} \backsim \mathfrak{x}, 1\right)\right\} = \bigvee \varrho\left(\mathfrak{x} \backsim \mathfrak{y}, 1\right)\\ &\succcurlyeq \varrho(\mathfrak{x}, 1) \,\overline{\curlywedge}\, \varrho(1, \mathfrak{y}) = \varrho(\mathfrak{x}, \mathfrak{y}) = m.\end{aligned}$$

Then $\mathfrak{x} \backsim \mathfrak{t}\varrho_{\mathfrak{m}}\mathfrak{y} \backsim \mathfrak{t}$. Similarly, $\mathfrak{y} \backsim \mathfrak{t}\varrho_{\mathfrak{m}}\mathfrak{y} \backsim \mathfrak{s}$. Since $\varrho_{\mathfrak{m}}$ is a regular relation on $\mathcal{L}$, we get $(\mathfrak{x} \backsim \mathfrak{s})\overline{\varrho_{\mathfrak{m}}}(\mathfrak{y} \backsim \mathfrak{t})$. Suppose $\vartheta^{\varrho}_{\mathfrak{z}} \in \vartheta^{\varrho}_{\mathfrak{x}} \backsim_{\varrho} \vartheta^{\varrho}_{\mathfrak{s}}$. Then there is $\mathfrak{x}' \in \mathfrak{x} \backsim \mathfrak{s}$ such that $\vartheta^{\varrho}_{\mathfrak{x}'} = \vartheta^{\varrho}_{\mathfrak{z}}$. Thus, $\mathfrak{z}\varrho_{\mathfrak{m}}\mathfrak{x}'$. On the other hand, $(\mathfrak{x} \backsim \mathfrak{s})\overline{\varrho_{\mathfrak{m}}}(\mathfrak{y} \backsim \mathfrak{t})$, so there exists $\mathfrak{y}' \in \mathfrak{y} \backsim \mathfrak{t}$ such that $\mathfrak{x}'\varrho_{\mathfrak{m}}\mathfrak{y}'$. Also, since $\varrho_{\mathfrak{m}}$ is transitive, and by Proposition 4.1 is an equivalence relation on $\mathcal{L}$, we have $\mathfrak{z}\varrho_{\mathfrak{m}}\mathfrak{y}'$. Then by Lemma 4.1, we get $\vartheta^{\varrho}_{\mathfrak{z}} = \vartheta^{\varrho}_{\mathfrak{y}'}$. This shows that $\vartheta^{\varrho}_{\mathfrak{z}} = \vartheta^{\varrho}_{\mathfrak{y}'} \in \vartheta^{\varrho}_{\mathfrak{y}} \backsim_{\varrho} \vartheta^{\varrho}_{\mathfrak{t}}$. Thus $\vartheta^{\varrho}_{\mathfrak{x}} \backsim_{\varrho} \vartheta^{\varrho}_{\mathfrak{s}} \subseteq \vartheta^{\varrho}_{\mathfrak{y}} \backsim_{\varrho} \vartheta^{\varrho}_{\mathfrak{t}}$. Similarly, we obtain $\vartheta^{\varrho}_{\mathfrak{y}} \backsim_{\varrho} \vartheta^{\varrho}_{\mathfrak{t}} \subseteq \vartheta^{\varrho}_{\mathfrak{x}} \backsim_{\varrho} \vartheta^{\varrho}_{\mathfrak{s}}$. Hence, $\backsim_{\varrho}$ is well-defined. Easily, $\overline{\curlywedge}_{\varrho}$ is well-defined, too. Now, we show that $\frac{\mathcal{L}}{\varrho} = \left\langle \frac{\mathcal{L}}{\varrho}; \backsim_{\varrho}, \overline{\curlywedge}_{\varrho}, \vartheta^{\varrho}_{1}\right\rangle$ is a hyper equality algebra.

($\mathcal{HE}1$) Clearly, ϑ_1^ϱ is the top element and $\langle \frac{\mathcal{L}}{\varrho}; \bar{\wedge}_\varrho, \vartheta_1^\varrho \rangle$ is a meet-semilattice.

($\mathcal{HE}2$) For all $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, since $\mathcal{L}$ is symmetric, we have

$$\vartheta_{\mathfrak{x}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{y}}^\varrho = \vartheta_{\mathfrak{x} \backsim \mathfrak{y}}^\varrho \preccurlyeq_\varrho \vartheta_{\mathfrak{y} \backsim \mathfrak{x}}^\varrho = \vartheta_{\mathfrak{y}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{x}}^\varrho. \tag{4.9}$$

($\mathcal{HE}3$) Assume $\mathfrak{x} \in \mathcal{L}$. Since $1 \in \mathfrak{x} \backsim \mathfrak{x}$, we have

$$\vartheta_1^\varrho \in \{\vartheta_{\mathfrak{z}}^\varrho : \mathfrak{z} \in \mathfrak{x} \backsim \mathfrak{x}\} = \vartheta_{\mathfrak{x} \backsim \mathfrak{x}}^\varrho.$$

Thus, $\vartheta_1^\varrho \in \vartheta_{\mathfrak{x} \backsim \mathfrak{x}}^\varrho = \vartheta_{\mathfrak{x}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{x}}^\varrho$.

($\mathcal{HE}4$) Simmilar to ($\mathcal{HE}3$), since for any $\mathfrak{x} \in 1 \backsim \mathfrak{x}$, we have $\vartheta_{\mathfrak{x}}^\varrho \in \vartheta_{1 \backsim \mathfrak{x}} = \vartheta_1 \backsim_\varrho \vartheta_{\mathfrak{x}}$.

($\mathcal{HE}5$) Suppose $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$ such that $\vartheta_{\mathfrak{x}}^\varrho \preccurlyeq_\varrho \vartheta_{\mathfrak{y}}^\varrho \preccurlyeq_\varrho \vartheta_{\mathfrak{z}}^\varrho$. Since $\vartheta_{\mathfrak{x}}^\varrho \preccurlyeq_\varrho \vartheta_{\mathfrak{y}}^\varrho$, we have $\vartheta_{\mathfrak{x}}^\varrho \bar{\wedge}_\varrho \vartheta_{\mathfrak{y}}^\varrho = \vartheta_{\mathfrak{x}}^\varrho$ and so $\vartheta_{\mathfrak{x} \bar{\wedge} \mathfrak{y}}^\varrho = \vartheta_{\mathfrak{x}}^\varrho$. By Lemma 4.1, $\varrho(\mathfrak{x} \bar{\wedge} \mathfrak{y}, \mathfrak{x}) = \bigvee_{\mathfrak{s}, \mathfrak{t} \in \mathcal{L}} \varrho(\mathfrak{s}, \mathfrak{t}) = \mathfrak{m}$. Thus $(\mathfrak{x} \bar{\wedge} \mathfrak{y}) \varrho_{\mathfrak{m}} \mathfrak{x}$. Since $\mathcal{L}$ is symmetric, by Proposition 2.2(P9), we get $\mathfrak{x} \lll \mathfrak{x} \backsim \mathfrak{z}$ and $\mathfrak{x} \bar{\wedge} \mathfrak{y} \preccurlyeq \mathfrak{y} \lll \mathfrak{y} \backsim \mathfrak{z}$. Then $\mathfrak{m} = \varrho(\mathfrak{x}, \mathfrak{x} \bar{\wedge} \mathfrak{y}) \preccurlyeq \varrho(\mathfrak{x} \backsim \mathfrak{z}, \mathfrak{y} \backsim \mathfrak{z})$, and so $(\mathfrak{x} \backsim \mathfrak{z}) \varrho_{\mathfrak{m}} (\mathfrak{y} \backsim \mathfrak{z})$. Consider $\vartheta_\gamma^\varrho \in \vartheta_{\mathfrak{x} \backsim \mathfrak{z}}^\varrho$. Then there exists $\alpha \in \mathfrak{x} \backsim \mathfrak{z}$ such that $\vartheta_\gamma^\varrho = \vartheta_\alpha^\varrho$. Thus $\gamma \varrho_{\mathfrak{m}} \alpha$. Since $(\mathfrak{x} \backsim \mathfrak{z}) \varrho_{\mathfrak{m}} (\mathfrak{y} \backsim \mathfrak{z})$, there exists $\beta \in \mathfrak{y} \backsim \mathfrak{z}$ such that $\beta \varrho_{\mathfrak{m}} \alpha$. From $\varrho_{\mathfrak{m}}$ has transitivity, we get $\gamma \varrho_{\mathfrak{m}} \beta$. Thus, $\varrho(\beta, \gamma) \succcurlyeq \mathfrak{m} = \bigvee_{\mathfrak{s}, \mathfrak{t} \in \mathcal{L}} \varrho(\mathfrak{s}, \mathfrak{t})$. Hence, $\vartheta_\gamma^\varrho = \vartheta_\beta^\varrho$ and so $\vartheta_\gamma^\varrho = \vartheta_\beta^\varrho \in \vartheta_{\mathfrak{y} \backsim \mathfrak{z}}^\varrho$. So $\vartheta_{\mathfrak{x} \backsim \mathfrak{z}}^\varrho \preccurlyeq_\varrho \vartheta_{\mathfrak{y} \backsim \mathfrak{z}}^\varrho$. The proof of other case is similar.

($\mathcal{HE}6$) For all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, we have $\mathfrak{x} \backsim \mathfrak{y} \lll (\mathfrak{x} \bar{\wedge} \mathfrak{z}) \backsim (\mathfrak{y} \bar{\wedge} \mathfrak{z})$. Then, for any $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, we have

$$\vartheta_{\mathfrak{x}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{y}}^\varrho = \vartheta_{\mathfrak{x} \backsim \mathfrak{y}}^\varrho \preccurlyeq_\varrho \vartheta_{(\mathfrak{x} \bar{\wedge} \mathfrak{z}) \backsim (\mathfrak{y} \bar{\wedge} \mathfrak{z})}^\varrho = \vartheta_{\mathfrak{x} \bar{\wedge} \mathfrak{z}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{y} \bar{\wedge} \mathfrak{z}}^\varrho = \left(\vartheta_{\mathfrak{x}}^\varrho \bar{\wedge}_\varrho \vartheta_{\mathfrak{z}}^\varrho\right) \backsim_\varrho \left(\vartheta_{\mathfrak{y}}^\varrho \bar{\wedge}_\varrho \vartheta_{\mathfrak{z}}^\varrho\right).$$

($\mathcal{HE}7$) For all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, we obtain $\mathfrak{x} \backsim \mathfrak{y} \lll (\mathfrak{x} \backsim \mathfrak{z}) \backsim (\mathfrak{y} \backsim \mathfrak{z})$. Then

$$\vartheta_{\mathfrak{x}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{y}}^\varrho = \vartheta_{\mathfrak{x} \backsim \mathfrak{y}}^\varrho \preccurlyeq_\varrho \vartheta_{(\mathfrak{x} \backsim \mathfrak{z}) \backsim (\mathfrak{y} \backsim \mathfrak{z})}^\varrho = \vartheta_{\mathfrak{x} \backsim \mathfrak{z}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{y} \backsim \mathfrak{z}}^\varrho = \left(\vartheta_{\mathfrak{x}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{z}}^\varrho\right) \backsim_\varrho \left(\vartheta_{\mathfrak{y}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{z}}^\varrho\right).$$

Thus, the above facts and (4.9) show that $\frac{\mathcal{L}}{\varrho}$ is a symmetric hyper equality algebra. □

5. Conclusions and Future Works

In this paper, the notion of strong fuzzy filter on hyper equality algebras is introduced and some equivalence definitions of it are investigated. Then by using this notion, a symmetric equality algebra is constructed and defined a special form of classes. By using these, the concept of a fuzzy hyper congruence relation on hyper equality algebra is defined and prove that the quotient is made by it is an equality algebra. Also, by using a fuzzy equivalence relation on hyper equality, a fuzzy hyper congruence relation is introduced and proved that this fuzzy hyper congruence is regular. Finally, it is proved that the quotient structure that is made by it is a symmetric hyper equality algebra.

For future work, we can define different kinds of fuzzy ideal on hyper equality algebras and investigate properties of them and study about fuzzy hyper congruence relation on hyper equality algebra and the quotient structure that is made by it.

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