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NESTED FUNCTIONS ON SOME GENERALIZATIONS OF WILKER AND HUYGENS'S INEQUALITIES

SEYYED HOSSEIN JAFARI PETROUDI[1], ARSALAN HOJJAT ANSARI[1,2,3], CHOONKIL PARK[4], AND AHMET DASDEMIR[5]

Abstract. This study focuses on Wilker and Huygens's inequalities. We provide generalizations of these inequalities, based on hyperbolic functions by various methods. Furthermore, this study presents a generalization of Cusa's inequality. We then investigate the relationship among Cusa's, Wilker's, and Huygens's inequalities and also provide generalizations of these inequalities, based on nested functions.

1. Introduction and Preliminaries

In recent years, many studies have been conducted in the field of inequalities, particularly, trigonometric inequalities, and interesting results have been obtained (see [4, 5, 8, 11–18, 20]). In particular, the following trigonometric inequalities, which are respectively known as Wilker's, Huygens's and Cusa's inequality, have attracted the attention of many researchers and various proofs for these inequalities have been presented with different methods, which can be found in papers [3, 6, 7] pointed out in this context:

$$\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}>2,\quad 0<|x|<\frac{\pi}{2},$$
$$2\frac{\sin(x)}{x}+\frac{\tan(x)}{x}>3,\quad 0<|x|<\frac{\pi}{2},$$
$$\frac{\sin(x)}{x}<\frac{2}{3}+\frac{1}{3}\cos(x),\quad 0<|x|<\frac{\pi}{2}.$$

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As a result, many efforts have been made to develop these trigonometric inequalities into other functions, such as lemniscate functions, Bessel functions, hyperbolic functions, exponential functions and weighted functions.

As an example, we can refer to the following inequalities based on hyperbolic functions from [19], which are respectively known as Wilker's first type inequality, Wilker's second type inequality, Cusa type inequality and Huygen's type inequality:

$$\left(\frac{\sinh(x)}{x}\right)^2 + \frac{\tanh(x)}{x} > 2, \quad x \neq 0,$$

$$\left(\frac{x}{\sinh(x)}\right)^2 + \frac{x}{\tanh(x)} > 2, \quad x \neq 0,$$

$$\frac{\sinh(x)}{x} - \frac{1}{3}\cosh(x) < \frac{2}{3}, \quad x \neq 0,$$

$$2\frac{\sinh(x)}{x} + \frac{\tanh(x)}{x} > 3, \quad x \neq 0.$$

In this paper, inspired by the work of Chen and Mortici [6], who established generalizations of the trigonometric inequalities of Wilker, Huygens, and Cusa, we present some generalizations of these inequalities based on hyperbolic functions and parameter $n \in \mathbb{N}$. Then we examine the trigonometric inequality of Cusa and prove a generalization of this inequality based on hyperbolic functions and the given parameter $n \in \mathbb{N}$. Finally, by initializing the given parameter $n \in \mathbb{N}$, we discuss the relationship between Wilker's inequality and Huygens's inequality based on the hyperbolic functions and also we provide generalizations of these inequalities, based on nested functions.

2. Main Results

We begin this section by stating an important but simple lemma about properties of hyperbolic functions. Then, we give a generalization of Wilker's inequality for hyperbolic functions.

Lemma 2.1. *For each $x \neq 0$, the following inequalities hold:*

$$\frac{\tanh(x)}{x} < 1, \quad \frac{\sinh(x)}{x} > 1.$$

Proof. The proof is easy. □

Theorem 2.1. *For each $x \neq 0$ and $n \geq 1$, the following inequality holds:*

$$\left(\frac{\sinh(x)}{x}\right)^n + \frac{n}{2} \cdot \frac{\tanh(x)}{x} > \frac{n+2}{2}.$$

Proof. By Lemma 2.1, $\frac{\tanh(x)}{x} < 1$ and $\frac{\sinh(x)}{x} > 1$. So, we get

$$f(x) := \left(\frac{\tanh(x)}{x} - 1\right)\left(1 - \frac{\sinh(x)}{x}\right) > 0.$$

Also, by the Huygens type inequality, we have

$$g(x) := \frac{\sinh(x)}{x} + \frac{\tanh(x)}{2x} - \frac{3}{2} > 0.$$

We prove this theorem by mathematical induction on n. For $n = 1$, the above inequality holds, because

$$\left(\frac{\sinh(x)}{x}\right)^1 + \frac{1}{2} \cdot \frac{\tanh(x)}{x} > \frac{3}{2},$$

which is Huygen's type inequality.

Now suppose that this inequality holds for $n = m$, that is, we have

$$\left(\frac{\sinh(x)}{x}\right)^m + \frac{m}{2} \cdot \frac{\tanh(x)}{x} > \frac{m+2}{2}.$$

We prove that this inequality also holds for $n = m + 1$. For $n = m + 1$, we have

$$\begin{aligned}
&\left(\frac{\sinh(x)}{x}\right)^{m+1} + \frac{m+1}{2} \cdot \frac{\tanh(x)}{x} - \frac{m+3}{2} \\
=&\frac{\sinh(x)}{x}\left(\frac{\sinh(x)}{x}\right)^m + \frac{m+1}{2} \cdot \frac{\tanh(x)}{x} \cdot \frac{m+3}{2} \\
>&\frac{\sinh(x)}{x}\left(\frac{m+2}{2} - \frac{m}{2} \cdot \frac{\tanh(x)}{x}\right) + \frac{m+1}{2} \cdot \frac{\tanh(x)}{x} - \frac{m+3}{2} \\
=&\frac{\sinh(x)}{x} \cdot \frac{m}{2} + \frac{\sinh(x)}{x} - \frac{m}{2} \cdot \frac{\tanh(x)}{x} \cdot \frac{\sinh(x)}{x} \\
&+ \frac{m}{2} \cdot \frac{\tanh(x)}{x} + \frac{\tanh(x)}{2x} - \frac{m}{2} - \frac{3}{2} \\
=&\left(\frac{m}{2}\left(\frac{\tanh(x)}{x} - 1\right)\left(1 - \frac{\sinh(x)}{x}\right)\right) + \left(\frac{\sinh(x)}{x} + \frac{\tanh(x)}{2x} - \frac{3}{2}\right) \\
=&\frac{m}{2}\ f(x) + g(x) \\
>&0.
\end{aligned}$$

So, this inequality also holds for $n = m + 1$. Thus, by mathematical induction, it holds for every $n \geq 1$. □

Theorem 2.2. *For each $x \neq 0$ and $n \geq 1$, the following inequality holds:*

$$\left(\frac{x}{\sinh(x)}\right)^n + \frac{n}{2} \cdot \frac{x}{\tanh(x)} > \frac{n+2}{2}.$$

Proof. By a similar way as in Theorem 2.1, one can prove this inequality. □

In the next theorem, we give a generalization of Huygen's inequality for the hyperbolic trigonometric functions.

Theorem 2.3. *For each $x \neq 0$ and $n \geq 2$, the following inequality holds:*

$$n\ \frac{\sinh(x)}{x} + \frac{\tanh(x)}{x} > n + 1.$$

Proof. We prove this theorem by mathematical induction on n. For $n = 2$, the above inequality holds, because it is Huygens's inequality:

$$2\ \frac{\sinh(x)}{x} + \frac{\tanh(x)}{x} > 3.$$

Now, suppose that this inequality holds for $n = m$, that is, we have

$$m\ \frac{\sinh(x)}{x} + \frac{\tanh(x)}{x} > m + 1.$$

We prove that this inequality also holds for $n = m + 1$. For $n = m + 1$, we have

$$\begin{aligned}(m+1)\left(\frac{\sinh(x)}{x}\right) + \frac{\tanh(x)}{x} &= m\frac{\sinh(x)}{x} + \frac{\sinh(x)}{x} + \frac{\tanh(x)}{x} \\ &> \frac{\sinh(x)}{x} + m + 1 > (m+1) + 1,\end{aligned}$$

since $\frac{\sinh(x)}{x} > 1$, by Lemma 2.1. Thus, this inequality also holds for $n = m+1$. Then, by mathematical induction on n, it holds for every $n \geq 2$. □

The following theorem is a generalization of Huygen's second type inequality.

Theorem 2.4. *For each $x \neq 0$ and $n \geq 1$, the following inequality holds:*

$$\frac{x}{\sinh(x)} + n\ \frac{x}{\tanh(x)} > n + 1.$$

Proof. We prove this theorem by mathematical induction on n. For $n = 1$, the above inequality holds, because it is Wilker's second type inequality:

$$\frac{x}{\sinh(x)} + \frac{x}{\tanh(x)} > 2.$$

Now, suppose that this inequality holds for $n = m$, that is, we have

$$\frac{x}{\sinh(x)} + m\ \frac{x}{\tanh(x)} > m + 1.$$

We prove that this inequality also holds for $n = m + 1$. For $n = m + 1$, we have

$$\begin{aligned}\frac{x}{\sinh(x)} + (m+1)\left(\frac{x}{\tanh(x)}\right) &= m\frac{x}{\tanh(x)} + \frac{x}{\sinh(x)} + \frac{x}{\tanh(x)} \\ &> \frac{x}{\tanh(x)} + m + 1 > (m+1) + 1,\end{aligned}$$

since $\frac{x}{\tanh(x)} > 1$, by Lemma 2.1. Thus this inequality also holds for $n = m+1$. Then, by mathematical induction on n, it holds for every $n \geq 1$. □

Lemma 2.2. *For each $x \neq 0$, the following inequality holds:*

$$\left(\frac{\sinh(x)}{x}\right)^2 - \frac{2}{3}\cosh(x) > \frac{1}{3}.$$

Proof. We define $f : \mathbb{R}^+ \to \mathbb{R}$ by

$$f(x) = \left(\frac{\sinh(x)}{x}\right)^2 - \frac{2}{3}\cosh(x) - \frac{1}{3}.$$

Then, we have

$$\begin{aligned} f'(x) &= 2\left(\frac{x\ \cosh(x) - \sinh(x)}{x^2}\right)\frac{\sinh(x)}{x} - \frac{2}{3}\ \sinh(x) \\ &= \left(3x\cosh(x) - 3\sinh(x) - x^3\right)\left(\frac{2\ \sinh(x)}{3\ x^3}\right). \end{aligned}$$

Now, we define $g : \mathbb{R}^+ \to \mathbb{R}$ by $g(x) = 3x\cosh(x) - 3\sinh(x) - x^3$. Then, we have

$$g'(x) = 3\cosh(x) + 3x\sinh(x) - 3\cosh(x) - 3x^2 = 3x\left(\sinh(x) - x\right) > 0.$$

By Lemma 2.1, $\sinh(x) - x > 0$. Thus g is increasing and since $g(0) = 0$, for $x > 0$, we deduce that $g(x) > g(0) = 0$. Consequently, we get $f'(x) > 0$. So, $f(x)$ is increasing for all $x > 0$. Also,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\left(\frac{\sinh(x)}{x}\right)^2 - \frac{2}{3}\cosh(x) - \frac{1}{3}\right) = 0.$$

Thus, $f(x)$ is positive on $(0, +\infty)$. Therefore, we obtain

$$\left(\frac{\sinh(x)}{x}\right)^2 - \frac{2}{3}\cosh(x) > \frac{1}{3}.$$

This completes the proof. □

Lemma 2.3. *For each $x \neq 0$, the following inequality holds:*

$$\frac{2}{3}\left(1 - \cosh(x)\right)\left(1 - \frac{\sinh(x)}{x}\right) + \left(\frac{\sinh(x)}{x} - \frac{\cosh(x)}{3} - \frac{2}{3}\right) > 0.$$

Proof. We define $f(x)$ by

$$f(x) = \frac{2}{3}\left(1 - \cosh(x)\right)\left(1 - \frac{\sinh(x)}{x}\right) + \left(\frac{\sinh(x)}{x} - \frac{\cosh(x)}{3} - \frac{2}{3}\right).$$

We will prove that

$$3f(x) = \frac{\sinh(x)}{x} + 2\cosh(x)\frac{\sinh(x)}{x} - 3\cosh(x) > 0.$$

Dividing both sides of this inequality by $\cosh(x)$, we get

$$\tanh(x)(1 + 2\cosh(x)) - 3x > 0.$$

So, we have to prove that

$$\tanh(x)(1+2\cosh(x))-3x=\tanh(x)+2\sinh(x)-3x>0.$$

We define $g:\mathbb{R}^+\to\mathbb{R}$ by $g(x)=\tanh(x)+2\sinh(x)-3x$. Then, we have

$$g'(x)=\frac{1}{\cosh^2(x)}+2\ \cosh(x)-3=\frac{1+2\ \cosh^3(x)-3\cosh^2(x)}{\cosh^2(x)}.$$

Now, we define $h:\mathbb{R}^+\to\mathbb{R}$ by $h(x)=1+2\cosh^3(x)-3\cosh^2(x)$. Then, we have

$$\begin{aligned}h'(x)&=6\sinh(x)\cos h^2(x)-6\cosh(x)\sinh(x)\\&=6\sinh(x)\cosh(x)\left(\cosh(x)-1\right)>0.\end{aligned}$$

Since $\cosh(x)>1$, h is increasing and since $h(0)=0$, $h(x)>h(0)=0$. Therefore, we get $g'(x)>0$. So, $g(x)$ is increasing. Since $g(0)=0$, $g(x)>g(0)=0$. Consequently, $3f(x)>0$ and we get

$$f(x)=\frac{2}{3}\left(1-\cosh(x)\right)\left(1-\frac{\sinh(x)}{x}\right)+\left(\frac{\sinh(x)}{x}-\frac{\cosh(x)}{3}-\frac{2}{3}\right)>0.$$

This completes the proof. □

In the next theorem, we give a generalization of Cusa's inequality.

Theorem 2.5. *For each $x\neq 0$ and $n\geq 2$, the following inequality holds:*

$$\left(\frac{\sinh(x)}{x}\right)^n-\frac{n}{3}\cosh(x)>1-\frac{n}{3}.$$

Proof. We prove this theorem by mathematical induction on n. For $n=2$, we have

$$\left(\frac{\sinh(x)}{x}\right)^2-\frac{2}{3}\cosh(x)>\frac{1}{3}\ .$$

The above inequality holds by Lemma 2.3. Now, suppose that this inequality holds for $n=m$, that is, we have

$$\left(\frac{\sinh(x)}{x}\right)^m-\frac{m}{3}\cosh(x)>1-\frac{m}{3}.$$

We prove that this inequality also holds for $n=m+1$. For $n=m+1$, we have

$$\begin{aligned}&\left(\frac{\sinh(x)}{x}\right)^{m+1}-\frac{m+1}{3}\cosh(x)-1+\frac{m+1}{3}\\&=\left(\frac{\sinh(x)}{x}\right)\left(\frac{\sinh(x)}{x}\right)^m-\frac{m+1}{3}\cosh(x)+\frac{m-2}{3}\\&>\left(\frac{\sinh(x)}{x}\right)\left(\frac{m}{3}\cosh(x)-\frac{m-3}{3}\right)-\frac{m+1}{3}\cosh(x)+\frac{m-2}{3}\\&=\frac{m}{3}\cdot\frac{\cosh(x)\sinh(x)}{x}-\frac{m}{3}\left(\frac{\sinh(x)}{x}\right)+\frac{\sinh(x)}{x}-\frac{m}{3}\ \cosh(x)-\frac{\cosh(x)}{3}-\frac{2}{3}\end{aligned}$$

$$=\frac{m}{3}\left(1-\cosh(x)\right)\left(1-\frac{\sinh(x)}{x}\right)+\left(\frac{\sinh(x)}{x}-\frac{\cosh(x)}{3}-\frac{2}{3}\right).$$

We have to prove that the above-mentioned statement is positive. It is enough to prove that expression is positive for $m=2$. We define $f(x)$ by

$$f(x)=\frac{2}{3}\left(1-\cosh(x)\right)\left(1-\frac{\sinh(x)}{x}\right)+\left(\frac{\sinh(x)}{x}-\frac{\cosh(x)}{3}-\frac{2}{3}\right).$$

According to Lemma 2.3, we know that $f(x)$ is positive. Thus this inequality also holds for $n=m+1$. Then, by mathematical induction on n, it holds for every $n\geq 2$. □

The next two corollaries present relations among Cusa, Wilker and Huygens's inequalities.

Corollary 2.1. (a) *For $x>0$,*

$$2\frac{\sinh(x)}{x}+\frac{\tanh(x)}{x}>3,$$

which is Huygens's type inequality.

(b) *For $x>0$,*

$$\left(\frac{\sinh(x)}{x}\right)^2+\frac{\tanh(x)}{x}>2,$$

which is Wilker's inequality of first type.

Proof. (a) If we set $n=1$ in the inequality of Theorem 2.1, then we obtain that

$$2\frac{\sinh(x)}{x}+\frac{\tanh(x)}{x}>3.$$

(a) If we set $n=2$ in the inequality of Theorem 2.1, then we find that

$$\left(\frac{\sinh(x)}{x}\right)^2+\frac{\tanh(x)}{x}>2.$$

This completes the proof. □

Corollary 2.2. (a) *For $x>0$,*

$$2\frac{x}{\sinh(x)}+\frac{x}{\tanh(x)}>3,$$

which is Cusa's type inequality.

(b) *For $x>0$,*

$$\left(\frac{x}{\sinh(x)}\right)^2+\frac{x}{\tanh(x)}>2,$$

which is Wilkmer's inequality of the second type.

Proof. (a) If we set $n = 1$ in the inequality of Theorem 2.2, then we obtain that

$$2\frac{x}{\sinh(x)} + \frac{x}{\tanh(x)} > 3.$$

(b) If we set $n = 2$ in the inequality of Theorem 2.2, then we find that

$$\left(\frac{x}{\sinh(x)}\right)^2 + \frac{x}{\tanh(x)} > 2.$$

This completes the proof. □

Lemma 2.4. *Let x and y be positive real numbers.*

(i) (*Mitrinovic et al.* [10]). *For $\mu \in [0,1]$, $\mu x + (1-\mu)\, y \geqslant x^{\mu} y^{1-\mu}$.*

(ii) (*Issa and Ibrahimov* [9]). *For $x \geq y$ and $\mu \in \left[\frac{1}{2}, 1\right]$,*

$$\mu x + (1-\mu)\, y \geqslant x^{1-\mu} y^{\mu} + (2\mu - 1)\,(x - y) \geqslant x^{\mu} y^{1-\mu}.$$

(iii) (*Issa and Ibrahimov* [9]). *For $x \geq y$ and $\mu \in \left[\frac{1}{2}, \frac{3}{4}\right]$,*

$$\mu x + (1-\mu)\, y \geqslant x^{\mu - \frac{1}{2}} y^{\frac{3}{2} - \mu} + \frac{(x-y)}{2} \geqslant x^{\mu} y^{1-\mu}.$$

3. H and T Nested Functions

In this section, as a special case of Mittag-Leffler functions, we consider H and T nested functions T_{pj} and H_{pj} in order to prove some results about the generalizations of the Wilker and Huygens type inequalities.

We begin this section with the definitions of T_{pj} and H_{pj} and give some properties of these special functions.

Definition 3.1 ([1, 2])**.** Let H and T be nested functions. $T_{pj}, H_{pj} : \mathbb{R} \to \mathbb{R}$, $j = 0, 1, 2, \ldots, p-1$, $p \in \mathbb{N}$, are defined as follows:

$$T_{pj}(t) = \sum_{n=0}^{+\infty} \frac{(-1)^n t^{pn+j}}{(pn+j)!}, \quad H_{pj}(t) = \sum_{n=0}^{+\infty} \frac{t^{pn+j}}{(pn+j)!}.$$

Theorem 3.1 ([1])**.** *For each $t \in \mathbb{R}$, we have*

$$\begin{aligned} T_{p0}^{'}(t) &= -T_{pp-1}(t), \quad H_{p0}^{'}(t) = H_{pp-1}(t), \\ T_{p1}^{'}(t) &= T_{p0}(t), \quad H_{p1}^{'}(t) = H_{p0}(t), \\ &\vdots \\ T_{pp-1}^{'}(t) &= T_{pp-2}(t), \quad H_{pp-1}^{'}(t) = H_{pp-2}(t). \end{aligned}$$

Remark 3.1. Let $\lambda^p = 1$, $\lambda \neq 1, w^p = -1$, $w \neq -1$. Then, we have

$$H_{p0}(t) = \frac{\sum_{j=0}^{p-1} e^{\lambda^j t}}{p}, \quad T_{p0}(t) = \frac{\sum_{j=0}^{p-1} e^{w^j t}}{p}, \quad j = 0, 1, \ldots, p-1,$$

$$H_{p1}(t)=\frac{\sum_{j=0}^{p-1}\lambda^{p-j}e^{\lambda^j t}}{p},\quad T_{p1}(t)=\frac{\sum_{j=0}^{p-1}w^{p-j}e^{w^j t}}{p},\quad j=0,1,\dots,p-1,$$
$$\vdots$$
$$H_{pp-1}(t)=\frac{\sum_{j=0}^{p-1}\lambda^{j}e^{\lambda^j t}}{p},\quad T_{pp-1}(t)=\frac{\sum_{j=0}^{p-1}w^{j}e^{w^j t}}{p},\quad j=0,1,\dots,p-1,$$

and

$$H_{p0}(t)=\frac{\sum_{j=0}^{p-1}e^{\lambda^j t}}{p},\quad j=0,1,\dots,p-1,$$
$$H_{p1}(t)=\frac{\sum_{j=0}^{p-1}\lambda^{p-j}e^{\lambda^j t}}{p},\quad j=0,1,\dots,p-1,$$
$$H_{p2}(t)=\frac{\sum_{j=0}^{p-1}\lambda^{p-2j}e^{\lambda^j t}}{p},\quad j=0,1,\dots,p-1,$$
$$\vdots$$
$$H_{pp-1}(t)=\frac{\sum_{j=0}^{p-1}\lambda^{j}e^{\lambda^j t}}{p},\quad j=0,1,\dots,p-1.$$

For example, for $p=3$,

$$H_{30}(t)=\frac{e^t+e^{\lambda t}+e^{\lambda^2 t}}{3},$$
$$H_{31}(t)=\frac{e^t+\lambda^2 e^{\lambda t}+\lambda e^{\lambda^2 t}}{3},$$
$$H_{32}(t)=\frac{e^t+\lambda e^{\lambda t}+\lambda^2 e^{\lambda^2 t}}{3}.$$

Remark 3.2 ([2]). For $p=3$, we have $w^p=-1, w\neq -1$, and so

$$T_{30}(t)=\frac{e^{-t}+e^{wt}+e^{-w^2 t}}{3},$$
$$T_{31}(t)=\frac{-e^{-t}-w^2 e^{wt}+we^{-w^2 t}}{3},$$
$$T_{32}(t)=\frac{e^{-t}-we^{wt}+w^2 e^{-w^2 t}}{3}.$$

Definition 3.2. The functions ${}_p\tan_{ij}$ and ${}_p\tanh_{ij}:\mathbb{R}\to\mathbb{R}$, $i,j=0,1,2,\dots,p-1$, $p\in\mathbb{N}, i\neq j$, are defined as follows:

$${}_p\tan_{ij}(t)=\frac{T_{pi}(t)}{T_{pj}(t)},\quad {}_p\tanh_{ij}(t)=\frac{H_{pi}(t)}{H_{pj}(t)}.$$

Remark 3.3. By Remark 3.1, for $p = 3$, we have

$$
\begin{aligned}
{}_3\tan_{10}(t) &= \frac{H_{31}(t)}{H_{30}(t)} = \frac{e^t + \lambda^2 e^{\lambda t} + \lambda e^{\lambda^2 t}}{e^t + e^{\lambda t} + e^{\lambda^2 t}},\\
{}_3\tanh_{21}(t) &= \frac{H_{32}(t)}{H_{31}(t)} = \frac{e^t + \lambda e^{\lambda t} + \lambda^2 e^{\lambda^2 t}}{e^t + \lambda^2 e^{\lambda t} + \lambda e^{\lambda^2 t}},\\
{}_3\tanh_{02}(t) &= \frac{H_{30}(t)}{H_{32}(t)} = \frac{e^t + e^{\lambda t} + e^{\lambda^2 t}}{e^t + \lambda e^{\lambda t} + \lambda^2 e^{\lambda^2 t}}.
\end{aligned}
$$

Lemma 3.1. *For each $x \neq 0$, the following inequalities hold:*

$$
\begin{aligned}
&\frac{{}_p\tanh_{10}(x)}{x} < 1, \quad \frac{H_{p1}(x)}{x} > 1,\\
&\frac{{}_p\tanh_{ii-1}(x)}{x} < 1, \quad i = 0, 1, 2, \ldots, p-1.
\end{aligned}
$$

Proof. We see that

$$
\frac{H_{p1}(x)}{x} = \frac{\sum_{n=0}^{+\infty} \frac{x^{pn+1}}{(pn+1)!}}{x} = 1 + \sum_{n=1}^{+\infty} \frac{x^{pn}}{(pn+1)!} > 1
$$

and

$$
\sum_{n=0}^{+\infty} \frac{x^{pn+1}}{(pn+1)!} = H_{p1}(x) < xH_{p0}(x) = \sum_{n=0}^{+\infty} \frac{x^{pn+1}}{(pn)!}.
$$

Thus, $\frac{H_{p1}(x)}{xH_{p0}(x)} < 1$ and so

$$
\frac{{}_p\tanh_{10}(x)}{x} = \frac{H_{p1}(x)}{xH_{p0}(x)} < 1.
$$

We have

$$
\sum_{n=0}^{+\infty} \frac{x^{pn+i}}{(pn+i)!} = H_{pi}(x) < xH_{pi-1}(x) = \sum_{n=0}^{+\infty} \frac{x^{pn+i}}{(pn+i)!}.
$$

Hence, $\frac{H_{pi}(x)}{xH_{pi-1}(x)} < 1$ and so

$$
\frac{{}_p\tanh_{10}(x)}{x} = \frac{H_{p1}(x)}{xH_{p0}(x)} < 1.
$$

This completes the proof. □

Lemma 3.2. *For each $x \neq 0$, the following inequality holds:*

$$
\left(\frac{H_{p1}(x)}{x}\right)^{p+1} > H_{p0}(x).
$$

Proof.

$$\begin{aligned}\left(\frac{H_{p1}(x)}{x}\right)^{p+1} - H_{p0}(x) &= \left(\sum_{n=0}^{+\infty}\frac{x^{pn}}{(pn+1)!}\right)^{p+1} - \sum_{n=0}^{+\infty}\frac{x^{pn}}{(pn)!} \\ &= \left(1+\frac{x^p}{p!}+\frac{x^{2p+1}}{(2p)!}+\cdots\right) - 1 - \frac{x^{2p+1}}{p!} - \frac{x^{3p+1}}{(2p)!} - \cdots \\ &> 0.\end{aligned}$$

This completes the proof. □

Lemma 3.3. *For each $x \neq 0$, the following inequality holds:*

$$p\frac{H_{p1}(x)}{x} + \frac{{}_p\tanh_{10}(x)}{x} > p+1.$$

Proof. Let $x > 0$. Then by Lemma 3.2, we have

$$1 < \left(\frac{H_{p1}(x)}{x}\right)^{\frac{p}{p+1}}\left(\frac{{}_p\tanh_{10}(x)}{x}\right)^{\frac{1}{p+1}} < \frac{p}{p+1}\cdot\frac{H_{p1}(x)}{x} + \frac{1}{p+1}\cdot\frac{{}_p\tanh_{10}(x)}{x}.$$

So, $p+1 < p\frac{H_{p1}(x)}{x} + \frac{{}_p\tanh_{10}(x)}{x}$. □

Problem 3.1 (Open problem). For each $x \neq 0$, the following inequality holds:

$$p\frac{T_{p1}(x)}{x} + \frac{{}_p\tan_{10}(x)}{x} > p+1.$$

Theorem 3.2. *For each $x \neq 0$ and $n \geq 1$, the following inequality holds:*

$$\left(\frac{H_{p1}(x)}{x}\right)^n + \frac{n}{p}\cdot\frac{{}_p\tanh_{10}(x)}{x} > \frac{n+p}{p}.$$

Proof. We prove this theorem by mathematical induction on n. For $n = 1$, the inequality holds, because

$$p\frac{H_{p1}(x)}{x} + \frac{{}_p\tanh_{10}(x)}{x} > p+1.$$

This is the inequality given in Lemma 3.3. Now suppose that this inequality holds for $n = m$, that is, we have

$$\left(\frac{H_{p1}(x)}{x}\right)^m + \frac{m}{p}\cdot\frac{{}_p\tanh_{10}(x)}{x} > \frac{m+p}{p}.$$

We prove that this inequality also holds for $n = m+1$. For $n = m+1$, we have

$$\begin{aligned}&\left(\frac{H_{p1}(x)}{x}\right)^{m+1} + \frac{m+1}{p}\cdot\frac{{}_p\tanh_{10}(x)}{x} - \frac{m+p+1}{p} \\ =&\frac{H_{p1}(x)}{x}\left(\frac{H_{p1}(x)}{x}\right)^m + \frac{m+1}{p}\cdot\frac{{}_p\tanh_{10}(x)}{x} - \frac{m+p+1}{p}\end{aligned}$$

$$
\begin{aligned}
&> \frac{H_{p1}(x)}{x}\left(\frac{m+p}{p} - \frac{m}{p}\cdot\frac{{}_p\tanh_{10}(x)}{x}\right) + \frac{m+1}{p}\cdot\frac{{}_p\tanh_{10}(x)}{x} - \frac{m+p+1}{p}\\
&= \frac{m}{p}\cdot\frac{H_{p1}(x)}{x} + \frac{H_{p1}(x)}{x} - \frac{m}{p}\cdot\frac{{}_p\tanh_{10}(x)}{x}\cdot\frac{H_{p1}(x)}{x}\\
&\quad + \frac{m}{p}\cdot\frac{{}_p\tanh_{10}(x)}{x} + \frac{{}_p\tanh_{10}(x)}{px} - \frac{m}{p} - \frac{p+1}{p}\\
&= \frac{m}{p}\left(\frac{{}_p\tanh_{10}(x)}{x} - 1\right)\left(1 - \frac{H_{p1}(x)}{x}\right) + \left(\frac{H_{p1}(x)}{x} + \frac{{}_p\tanh_{10}(x)}{px} - \frac{p+1}{p}\right)\\
&= \frac{m}{p}f(x) + g(x)\\
&> 0.
\end{aligned}
$$

By Lemma 3.1, $\frac{{}_p\tanh_{10}(x)}{x} < 1$ and so $\frac{H_{p1}(x)}{x} > 1$. Thus, we get

$$f(x) = \left(\frac{{}_p\tanh_{10}(x)}{x} - 1\right)\left(1 - \frac{H_{p1}(x)}{x}\right) > 0.$$

Also, by Lemma 3.3, we have

$$g(x) = \frac{H_{p1}(x)}{x} + \frac{{}_p\tanh_{10}(x)}{x} - \frac{p+1}{p} > 0.$$

So, this inequality also holds for $n = m+1$. Thus, by mathematical induction, it holds for every $n \geq 1$. □

Theorem 3.3. *Let $x > 0$, $a > 0$, $b > 0$, $p \geq 2$, $p \in \mathbb{N}$ and $m \geq \frac{pnb}{a}$. Then, for $n > 0$, the following inequality holds:*

$$\frac{a}{a+b}\left(\frac{H_{p1}(x)}{x}\right)^m + \frac{b}{a+b}\left(\frac{{}_p\tanh_{10}(x)}{x}\right)^n > 1.$$

Proof. By Lemma 2.4, we have

$$
\begin{aligned}
&\frac{a}{a+b}\left(\frac{H_{p1}(x)}{x}\right)^m + \frac{b}{a+b}\left(\frac{{}_p\tanh_{10}(x)}{x}\right)^n\\
&\geq \left(\frac{H_{p1}(x)}{x}\right)^{\frac{ma}{a+b}}\left(\frac{{}_p\tanh_{10}(x)}{x}\right)^{\frac{nb}{a+b}}\\
&= \left(\frac{H_{p1}(x)}{x}\right)^{\frac{ma}{a+b}}\left(\frac{H_{p1}(x)}{x}\right)^{\frac{nb}{a+b}}\left(\frac{1}{H_{p0}(x)}\right)^{\frac{nb}{a+b}}\\
&> \left(\frac{H_{p1}(x)}{x}\right)^{\frac{ma}{a+b}}\left(\frac{H_{p1}(x)}{x}\right)^{\frac{nb}{a+b}}\left(\frac{H_{p1}(x)}{x}\right)^{\frac{-(p+1)nb}{a+b}}\\
&= \left(\frac{H_{p1}(x)}{x}\right)^{\frac{ma-pnb}{a+b}} > 1.
\end{aligned}
$$

This completes the proof. □

Lemma 3.4. *For each $x \neq 0$, the following inequalities hold:*

$$\frac{H_{pi}(x)}{x^i} > 1, \quad i = 0, 1, 2, \ldots, p-1,$$
$$\frac{{}_p\tanh_{i0}(x)}{x^i} < 1, \quad i = 0, 1, 2, \ldots, p-1,$$
$$\frac{{}_p\tanh_{ii-1}(x)}{x} < 1.$$

Proof. We have

$$\frac{H_{pi}(x)}{x^i} = \frac{\sum_{n=0}^{+\infty} \frac{x^{pn+i}}{(pn+i)!}}{x^i} = 1 + \sum_{n=1}^{+\infty} \frac{x^{pn}}{(pn+i)!} > 1$$

and since

$$\sum_{n=0}^{+\infty} \frac{x^{pn+i}}{(pn+i)!} = H_{pi}(x) < x^i H_{p0}(x) = \sum_{n=0}^{+\infty} \frac{x^{pn+i}}{(pn)!},$$

we have

$$\frac{H_{pi}(x)}{x^i H_{p0}(x)} < 1.$$

Thus,

$$\frac{{}_p\tanh_{i0}(x)}{x^i} = \frac{H_{pi}(x)}{x^i H_{p0}(x)} < 1$$

and since

$$\sum_{n=0}^{+\infty} \frac{x^{pn+i}}{(pn+i)!} = H_{pi}(x) < x H_{pi-1}(x) = \sum_{n=0}^{+\infty} \frac{x^{pn+i}}{(pn+i-1)!},$$

we have

$$\frac{H_{pi}(x)}{x H_{pi-1}(x)} < 1.$$

This completes the proof. □

Lemma 3.5. *For each $x \neq 0$, the following inequality holds:*

$$\left(\frac{H_{pi}(x)}{x}\right)^{p+i} > H_{pi-1}(x), \quad p \geq 2, i = 1, 2, \ldots, p-1.$$

Proof. We have

$$\begin{aligned}
\left(\frac{H_{pi}(x)}{x}\right)^{p+1} - H_{pi-1}(x) &= \left(\sum_{n=0}^{+\infty} \frac{x^{pn+i-1}}{(pn+i)!}\right)^{p+i} - \sum_{n=0}^{+\infty} \frac{x^{pn+i-1}}{(pn+i-1)!} \\
&= \left(1 + \frac{x^{p+i-1}}{(p+i-1)!} + \cdots\right) - 1 - \frac{x^{pn+i-1}}{(pn+i-1)!} - \cdots \\
&> 0.
\end{aligned}$$

This completes the proof. □

Lemma 3.6. *For each $x \neq 0$, the following inequality holds:*

$$p\frac{H_{pi}(x)}{x} + \frac{{}_p\tanh_{ii-1}(x)}{x} > p+1.$$

Proof. Let $x > 0$. Then, by Lemma 3.5, we have

$$1 < \left(\frac{H_{pi}(x)}{x}\right)^{\frac{p}{p+i}} \left(\frac{{}_p\tanh_{ii-1}(x)}{x}\right)^{\frac{i}{p+i}} < \frac{p}{p+i}\cdot\frac{H_{pi}(x)}{x} + \frac{i}{p+i}\cdot\frac{{}_p\tanh_{ii-1}(x)}{x}.$$

So,

$$p+i < p\frac{H_{pi}(x)}{x} + i\frac{{}_p\tanh_{ii-1}(x)}{x}.$$

This completes the proof. □

Theorem 3.4. *For each $x \neq 0$, $i = 1, 2, \ldots, p-1$ and $n \geq 1$, the following inequality holds:*

$$\left(\frac{H_{pi}(x)}{x}\right)^n + \frac{n}{p}\cdot\frac{{}_p\tanh_{ii-1}(x)}{x} > \frac{n+p}{p}.$$

Proof. Let $x > 0$. Then, by Lemma 3.5, we have

$$1 < \left(\frac{H_{pi}(x)}{x}\right)^{\frac{p}{p+n}} \left(\frac{{}_p\tanh_{ii-1}(x)}{x}\right)^{\frac{n}{p+n}} < \frac{p}{p+n}\cdot\frac{H_{pi}(x)}{x} + \frac{n}{p+n}\cdot\frac{{}_p\tanh_{ii-1}(x)}{x}.$$

So,

$$p+n < p\frac{H_{pi}(x)}{x} + n\frac{{}_p\tanh_{ii-1}(x)}{x}.$$

This completes the proof. □

Theorem 3.5. *Let $x > 0$, $a > 0$, $b > 0$, $p \geq 2$, $p \in \mathbb{N}$, $i = 1, 2, \ldots, p-1$ and $m \geq \frac{(p+i-1)nb}{a}$. Then, for $n > 0$, the following inequality holds:*

$$\frac{a}{a+b}\left(\frac{H_{pi}(x)}{x}\right)^m + \frac{b}{a+b}\left(\frac{{}_p\tanh_{ii-1}(x)}{x}\right)^n > 1.$$

Proof. By Lemma 2.4, we have

$$\begin{aligned}
&\frac{a}{a+b}\left(\frac{H_p(x)}{x}\right)^m + \frac{b}{a+b}\left(\frac{{}_p\tanh_{ii-1}(x)}{x}\right)^n \\
&\geq \left(\frac{H_{pi}(x)}{x}\right)^{\frac{ma}{a+b}} \left(\frac{{}_p\tanh_{ii-1}(x)}{x}\right)^{\frac{nb}{a+b}} \\
&= \left(\frac{H_{pi}(x)}{x}\right)^{\frac{ma}{a+b}} \left(\frac{H_{pi}(x)}{x}\right)^{\frac{nb}{a+b}} \left(\frac{1}{H_{pi-1}(x)}\right)^{\frac{nb}{a+b}} \\
&> \left(\frac{H_{pi}(x)}{x}\right)^{\frac{ma}{a+b}} \left(\frac{H_{pi}(x)}{x}\right)^{\frac{nb}{a+b}} \left(\frac{H_{pi}(x)}{x}\right)^{\frac{-(p+i)nb}{a+b}}
\end{aligned}$$

$$= \left(\frac{H_{pi}(x)}{x^i}\right)^{\frac{ma-(p+i-1)nb}{a+b}} > 1.$$

This completes the proof. □

Lemma 3.7. *Let $x > 0$. Then, the following inequality holds:*

$$\left(\frac{H_{p1}(x)}{x}\right)^p > (p+1)(H_{p0}(x) - 1), \quad p \geq 2.$$

Proof. The proof is simply calculated. □

Lemma 3.8. *Let $x > 0$. Then, the following inequality holds:*

$$\left(\frac{H_{p1}(x)}{x}\right)^p - \frac{p}{p+1} H_{p0}(x) > \frac{1}{p+1}.$$

Proof. We have

$$\begin{aligned}
\left(\frac{H_{p1}(x)}{x}\right)^p - \frac{p}{p+1} H_{p0}(x) &= \left(\sum_{n=0}^{+\infty} \frac{x^{pn}}{(pn+1)!}\right)^p - \frac{p}{p+1} \sum_{n=0}^{+\infty} \frac{x^{pn}}{(pn)!} \\
&= \left(1 + \frac{p}{p+1} \cdot \frac{x^p}{(p)!} + \frac{x^{2p}}{(2p)!} + \cdots\right) \\
&\quad - \frac{p}{p+1} - \frac{p}{p+1} \cdot \frac{x^p}{(p)!} - \frac{p}{p+1} \cdot \frac{x^{2p}}{(2p)!} - \cdots \\
&> \frac{1}{p+1}.
\end{aligned}$$

This completes the proof. □

Lemma 3.9. *Let $x > 0$. Then, the following inequality holds:*

$$\frac{p}{p+1}\left(1 - H_{p0}(x)\right)\left(1 - \frac{H_{p1}(x)}{x}\right) + \left(\frac{H_{p1}(x)}{x} - \frac{H_{p0}(x)}{p+1} - \frac{p}{p+1}\right) > 0.$$

Proof. We define $f(x)$ by

$$f(x) = \frac{p}{p+1}\left(1 - H_{p0}(x)\right)\left(1 - \frac{H_{p1}(x)}{x}\right) + \left(\frac{H_{p1}(x)}{x} - \frac{H_{p0}(x)}{p+1} - \frac{p}{p+1}\right).$$

Then, we have

$$\begin{aligned}
(p+1)f(x) &= \frac{H_{p1}(x)}{x} + pH_{p0}(x)\frac{H_{p1}(x)}{x} - (p+1)H_{p0}(x) \\
&= \frac{H_{p1}(x)(1 + pH_{p0}(x))}{x} - (p+1)H_{p0}(x) \\
&= (1 + pH_{p0}(x)) \sum_{n=0}^{+\infty} \frac{x^{pn}}{(pn+1)!} - (p+1) \sum_{n=0}^{+\infty} \frac{x^{pn}}{(pn)!}
\end{aligned}$$

$$
\begin{aligned}
&=(1+pH_{p0}(x))-(p+1)+(1+pH_{p0}(x))\sum_{n=1}^{+\infty}\frac{x^{pn}}{(pn+1)!}\\
&\quad-(p+1)\sum_{n=1}^{+\infty}\frac{x^{pn}}{(pn)!}\\
&=pH_{p0}(x))-p+(1+p\ H_{p0}(x))\sum_{n=1}^{+\infty}\frac{x^{pn}}{(pn+1)!}-(p+1)\sum_{n=1}^{+\infty}\frac{x^{pn}}{(pn)!}\\
&=(1+pH_{p0}(x))\sum_{n=1}^{+\infty}\frac{x^{pn}}{(pn+1)!}-\sum_{n=1}^{+\infty}\frac{x^{pn}}{(pn)!}\\
&=(1+pH_{p0}(x))\sum_{n=1}^{+\infty}\frac{x^{pn}}{(pn+1)!}+1-H_{p0}(x)\\
&=1+\sum_{n=1}^{+\infty}\frac{x^{pn}}{(pn+1)!}+H_{p0}(x)\left(p\sum_{n=1}^{+\infty}\frac{x^{pn}}{(pn+1)!}-1\right)\\
&>0.
\end{aligned}
$$

This completes the proof. □

From above, we have

$$(1+pH_{p0}(x))\sum_{n=1}^{+\infty}\frac{x^{pn}}{(pn+1)!}+1-H_{p0}(x)>0,$$

that is, $(1+pH_{p0}(x))(H_{p1}(x)-x^p)>x+xH_{p0}(x)$. So,

$$(1+pH_{p0}(x))H_{p1}(x)>x+xH_{p0}(x)+x^p(1+pH_{p0}(x)).$$

4. Conclusion

In this paper, we studied Wilker's and Huygens's inequalities and expressed some generalizations of these inequalities based on trigonometric hyperbolic functions. Then, we studied the relationship between these inequalities by initializing the value of the given parameter n. In the end, we investigated the trigonometric inequality of Cusa and proved a generalization of this inequality.

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[1]Department of Mathematics,
Payame Noor University,
P. O. Box 19395-3697, Tehran, Iran
Email address: petroudi@pnu.ac.ir

[2]Department of Mathematics,
Islamic Azad University,
Karaj, Iran
[3]Department of Mathematics and Applied Mathematics,
Sefako Makgatho Health Sciences University,
Medunsa-0204, South Africa
Email address: analsisamirmath2@gmail.com
ORCID iD: https://orcid.org/0000-0002-5527-1284

[4]Research Institute for Convergence of Basic Science,
YK Intercollege, Hanyang University,
Seoul 04763, Korea
Email address: baak@hanyang.ac.kr
ORCID iD: https://orcid.org/0000-0001-6329-8228

[5]Department of Mathematics,
Kastamonu University,
Kastamonu, Turkey
Email address: ahmetdasdemir37@gmail.com
ORCID iD: https://orcid.org/0009-0006-2890-4046