ON GRADED 2-NIL-GOOD RINGS

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Abstract. In this paper we introduce and study the notion of a graded 2-nil-good ring which is graded by a group. We discuss graded group ring and graded matrix ring extensions of graded 2-nil-good rings. The question of when the 2-nil-good property of the component, which corresponds to the identity element of the grading group, implies the graded 2-nil-good property of the whole graded ring is also examined.

1. Introduction

Ever since the introduction of clean rings in [20] as rings in which every element can be written as a sum of an idempotent and a unit, many papers have been written discussing the ring structure depending on the various ring element properties. In particular, many results are obtained concerning nil clean rings introduced in [6]. For instance, study of matrix rings over nil clean rings is related to the famous Köthe’s Conjecture (see [16] and references therein). Nil-cleaness of group rings has also attracted attention (see [17, 21]).

Theory of graded rings has also been studied by many authors (see [13, 19]). Graded nil clean rings are introduced in [10], and in this paper we continue with studying rings determined by various properties defined elementwise from the graded ring theory point of view. Namely, we introduce and study graded 2-nil-good rings as a graded version of the notion introduced recently in [1]. In [1], a 2-nil-good ring is defined as a ring whose every element is a sum of two units and a nilpotent and the main results deal with the question of when the matrix rings are 2-nil-good.

Here, by a graded 2-nil-good ring we mean a group graded ring whose every homogeneous element can be written as a sum of two homogeneous units and a homogeneous

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nilpotent. We start by giving the basic properties of graded 2-nil-good rings which represent graded versions of results concerning 2-nil-good rings. We are also interested in the question of when the (graded) group ring is (graded) 2-nil-good, which is the content of several theorems. These yield an interesting question of how the graded 2-nil-good property of a group graded ring depends on the 2-nil-good property of the component which corresponds to the identity element of the grading group. It is shown that 2-nil-good property of the component corresponding to the identity element of the grading group does not imply the graded 2-nil-good property of the whole graded ring in general. However, under some additional assumptions, this implication does hold true. Finally, we prove that the graded matrix ring over a crossed product, which is graded 2-nil-good, is also a graded 2-nil-good ring.

2. Preliminaries

All rings are assumed to be associative with identity. If \( R \) is a ring, then, as usual, \( J(R) \) denotes the Jacobson radical of \( R \), and \( U(R) \) stands for the multiplicative group of units of \( R \).

Next we recall the notions of a group graded ring and module, and how the group ring and the matrix ring over a group graded ring can be graded. For other graded ring theory notions and further details, we refer to [13,19].

Let \( R \) be a ring, \( G \) a group with the identity element \( e \), and let \( \{R_g\}_{g \in G} \) be a family of additive subgroups of \( R \). \( R \) is said to be \( G \)-graded if \( R = \bigoplus_{g \in G} R_g \) and \( R_g R_h \subseteq R_{gh} \) for all \( g, h \in G \). The set \( H = \bigcup_{g \in G} R_g \) is called the homogeneous part of \( R \), elements of \( H \) are called homogeneous, and subgroups \( R_g \ (g \in G) \) are called components. If \( a \in R_g \), then we say that \( a \) has the degree \( g \).

A \( G \)-graded ring \( R = \bigoplus_{g \in G} R_g \) is called a crossed product if \( U(R) \cap R_g \neq \emptyset \) for all \( g \in G \).

A right ideal (left, two-sided) \( I \) of a \( G \)-graded ring \( R = \bigoplus_{g \in G} R_g \) is called homogeneous or graded if \( I = \bigoplus_{g \in G} I \cap R_g \). If \( I \) is a two-sided homogeneous ideal (homogeneous ideal in the rest of the paper), then \( R/I \) is a \( G \)-graded ring with components \((R/I)_g = R_g/I \cap R_g \). A graded ring \( R \) is graded-nil if every homogeneous element of \( R \) is nilpotent.

Let \( R = \bigoplus_{g \in G} R_g \) be a \( G \)-graded ring, and observe the group ring \( R[G] \). According to [18], we have that \( R[G] \) is \( G \)-graded with the \( g \)-component \((R[G])_g = \sum_{h \in G} R_{gh^{-1}}h \) and with the multiplication defined via the rule \((r_g g')(r_h h') = r_g r_h (h^{-1} g' h h')\), where \( g, g', h, h' \in G \) and \( r_g \in R_g, \ r_h \in R_h \).

If \( H \) is a normal subgroup of \( G \), then, according to [19], we may observe \( R[H] \) as a \( G \)-graded ring \( \bigoplus_{g \in G} (R[H])_g \), where \((R[H])_g = \bigoplus_{h \in H} R_{gh^{-1}}h \), and where the multiplication is given by \((r_g g')(r_h h') = r_g r_h (h^{-1} g' h h')\), where \( g, h \in G, \ g', h' \in H \) and \( r_g \in R_g, \ r_h \in R_h \).

All of the group rings in this paper, if observed as graded rings, are assumed to be graded in one of the above described ways.

If \( R \) is a \( G \)-graded ring and \( n \) a natural number, then the matrix ring \( M_n(R) \) can
be made into a $G$-graded ring in the following manner. Let $\sigma = (g_1, \ldots, g_n) \in G^n$, $\lambda \in G$ and $M_n(R)_{\lambda}(\sigma) = (a_{ij})_{n \times n}$, where $a_{ij} \in R_{g_i\lambda g_j^{-1}}$, $i, j \in \{1, 2, \ldots, n\}$. Then $M_n(R) = \bigoplus_{\lambda \in G} M_n(R)_{\lambda}(\sigma)$ is a $G$-graded ring with respect to the usual matrix addition and multiplication. This ring is usually denoted by $M_n(R)(\sigma)$.

If $R = \bigoplus_{g \in G} R_g$ is a $G$-graded ring, then a right $G$-graded $R$-module is a right $R$-module $M$ such that $M = \bigoplus_{x \in G} M_x$, where $M_x$ are additive subgroups of $M$, and such that $M_x R_g \subseteq M_{xg}$ for all $x, g \in G$. A submodule $N$ of a $G$-graded $R$-module $M = \bigoplus_{x \in G} M_x$ is called homogeneous if $N = \bigoplus_{x \in G} N \cap M_x$.

A right $G$-graded $R$-module $M$ is said to be graded irreducible if $MR \neq 0$ and if the only homogeneous submodules of $M$ are trivial submodules. The graded Jacobson radical $J^g(R)$ of a $G$-graded ring $R$ is defined to be the intersection of annihilators of all graded irreducible graded $R$-modules. It is known that $J^g(R)$ coincides with the intersection of all maximal homogeneous right ideals of $R$, and that it is left-right symmetric.

3. Graded 2-Nil-Good Rings

Let $G$ be a group with the identity element $e$.

**Definition 3.1.** A homogeneous element of a $G$-graded ring is said to be graded 2-nil-good if it can be written as a sum of two homogeneous units and a homogeneous nilpotent. A $G$-graded ring is said to be graded 2-nil-good if every of its homogeneous elements is graded 2-nil-good.

**Example 3.1.** Let $p > 2$ be a prime number, $G = \{e, g\}$ a cyclic group of order 2, and $R = \left(\begin{array}{cc} \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{array}\right)$. The ring $\mathbb{Z}_p$ is a 2-nil-good ring (see [1, Example 2.1]). Then $R = \left(\begin{array}{cc} \mathbb{Z}_p & 0 \\ 0 & \mathbb{Z}_p \end{array}\right) \bigoplus \left(\begin{array}{cc} 0 & \mathbb{Z}_p \\ \mathbb{Z}_p & 0 \end{array}\right)$ is a $G$-graded 2-nil-good ring.

**Remark 3.1.** Let us notice that if $R = \bigoplus_{g \in G} R_g$ is a $G$-graded ring which is graded 2-nil-good, then $R_e$ is a 2-nil-good ring. Namely, even if $a \in R_e$ is a nilpotent, we can always write $a = 1 + (-1) + a$, and $1 \in R_e$. If $g \neq e$, for a nilpotent element $a \in R_g$, we may have a different situation, that is, it may be the case that $a$ is written as a sum of itself and of two units which are not of degree $g$, for instance $a = 1 + (-1) + a$. However, if we assume that $R$ is a crossed product, then every homogeneous element can be written as a sum of two homogeneous units and a homogeneous nilpotent, all of which are of the same degree.

In [22], 2-good rings are defined as rings in which every element can be written as a sum of two units. Since 2-good rings are closely related to 2-nil-good rings, as every 2-good ring is 2-nil-good, it is natural to introduce the following notion as well.

**Definition 3.2.** A homogeneous element of a $G$-graded ring is said to be graded 2-good if it can be written as a sum of two homogeneous units. A $G$-graded ring is said to be graded 2-nil-good if every of its homogeneous elements is graded 2-good.
Remark 3.2. Let us notice that all graded 2-good rings are crossed products. Also, obviously, every graded 2-good ring is graded 2-nil-good. Example 3.1 also serves as an example of a graded 2-good ring.

In [1] it is proved that $R$ is a 2-nil-good ring if and only if $R/I$ is 2-nil-good, whenever $I$ is a nil ideal of $R$. Here we have the following result.

**Theorem 3.1.** Let $R$ be a $G$-graded ring and $I$ a graded-nil ideal of $R$. Then $R$ is graded 2-nil-good if and only if $R/I$ is graded 2-nil-good.

**Proof.** If $R$ is graded 2-nil-good, then $R/I$ is also graded 2-nil-good as a graded homomorphic image of $R$.

Conversely, let $R/I$ be a graded 2-nil-good ring and let $\bar{x} = x + I \in R_g/I_g$, where $g \in G$. Then $\bar{x} = \bar{u} + \bar{v} + \bar{w}$, where $\bar{u}, \bar{v}$ are homogeneous units of $R/I$, and $\bar{w}$ is a nilpotent element of degree $g$ in $R/I$. Since $I$ is graded-nil, we have that $w$ is a homogeneous nilpotent of degree $g$ in $R$. Also, since $I$, as a graded-nil ideal, is contained in the graded Jacobson radical $J^g(R)$, homogeneous units lift modulo $I$ (see [19, Proposition 2.9.1]), and the claim follows. $\square$

**Corollary 3.1.** Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring, where $G$ is a finite group, and $R_e$ is a PI-ring. Also, let $I \subseteq J(R)$ be a homogeneous ideal of $R$ such that $I_e$ is nil. Then $R$ is graded 2-nil-good if and only if $R/I$ is graded 2-nil-good.

**Proof.** Since $G$ is finite and $R_e$ is a PI-ring, by [12] we know that $R$ is also a PI-ring. This and the fact that $I \subseteq J(R)$ is a homogeneous ideal with $I_e$ nil together imply that $I$ is nil by [14, Lemma 5]. In particular, $I$ is graded-nil, and the claim follows by the previous theorem. $\square$

**Definition 3.3 ([10]).** A homogeneous element $a$ of a $G$-graded ring is said to be graded strongly $\pi$-regular if it can be written as a sum of a homogeneous idempotent element $f$ and a homogeneous unit $u$ such that $fa = af$ and $faf$ is nilpotent.

Naturally, by a graded strongly $\pi$-regular ring we mean a $G$-graded ring whose every homogeneous element is graded strongly $\pi$-regular.

The following result represents a graded version of [1, Theorem 2.1].

**Theorem 3.2.** Let $R = \bigoplus_{g \in G} R_g$ be a graded strongly $\pi$-regular ring. The following statements are equivalent:

i) $R$ is graded 2-nil-good;

ii) $1 = u + v$ for some units $u, v$ from $R_e$.

**Proof.** i)⇒ii) If $R$ is a graded 2-nil-good ring, it follows that $R_e$ is 2-nil-good. Since $1 \in R_e$, the claim follows by [1, Theorem 2.1] applied to the ring $R_e$.

ii)⇒i) Again, if we apply [1, Theorem 2.1] to the ring $R_e$, we have that $R_e$ is 2-nil-good. Let $0 \neq x \in R_g$, where $g \neq e$. Then, since $R$ is graded strongly $\pi$-regular, it follows that $x$ is a unit. Since $1 = u + v$, with $u, v \in U(R_e)$, we have that $x = 1x = ux + vx + 0$, hence $x$ is graded 2-nil-good. $\square$
By [3, Proposition 10], if $R$ is a clean ring with $2 \in U(R)$, then $R$ is 2-good. We end this section with a graded version of this result.

**Theorem 3.3.** Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring. If $R$ is graded clean and $2 \in U(R)$, then $R$ is graded 2-good.

**Proof.** By assumption, $R$ is graded clean, which means that $R_e$ is clean (see [10]). Since $2 \in R_e$, we have that $R_e$ is 2-good by [3, Proposition 10]. Now, let $0 \neq x \in R_g$, where $g \neq e$. Since $R$ is by assumption graded clean, we have that $x$ is a unit. Therefore $x/2 \neq 0$ is a homogeneous unit $u$ of degree $g$. Hence $x = 2u = u + u$, and so, $R$ is graded 2-good.  

4. **Extensions of Graded 2-Nil-Good Rings**

4.1. **Group rings.** In this subsection we investigate graded 2-nil-good property of graded group rings. However, we first establish some sufficient conditions for a group ring to be 2-nil-good.

**Theorem 4.1.** Let $R$ be a 2-nil-good ring, and let $p$ be a prime number which is nilpotent in $R$. If $G$ is a locally finite $p$-group, then $R[G]$ is a 2-nil-good ring.

**Proof.** As in the proof of [21, Theorem 2.3], we may assume that $G$ is a finite $p$-group. Since $p$ is nilpotent, by [5, Theorem 9], we have that the augmentation ideal $\Delta(R[G])$ is nilpotent. Since $R[G]/\Delta(R[G])$ and $R$ are isomorphic as rings, by [1, Theorem 2.2], we then have that $R[G]$ is a 2-nil-good ring.

**Remark 4.1.** One example of a 2-nil-good ring satisfying the assumptions of the previous theorem is $\mathbb{Z}_p$, where $p > 2$ is a prime number.

**Theorem 4.2.** Let $R$ be a clean ring with $2 \in U(R)$. If $p > 2$ is a prime number belonging to $J(R)$, and $G$ a locally finite $p$-group, then $R[G]$ is 2-nil-good.

**Proof.** Since $R$ is clean, $G$ a locally finite $p$-group and $p \in J(R)$, according to [24, Theorem 4], we have that $R[G]$ is clean. Also, since 2 is a unit in $R$, it is also a unit in $R[G]$. By [3, Proposition 10], $R[G]$ is 2-good, and therefore, 2-nil-good.  

**Theorem 4.3.** Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring, where $G$ is a finite group. If $R$ is a semilocal ring with $2 \in U(R)$, then $R[G]$ is a 2-nil-good ring.

**Proof.** If $R$ is semilocal with $2 \in U(R)$, then by [23, Proposition 2.10] we have that $R$ is 2-good, and therefore 2-nil-good. Now, by [18, Proposition 2.1(4)], we have that $(R[G])_e$ and $R$ are isomorphic as rings. Therefore $(R[G])_e$ is a semilocal ring. According to [2], we have that $R[G]$ is semilocal too. Also, as 2 is a unit in $R$ it is also a unit in $R[G]$. Hence, $R[G]$ is 2-good, and therefore 2-nil-good.

Next we deal with the graded 2-nil-good property of graded group rings. It is convenient now to recall that if $G$ is a group, and $H$ a normal subgroup of $G$, then a $G$-graded ring $R = \bigoplus_{g \in G} R_g$ can be viewed as a $G/H$-graded ring $R = \bigoplus_{C \in G/H} R_C$, where $R_C = \bigoplus_{x \in C} R_x$ (see, for instance, [13,19]).
Theorem 4.4. Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring, where $G$ is a locally finite $p$-group, and let $H$ be a normal subgroup of $G$. Also, let us assume that $p$ is nilpotent in $R$. If $R$ is graded 2-nil-good as a $G/H$-graded ring, then $R[H]$ is graded 2-nil-good as a $G/H$-graded ring.

Proof. Again, as in the proof of [21, Theorem 2.3], we may assume that $H$ is a finite $p$-group. We know from [19], page 180, that $R[H]/\Delta(R[H])$ and $R$ are graded isomorphic as $G/H$-graded rings. Since $p$ is nilpotent, according to [5, Theorem 9], we have that $\Delta(R[H])$ is nilpotent, and in particular, graded-nil. Hence, by Theorem 3.1, $R[H]$ is graded 2-nil-good as a $G/H$-graded ring. \qed

Theorem 4.5. Let $R$ be a $G$-graded ring and $H$ a normal subgroup of $G$. Also, let $R$ be graded clean as a $G/H$-graded ring with $2 \in U(R)$. If $p > 2$ is a prime number belonging to the $H$-component of the graded Jacobson radical $J^{G/H}(R)$ of $R$, regarded as a $G/H$-graded ring, and $G$ a locally finite $p$-group, then $R[H]$ is graded 2-nil-good as a $G/H$-graded ring.

Proof. According to our assumptions, since $R$ is graded clean as a $G/H$-graded ring, we have that $R[H]$ is graded clean as a $G/H$-graded ring. This follows by Theorem 4.1 in [E. Ilić-Georgijević, On graded clean group rings, preprint]. We give here a short proof for readers’ convenience. We may assume that $G$ is a finite $p$-group. Since $R[H]/\Delta(R[H])$ and $R$ are graded isomorphic as $G/H$-graded rings, we have that $R[H]/\Delta(R[H])$ is graded clean. In particular, $(R[H]/\Delta(R[H]))^H$ is clean, that is, $R[H]^H \cong R_H$ is clean by [19, Proposition 6.2.1]. Since $G$ is finite, $J^{G/H}(R)^H \subseteq J^G(R) \subseteq J(R)$, by [4, Theorem 4.4]. Now, by [24, Lemma 2], we have that $\Delta(R[H])$ is contained in $J(R[H])$. By [4, Theorem 4.4], we have that $\Delta(R[H]) \subseteq J^{G/H}(R[H])$. This, together with the fact that $R[H]^H \cong R_H$, and [19, Proposition 2.9.1vi)], implies that $R[H]$ is a graded clean $G/H$-graded ring. Now, as 2 is a unit in $R$, we have that 2 is also a unit in $R[H]$. Therefore by Theorem 3.3, $R[H]$ is graded 2-nil-good as a $G/H$-graded ring. \qed

Theorem 4.6. Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring, where $G$ is a locally finite $2$-group, and let $H$ be a normal subgroup of $G$. Also, let us assume that $R_e$ is a nil clean ring. If $R$ is graded 2-nil-good as a $G/H$-graded ring, then $R[H]$ is graded 2-nil-good as a $G/H$-graded ring.

Proof. We again can assume that $H$ is finite. We know from [19], page 180, that $R[H]/\Delta(R[H])$ and $R$ are graded isomorphic as $G/H$-graded rings. Since $R_e$ is by assumption nil clean, we have that 2 is nilpotent by [6, Proposition 3.14] applied to $R_e$. Hence, according to [5, Theorem 9], $\Delta(R[H])$ is nilpotent, and in particular, graded-nil. Now, by Theorem 3.1, it follows that $R[H]$ is graded 2-nil-good ring as a $G/H$-graded ring. \qed

Let us return to Theorem 4.1 for a moment. Since $R$ and $(R[G])_e$ are isomorphic as rings, we have that $(R[G])_e$ is 2-nil-good. If we moreover assume that the units and
nilpotents of $R[G]$ are all homogeneous, then we of course get that $R[G]$ is graded 2-nil-good. Also, let us take a look at the following example.

**Example 4.1.** Let $S$ be a 2-nil-good ring, $G = \{e, g\}$ a cyclic group of order 2, and $R = \begin{pmatrix} S & S \\ 0 & S \end{pmatrix}$. Then $R = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \oplus \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ is a $G$-graded ring whose $e$-component $R_e = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}$ is a 2-nil-good ring, and also $R$ is a graded 2-nil-good ring since elements of $R_g = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ are nilpotent and therefore, graded 2-nil-good, as every $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ can be written as $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$.

These observations lead to the question of when the following implication holds true

\[(4.1) \quad R_e \text{ is 2-nil-good } \Rightarrow R = \bigoplus_{g \in G} R_g \text{ is graded 2-nil-good.}\]

The following example proves that the above implication does not hold in general.

**Example 4.2.** Let $R$ be a commutative 2-good ring which is moreover reduced, and let $R[x]$ be a polynomial ring with indeterminate $x$. Then $R[x]$ is $\mathbb{Z}$-graded with $i$-component $Rx^i$ if $i \geq 0$ and $0$ if $i < 0$ (see for instance [19]). Then $R_0 = R$ is a 2-nil-good ring. Also, since $R$ is reduced, we have that $U(R[x]) = U(R)$ by [11, Corollary 1.7]. Hence if $x$ is graded 2-nil-good, then $x = u + v + w$, where $u, v \in U(R)$ and $w \in N(R[x])$. In other words, $x - u - v$ is nilpotent, which is impossible. Therefore $R[x]$ is not graded 2-nil-good.

**Theorem 4.7.** Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded PI-ring which is graded local, that is, it has a unique maximal homogeneous right ideal, and let $G$ be a finite group such that the order of $G$ is a unit in $R$. Also, let $R_g R_{g^{-1}} = 0$ for every $g \in G \setminus \{e\}$. If $R_e$ is 2-nil-good ring with nil Jacobson radical $J(R_e)$, then $R$ is graded 2-nil-good.

**Proof.** Assumptions on $R_e$ imply that $R_e/J(R_e)$ is a 2-nil-good ring. Further, [4, Corollary 4.2, Theorem 4.4] and [14, Theorem 3] together imply that $J(R)$ is a graded-nil ideal of $R$. According to [10, Theorem 3.27] (see also the proof of [9, Theorem 3.2]), we have that every homogeneous element of $R/J(R)$ is a 2-nil-good element of $R_e/J(R_e)$. Hence $R/J(R)$ is graded 2-nil-good, and thus by Theorem 3.1, $R$ is graded 2-nil-good. $\square$

**Theorem 4.8.** Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring of finite support, where $G$ is a torsion free group. Also, let $R$ be a semiprimary ring with $R_e$ local and $2 \in U(R)$. Then $R$ is graded 2-nil-good.

**Proof.** Since $2 \in R_e$, and $R_e$ is local, by [23, Proposition 2.10] we know that $R_e$ is 2-nil-good. It follows that $R_e/J(R_e)$ is 2-nil-good. By [19, Proposition 9.6.4], we have
that \( J^2(R) = J(R) \) and that \( R/J(R) = R_e/J(R_e) \). Since \( R \) is semiprimary, \( J(R) \) is nil and so the claim follows by Theorem 3.1. \( \square \)

4.2. Matrix rings. Since graded 2-good rings are graded 2-nil-good, let us start with the question of whether the graded matrix ring over a graded 2-good ring is also a graded 2-good ring. In [23] this is answered in affirmative for the classical, that is, ungraded case. Their proof relies on the technique which can be seen in the proof of [8, Lemma], that is, they prove (see [23, Proposition 3.6]) that a ring \( R \) is 2-good if the corner rings, with respect to some idempotent of a ring, are 2-good. The following theorem represents a graded version of that result.

**Theorem 4.9.** Let \( R = \bigoplus_{g \in G} R_g \) be a \( G \)-graded ring and \( f \in R_e \) an idempotent. Let us write \( \bar{f} = 1 - f \). If \( fRf \) and \( \bar{f}R\bar{f} \) are graded 2-good rings, then \( R \) is also a graded 2-good ring.

**Proof.** Since graded 2-good rings are crossed products, the proof of [23, Proposition 3.6] can be easily modified to our setting. We give a sketch of the proof. Let \( R = \left( \begin{array}{cc} fRf & fR\bar{f} \\ \bar{f}Rf & \bar{f}R\bar{f} \end{array} \right) \) be the Pierce decomposition of \( R \), and let \( A = \left( \begin{array}{cc} a & x \\ y & b \end{array} \right) \in R_g \), where \( g \in G \). Since \( fRf \) is by assumption graded 2-good ring, and graded 2-good rings are crossed products, there exist \( u_1, u_2 \in U(fRf) \cap R_g \) such that \( a = u_1 + u_2 \). Now, \( b - yu_2^{-1}x \in \bar{f}R\bar{f} \). Again, by assumption, \( \bar{f}R\bar{f} \) is a graded 2-good ring, and as it is a crossed product, there exist \( v_1, v_2 \in U(\bar{f}R\bar{f}) \cap R_g \) such that \( b - yu_2^{-1}x = v_1 + v_2 \). The rest of the proof goes as in the proof of [23, Proposition 3.6] (see also [8, Lemma]). \( \square \)

This theorem by mathematical induction implies the following corollaries.

**Corollary 4.1.** Let \( R = \bigoplus_{g \in G} R_g \) be a \( G \)-graded ring. If \( 1 = f_1 + \cdots + f_n \) in \( R \), where \( f_i \in R_e \) are orthogonal idempotents and each \( f_iRf_i \) is graded 2-good, then \( R \) is graded 2-good.

**Corollary 4.2.** Let \( R = \bigoplus_{g \in G} R_g \) be a \( G \)-graded ring. If \( R \) is graded 2-good and \( n \) a natural number, then \( M_n(R) \) is graded 2-good for every \( \sigma \in G^n \).

In order to obtain a similar result for graded 2-nil-good rings, we first give a graded version of [1, Theorem 4.1].

However, let us first recall from [7] what a \( G \)-graded Morita context is. So, let \( A = \bigoplus_{g \in G} A_g \) and \( B = \bigoplus_{g \in G} B_g \) be \( G \)-graded rings, and let \( V = \bigoplus_{g \in G} V_g \) and \( W = \bigoplus_{g \in G} W_g \) be \( G \)-graded \( A \rightarrow B \) and \( B \rightarrow A \)-bimodules, respectively. Then a quadruple \((A, V, W, B)\) is a \( G \)-graded Morita context if \((A, V, W, B)\) is a Morita context and if \( V_gW_h \subseteq A_{gh} \) and \( W_hV_g \subseteq B_{hg} \) for all \( g, h \in G \) (see [7]). The ring \( R = \left( \begin{array}{cc} A & V \\ W & B \end{array} \right) \) can be \( G \)-graded with respect to any \( \sigma \in G^2 \) as it is described in Preliminaries (see [19]) and then it will be denoted by \( R(\sigma) = \bigoplus_{\lambda \in G} R_{\lambda}(\sigma) \).
Theorem 4.10. Let \((A, V, W, B)\) be a \(G\)-graded Morita context. If \(A\) and \(B\) are graded 2-nil-good rings which are crossed products, then \(R(\sigma)\) is a graded 2-nil-good ring for every \(\sigma = (g_1, g_2) \in G \times G\).

Proof. First, let us notice that, since \(A\) and \(B\) are crossed products, that \(R(\sigma)\) is also a crossed product. Let \(M \in R(\sigma)), where \(\lambda \in G\). Then \(M = \left( \begin{array}{cc} a & x \\ y & b \end{array} \right), \) where \(a \in A_{g_1\lambda g_1^{-1}}, x \in V_{g_1\lambda g_2^{-1}}, b \in B_{g_2\lambda g_2^{-1}}.\) Since \(A\) and \(B\) are moreover crossed products, there exist \(u^a_1, u^a_2 \in U(A) \cap A_{g_1\lambda g_1^{-1}},\) and \(u^b_1, u^b_2 \in U(B) \cap B_{g_2\lambda g_2^{-1}}\) such that \(a = u^a_1 + u^a_2 + n^a\) and \(b = u^b_1 + u^b_2 + n^b\) for some nilpotents \(n^a \in A_{g_1\lambda g_1^{-1}}, n^b \in B_{g_2\lambda g_2^{-1}}.\) Therefore \(M = \left( \begin{array}{cc} u^a_1 & x \\ 0 & u^a_1 \end{array} \right) + \left( \begin{array}{cc} u^b_1 & 0 \\ y & u^b_1 \end{array} \right) + \left( \begin{array}{cc} n^a & 0 \\ 0 & n^b \end{array} \right)\) is a graded 2-nil-good element. \(\square\)

Corollary 4.3. Let \(R = \bigoplus_{g \in G} R_g\) be a crossed product. If \(R\) is graded 2-nil-good and \(n\) a natural number, then \(M_n(R)(\sigma)\) is graded 2-nil-good for every \(\sigma \in G^n.\)

Proof. This follows by the previous theorem by using mathematical induction (cf. [1, Corollary 4.2]). \(\square\)

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