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# ON THE LAPLACIAN COEFFICIENTS OF TREES

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ABSTRACT. Let $G$ be a finite simple graph with Laplacian polynomial $\psi(G,\lambda) = \sum_{k=0}^{n}(-1)^{n-k}c_k(G)\lambda^k$. In an earlier paper, we computed the coefficient of $c_{n-4}$ for trees with respect to some degree-based graph invariant. The aim of this paper is to continue this work by giving an exact formula for the coefficient $c_{n-5}$ in the polynomial $\psi(G,\lambda)$. As a consequence of this work, the Laplacian coefficients $c_{n-k}$, $k = 2, 3, 4, 5$, for some know trees were computed.

## 1. DEFINITIONS AND NOTATIONS

Throughout this paper, our graphs will be assumed to be simple, connected and undirected, and the standard notation for such a graphs is used. The notations $n(G)$ and $m(G)$ stand for the number of elements in the vertex set $V(G)$ and the edge set $E(G)$, respectively. The degree of a vertex $v$ in $G$, $\deg_G(v)$, is the number of edges in $G$ with one end point $v$ and the degree of an edge $e$ in $G$, $\deg_G(e)$, is the degree of vertex $e$ in the line graph of $G$. It is easy to see that $\deg_G(e) = \deg_G(u) + \deg_G(v) - 2$. The *distance* between two vertices $u$ and $v$ is defined as the length of a sort path connecting them. If $Z \subseteq V(G)$, then the *induced subgraph* $G[Z]$ is the graph with vertex set $Z$ and edge set $\{uv \in E(G) \mid \{u,v\} \subseteq Z\}$.

Suppose $G$ is a graph. The *subdivision graph* $S(G)$ is a graph obtained from $G$ by inserting a new vertex on each edge of $G$. It is clear from this definition that $n(S(G)) = n(G) + m(G)$ and $m(S(G)) = 2m(G)$.

Suppose $e = xy$ and $f$ are two edges of a graph $G$ and $v \in V(G)$, where $v \neq x, y$. The *common vertex* of $e$ and $f$ is denoted by $e \cap f$ and $e \cap f = \emptyset$ means that $e$ and $f$ are not incident. If $e \cap f = \emptyset$ then $e$ and $f$ are said to be independent. A subset $M$

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of $E(G)$ is called a matching if all pairs of distinct edges in $M$ are independent. Note that $M$ is a matching in $G$ if $|\{u \mid u \text{ is an end point of an edge in } M\}| = 2|M|$. If $M$ is a matching of size $k$ then we say $M$ is a $k-$matching. Furthermore, the notation $p(G;k)$, $1 \leq k \leq \frac{n}{2}$, is used for the number of distinct $k-$matchings in $G$. The matchings polynomial of $G$ was first introduced by Godsil and Gutman in [4]. This polynomial is defined as $p(G;0) = 1$ and for other values of $x$, $\alpha(G,x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k p(G;k)x^{n-2k}$.

Suppose $G$ is a simple graph with vertex set $\{a_1, \ldots, a_n\}$. The $0-1$ matrix $A(G) = (a_{ij})$ such that $a_{ij} = 1$ if and only if $v_iv_j \in E(G)$ is called the adjacency matrix of $G$. The Laplacian matrix of $G$ is another $n \times n$ matrix defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of $G$ whose diagonal entry $dii$ is the degree of $a_i$ in $G$. It is well-known that the eigenvalues of $L(G)$ are non-negative real numbers with 0 as the smallest eigenvalue. The characteristic polynomial of $L(G)$ is called the Laplacian polynomial of $G$ and its roots are Laplacian eigenvalues of $G$. In this paper we write this polynomial in the form of $\psi(G,x) = \det(xI_n - L(G)) = \sum_{k=0}^{n}(-1)^{n-k}c_k(G)x^k$.

The first and second Zagreb indices of a graph $G$ are two important degree-based graph invariants that was introduced by two pioneers of *Chemical Graph Theory* Gutman and Trinajstić [6]. These invariants are defined as $M_1(G) = \sum_{v\in V(G)} \deg_G(v)^2$ and $M_2(G) = \sum_{uv\in E(G)} \deg_G(u)\deg_G(v)$. We encourage the interested readers to consult the interesting papers [7] and [14], for more information about mathematical properties and chemical applications of these invariants.

Following Milićević et al. [11], the edge counterpart of the first and second Zagreb indices of a graph $G$ are defined as $EM_1(G) = \sum_{e\sim f}(\deg_G(e) + \deg_G(f)) = \sum_{e\in E(G)} \deg_G(e)^2$ and $EM_2(G) = \sum_{e\sim f} \deg_G(e)\deg_G(f)$, where for $e = uv$, $\deg_G(e) = \deg_G(u) + \deg_G(v) - 2$ denotes the degree of the edge $e$, and $e \sim f$ means that the edges $e$ and $f$ are incident.

Furtula and Gutman [5] studied in details the sum of cubes of degrees of vertices in a graph $G$ and used the name *forgotten index* for this invariant. They defined $F(G) = \sum_{v\in V(G)} \deg_G(v)^3 = \sum_{e=uv\in E(G)}(\deg_G(u)^2 + \deg_G(v)^2)$. The first Zagreb index and the forgotten index can be generalized in the form of $M_1^{\alpha}(G) = \sum_{u\in V(G)} \deg_G(u)^{\alpha}$, where $\alpha \neq 0, 1$ is a real number. Zhang and Zhang [17] obtained some extremal values of this invariant in the class of all unicyclic graphs of a given order. An interesting survey of these degree-based indices is given in [8].

Let $T$ be a tree with Laplacian polynomial

$$\psi(T,x) = \det(xI_n - L(T)) = \sum_{k=0}^{n}(-1)^{n-k}c_k(T)x^k.$$

Merris [12] and Mohar [13] proved that $c_0(T) = 0$, $c_1(T) = n$, $c_n = 1$ and $c_{n-1}(T) = 2(n-1)$. In [16], it is proved that $c_2(T) = W(T)$ and in the paper [15], the authors proved that $c_{n-2}(T) = 2n^2 - 5n + 3 - \frac{1}{2}M_1(T)$ and $c_{n-3}(T) = \frac{1}{3}(4n^3 - 18n^2 + 24n - 10 + F(T) - 3(n-2)M_1(T))$.

Suppose $\lambda$ is an arbitrary real number. We now define three invariants which is useful in simplifying formulas in our results. These are:

$$\alpha_\lambda(G) = \sum_{uv\in E(G)} \deg_G(u)\,\deg_G(v)\left(\deg_G(u)^\lambda + \deg_G(v)^\lambda\right),$$
$$\beta(G) = \sum_{e\sim f} \deg_G(e\cap f)\left(\deg_G(e) + \deg_G(f)\right),$$
$$M_2^\lambda(G) = \sum_{uv\in E(G)} \left(\deg_G(u)\deg_G(v)\right)^\lambda.$$

Note that the second Zagreb index is just the case of $\lambda = 1$ in $M_2^\lambda(G)$.

The girth of a graph $G$, $g(G)$, is defined as the length of a shortest cycle of $G$. In a recent paper [3], Das et al. proved the following result.

**Theorem 1.1.** *Let $G$ be a graph with $m$ edges and $g(G) \geq 5$. Then*

$$\begin{aligned}
5p(G;5) =& \frac{1}{24}m(m^4 + 10m^3 + 43m^2 + 54m - 328) + \frac{5}{4}(M_1(G))^2 - \frac{1}{2}\alpha_1(G)(m-7)\\
&- \frac{5}{6}\alpha_2(G) - \frac{1}{12}M_1(G)(2m^3 + 30m^2 + 61m - 225) + \frac{1}{2}\beta(G)\\
&+ \frac{1}{12}M_2(G)(6m^2 + 66m - 239) + \frac{1}{24}F(G)(6m^2 + 24m - 149)\\
&+ \frac{1}{12}M_1^4(G)(m+10) + \frac{1}{4}M_2^2(G) - EM_2(G) - \frac{5}{24}M_1^5(G)\\
&+ \frac{1}{8}\sum_{uv\in E(G)}(M_1(G-\{u,v\}))^2 + \frac{1}{3}\sum_{uv\in E(G)} m(G-\{u,v\})F(G-\{u,v\})\\
&- \frac{1}{4}\sum_{uv\in E(G)} m^2(G-\{u,v\})M_1(G-\{u,v\}) - \sum_{uv\in E(G)} EM_2(G-\{u,v\})\\
&+ \sum_{uv\in E(G)} m(G-\{u,v\})M_2(G-\{u,v\}).
\end{aligned}$$

The present authors [1,2] proved the following formulas for the coefficient $c_{n-4}(T)$, when $T$ is a tree:

$$\begin{aligned}
c_{n-4}(T) =& (n-1)\left(\frac{16}{24}n^3 - 4n^2 + \frac{348}{24}n - \frac{532}{6}\right) + \frac{17}{8}M_1(T)^2\\
&+ \left(\frac{4}{6}n - \frac{412}{24}\right)F(T) + \frac{39}{2}EM_1(T) - \frac{108}{48}M_1^4(T) - 40M_2(T)\\
&- \left(n^2 + \frac{7}{2}n - \frac{1920}{24}\right)M_1(T) - 16\sum_{\{u,v\}\subset V(T)}\binom{\deg_T(u)}{2}\binom{\deg_T(v)}{2}\\
=& \frac{1}{6}(n-1)(4n^3 - 24n^2 + 39n - 16) + \frac{1}{3}F(G)(2n-5)\\
&+ \frac{1}{8}M_1(T)(-8n^2 + M_1(T) + 36n - 32) - \frac{1}{4}M_1^4(T) - M_2(T).
\end{aligned}$$

In this paper, an exact formula for computing the coefficient $c_{n-5}(T)$, $T$ is a tree, with respect to some degree-based topological indices is presented.

## 2. Main Results

The aim of this section is to present a closed formula for $c_{n-5}(T)$, when $T$ is tree. To do this, we first define five invariants $\chi_1, \chi_2, \chi_3, \chi_4$ and $\chi_5$ with respect to the subdivision graph as follows:

$$\begin{aligned}
\chi_1(S(G)) &= \sum_{uv\in E(S(G))} (M_1(S(G)-\{u,v\}))^2,\\
\chi_2(S(G)) &= \sum_{uv\in E(S(G))} m(S(G)-\{u,v\})F(S(G)-\{u,v\}),\\
\chi_3(S(G)) &= \sum_{uv\in E(S(G))} m^2(S(G)-\{u,v\})M_1(S(G)-\{u,v\}),\\
\chi_4(S(G)) &= \sum_{uv\in E(S(G))} EM_2(S(G)-\{u,v\}),\\
\chi_5(S(G)) &= \sum_{uv\in E(S(G))} m(S(G)-\{u,v\})M_2(S(G)-\{u,v\}).
\end{aligned}$$

**Lemma 2.1.** *Let $G$ be a graph with $m$ edges. Then*

$$\begin{array}{ll}
M_1(S(G)) = M_1(G)+4m, & F(S(G)) = F(G)+8m,\\
M_1^4(S(G)) = M_1^4(G)+16m, & M_1^5(S(G)) = M_1^5(G)+32m,\\
\alpha_1(S(G)) = 4M_1(G)+2F(G), & \alpha_2(S(G)) = 8M_1(G)+2M_1^4(G),\\
\beta(S(G)) = 2M_1(G)+M_1^4(G)-F(G), & M_2(S(G)) = 2M_1(G),\\
EM_2(S(G)) = M_2(G)+\frac{1}{2}M_1^4(G)-\frac{1}{2}F(G), & M_2^2(S(G)) = 4F(G).
\end{array}$$

*Proof.* By definition of subdivision graph, we have:

$$\begin{aligned}
M_1(S(G)) &= \sum_{v\in V(G)} \deg_G(v)^2 + \sum_{uv\in E(G)} 4 = M_1(G)+4m,\\
F(S(G)) &= \sum_{v\in V(G)} \deg_G(v)^3 + \sum_{uv\in E(G)} 8 = F(G)+8m,\\
M_1^4(S(G)) &= \sum_{v\in V(G)} \deg_G(v)^4 + \sum_{uv\in E(G)} 16 = M_1^4(G)+16m,\\
M_1^5(S(G)) &= \sum_{v\in V(G)} \deg_G(v)^5 + \sum_{uv\in E(G)} 32 = M_1^5(G)+32m,\\
\alpha_1(S(G)) &= \sum_{v\in V(G)} \sum_{uv\in E(G)} 2\deg_G(v)(2+\deg_G(v))\\
&= \sum_{v\in V(G)} \sum_{uv\in E(G)} (4\deg_G(v)+2\deg_G(v)^2)\\
&= \sum_{v\in V(G)} \deg_G(v)(4\deg_G(v)+2\deg_G(v)^2) = 4M_1(G)+2F(G),
\end{aligned}$$

$$
\begin{aligned}
\alpha_2(S(G)) &= \sum_{v\in V(G)} \sum_{uv\in E(G)} 2\deg_G(v)(4+\deg_G(v)^2) \\
&= \sum_{v\in V(G)} \sum_{uv\in E(G)} (8\deg_G(v)+2\deg_G(v)^3) \\
&= \sum_{v\in V(G)} \deg_G(v)(8\deg_G(v)+2\deg_G(v)^3) = 8M_1(G)+2M_1^4(G), \\
\beta(S(G)) &= \sum_{uv\in E(G)} 2(\deg_G(u)+\deg_G(v)) \\
&\quad + \sum_{v\in V(G)} \binom{\deg_G(v)}{2} \deg_G(v)(\deg_G(v)+\deg_G(v)) \\
&= 2M_1(G) + \sum_{v\in V(G)} \deg_G(v)(\deg_G(v)-1)\deg_G(v)^2 \\
&= 2M_1(G)+M_1^4(G)-F(G), \\
M_2(S(G)) &= \sum_{v\in V(G)} \sum_{uv\in E(G)} 2\deg_G(v) = \sum_{v\in V(G)} 2\deg_G(v)^2 = 2M_1(G), \\
M_2^2(S(G)) &= \sum_{v\in V(G)} \sum_{uv\in E(G)} 4\deg_G(v)^2 = \sum_{v\in V(G)} 4\deg_G(v)^3 = 4F(G), \\
EM_2(S(G)) &= \sum_{uv\in E(G)} \deg_G(u)\deg_G(v) + \sum_{v\in V(G)} \binom{\deg_G(v)}{2}\deg_G(v)^2 \\
&= M_2(G) + \frac{1}{2}\sum_{v\in V(G)} \deg_G(v)(\deg_G(v)-1)\deg_G(v)^2 \\
&= M_2(G) + \frac{1}{2}M_1^4(G) - \frac{1}{2}F(G),
\end{aligned}
$$

proving the lemma. □

**Lemma 2.2.** $\chi_1(S(G)) = (2m-10)(M_1(G))^2+(16m^2-2F(G)-40m)M_1(G) + 32m^3 - 8mF(G) + 13F(G) + 6M_1^4(G) + M_1^5(G) + 24M_2(G) + 4\alpha_1(G)$.

*Proof.* By definition of the graph $S(G)$,

$$
\begin{aligned}
\chi_1(S(G)) &= \sum_{v\in V(G)} \sum_{uv\in E(G)} (M_1(S(G)) - \deg_G(v)^2 - 3\deg_G(v) - 2\deg_G(u))^2 \\
&= \sum_{v\in V(G)} \sum_{uv\in E(G)} (\deg_G(v)^4 + 6\deg_G(v)^3 + 9\deg_G(v)^2) \\
&\quad - M_1(S(G)) \sum_{v\in V(G)} \sum_{uv\in E(G)} (2\deg_G(v)^2 + 6\deg_G(v)) \\
&\quad + \sum_{v\in V(G)} \sum_{uv\in E(G)} M_1(S(G))^2 \\
&\quad + \sum_{v\in V(G)} \sum_{uv\in E(G)} (4\deg_G(u)\deg_G(v)^2 + 12\deg_G(u)\deg_G(v))
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{v\in V(G)}\sum_{uv\in E(G)}(-4M_1(S(G))\deg_G(u)+4\deg_G(u)^2)\\
=&M_1^5(G)+6M_1^4(G)+9F(G)-2M_1(S(G))F(G)-6M_1(S(G))M_1(G)\\
&+2mM_1(S(G))^2+\sum_{uv\in E(G)}(4\deg_G(u)\deg_G(v)^2+4\deg_G(u)^2\deg_G(v)\\
&+24\deg_G(u)\deg_G(v))+\sum_{uv\in E(G)}(-4M_1(S(G))\deg_G(u)+4\deg_G(u)^2\\
&-4M_1(S(G))\deg_G(v)+4\deg_G(v)^2)\\
=&M_1^5(G)+6M_1^4(G)+13F(G)-2M_1(S(G))F(G)-10M_1(S(G))M_1(G)\\
&+2mM_1(S(G))^2+4\alpha_1(G)+24M_2(G).
\end{aligned}
$$

We now apply Lemma 2.1 to deduce that

$$
\begin{aligned}
\chi_1(S(G)) =&(2m-10)(M_1(G))^2+(16m^2-2F(G)-40m)M_1(G)+32m^3-8mF(G)\\
&+13F(G)+6M_1^4(G)+M_1^5(G)+24M_2(G)+4\alpha_1(G),
\end{aligned}
$$

which completes the proof. □

**Lemma 2.3.** $\chi_2(S(G)) = 32m^3+(4F(G)-24)m^2-(8F(G)+16M_1(G)+2M_1^4(G))m + 4m-(M_1(G)-10)F(G)+6M_1(G)+M_1^4(G)+M_1^5(G)-6M_2+3\alpha_1(G)$.

*Proof.* It is easy to see that $\chi_2(S(G)) = \sum_{v\in V(G)}\sum_{uv\in E(G)}(2m-\deg_G(v)-1)(F(S(G)) - \deg_G(v)^3-3\deg_G(u)^2+3\deg_G(u)-7\deg_G(v)-2)$. If we expand the summation, this becomes:

$$
\begin{aligned}
\chi_2(S(G)) =&\sum_{v\in V(G)}\sum_{uv\in E(G)}(\deg_G(v)^4+\deg_G(v)^3+7\deg_G(v)^2+9\deg_G(v)-F(S(G))\\
&+2(1-m\deg_G(v)^3-7m\deg_G(v)-2m+mF(S(G)))-F(S(G))\deg_G(v))\\
&+\sum_{v\in V(G)}\sum_{uv\in E(G)}(-6m\deg_G(u)^2+6m\deg_G(u)+3\deg_G(u)^2-3\deg_G(u))\\
&+\sum_{v\in V(G)}\sum_{uv\in E(G)}(3\deg_G(u)^2\deg_G(v)-3\deg_G(u)\deg_G(v))\\
=&M_1^5(G)+M_1^4(G)+7F(G)+9M_1(G)+4m-2mM_1^4(G)-14mM_1(G)\\
&-8m^2+4m^2F(S(G))-F(S(G))M_1(G)-2mF(S(G))\\
&+\sum_{uv\in E(G)}(-6m\deg_G(u)^2+6m\deg_G(u)+3\deg_G(u)^2-3\deg_G(u)\\
&-6m\deg_G(v)^2+6m\deg_G(v)+3\deg_G(v)^2-3\deg_G(v))\\
&+\sum_{uv\in E(G)}(3\deg_G(u)^2\deg_G(v)-6\deg_G(u)\deg_G(v)+3\deg_G(v)^2\deg_G(u))\\
=&M_1^5(G)+M_1^4(G)+10F(G)+6M_1(G)+4m-2mM_1^4(G)-8mM_1(G)\\
&-8m^2+4m^2F(S(G))-F(S(G))M_1(G)-2mF(S(G))-6mF(G)
\end{aligned}
$$

$$+ 3\alpha_1(G) - 6M_2(G).$$

Now by Lemma 2.1,

$$\begin{aligned}\chi_2(S(G)) =& 32m^3 + (4F(G) - 24)m^2 - (8F(G) + 16M_1(G) + 2M_1^4(G))m + 4m \\ & - (M_1(G) - 10)F(G) + 6M_1(G) + M_1^4(G) + M_1^5(G) - 6M_2 + 3\alpha_1(G).\end{aligned}$$

This completes the proof. □

**Lemma 2.4.** $\chi_3(S(G)) = 32m^4 + (8M_1(G) - 32)m^3 - (4F(G) + 44M_1(G) - 8)m^2 + (20F(G) - 4(M_1(G))^2 + 30M_1(G) + 4M_1^4(G) + 16M_2(G))m + F(G)M_1(G) + 2(M_1(G))^2 - 7F(G) - 5M_1(G) - 5M_1^4(G) - M_1^5(G) - 8M_2(G) - 2\alpha_1(G)$.

*Proof.* The degree sequence of subdivision graph $S(G)$ shows that $\chi_3(S(G)) = \sum_{v\in V(G)} \sum_{uv\in E(G)} (2m - \deg_G(v) - 1)^2 (M_1(S(G)) - \deg_G(v)^2 - 3\deg_G(v) - 2\deg_G(u))$. By expanding this summation,

$$\begin{aligned}\chi_3(S(G)) =& \sum_{v\in V(G)} \sum_{uv\in E(G)} (4\deg_G(v)^3 m - \deg_G(v)^4 - 4\deg_G(v)^2 m^2 \\ & + M_1(S(G))\deg_G(v)^2 - 4M_1(S(G))\deg_G(v)m + 4M_1(S(G))m^2 \\ & - 5\deg_G(v)^3 + 16\deg_G(v)^2 m - 12\deg_G(v)m^2 + 2M_1(S(G))\deg_G(v) \\ & - 4M_1(S(G))m - 7\deg_G(v)^2 + 12\deg_G(v)m + M_1(S(G)) - 3\deg_G(v)) \\ & + \sum_{v\in V(G)} \sum_{uv\in E(G)} (8\deg_G(u)\deg_G(v)m - 2\deg_G(u)\deg_G(v)^2 - 8\deg_G(u)m^2 \\ & - 4\deg_G(u)\deg_G(v) + 8\deg_G(u)m - 2\deg_G(u)) \\ =& 4M_1^4(G)m - M_1^5(G) - 4F(G)m^2 + M_1(S(G))F(G) \\ & - 4M_1(S(G))M_1(G)m + 8M_1(S(G))m^3 - 5M_1^4(G) + 16F(G)m \\ & - 12M_1(G)m^2 + 2M_1(S(G))M_1(G) - 8M_1(S(G))m^2 - 7F(G) \\ & + 12M_1(G)m + 2mM_1(S(G)) - 3M_1(G) \\ & + \sum_{uv\in E(G)} (16\deg_G(u)\deg_G(v)m - 2\deg_G(u)\deg_G(v)^2 - 2\deg_G(v)\deg_G(u)^2 \\ & - 8\deg_G(u)m^2 - 8\deg_G(v)m^2 - 8\deg_G(u)\deg_G(v) + 8\deg_G(u)m \\ & + 8\deg_G(v)m - 2\deg_G(u) - 2\deg_G(v)) \\ =& 4M_1^4(G)m - 4F(G)m^2 + M_1(S(G))F(G) - 4M_1(S(G))M_1(G)m \\ & + 8M_1(S(G))m^3 + 16F(G)m - 20M_1(G)m^2 + 2M_1(S(G))M_1(G) \\ & - 8M_1(S(G))m^2 - 7F(G) + 20M_1(G)m + 2mM_1(S(G)) - 5M_1(G) \\ & + 16mM_2(G) - 2\alpha_1(G) - 8M_2(G) - M_1^5(G) - 5M_1^4(G).\end{aligned}$$

Now by Lemma 2.1,

$$\begin{aligned}\chi_3(S(G)) =& 32m^4 + (8M_1(G) - 32)m^3 - (4F(G) + 44M_1(G) - 8)m^2 \\ & + (20F(G) - 4(M_1(G))^2 + 30M_1(G) + 4M_1^4(G) + 16M_2(G))m\end{aligned}$$

$$+ F(G)M_1(G) + 2(M_1(G))^2 - 7F(G) - 5M_1(G) - 5M_1^4(G) - M_1^5(G) - 8M_2(G) - 2\alpha_1(G).$$

Hence, the result follows. □

**Lemma 2.5.** $\chi_4(S(G)) = \frac{1}{2}m(4M_2(G) - 2F(G) + 2M_1^4(G) + 4) + \frac{11}{2}F(G) - 2\alpha_1(G) - \frac{7}{2}M_1(G) - \frac{3}{2}M_1^4(G) - \frac{1}{2}M_1^5(G)$.

*Proof.* By relation between adjacencies in $G$ and $S(G)$, we can see that

$$\begin{aligned}
\chi_4(S(G)) =& 2mEM_2(S(G)) - \sum_{v\in V(G)} \left( \binom{\deg_G(v)}{2} \deg_G(v)^3 + (\deg_G(v)-1)\deg_G(v)^3 \right.\\
&\left. + \binom{\deg_G(v)-1}{2} \deg_G(v)^3 - \binom{\deg_G(v)-1}{2} (\deg_G(v)-1)^2 \deg_G(v) \right)\\
&- \sum_{v\in V(G)} \sum_{uv\in E(G)} (\deg_G(u)-1)(\deg_G(v)^2(\deg_G(v)-1) - \deg_G(v)\\
&\times (\deg_G(v)-1)^2) - \sum_{uv\in E(G)} (\deg_G(u)\deg_G(v)(\deg_G(u)+\deg_G(v))\\
&+ \deg_G(u)\deg_G(v)(\deg_G(u)+\deg_G(v)-2)\\
&- (\deg_G(u)-1)^2\deg_G(v) - \deg_G(u)(\deg_G(v)-1)^2)\\
=& 2mEM_2(S(G)) - \sum_{v\in V(G)} \left( \frac{1}{2}\deg_G(v)^5 + \frac{3}{2}\deg_G(v)^4 - \frac{9}{2}\deg_G(v)^3 \right.\\
&\left. + \frac{7}{2}\deg_G(v)^2 - \deg_G(v) \right) - \sum_{uv\in E(G)} (\deg_G(u)\deg_G(v)^2 + \deg_G(v)\deg_G(u)^2\\
&- 2\deg_G(u)\deg_G(v) - \deg_G(u)^2 - \deg_G(v)^2 + \deg_G(u) + \deg_G(v))\\
&- \sum_{uv\in E(G)} (\deg_G(u)^2\deg_G(v) + \deg_G(u)\deg_G(v)^2 + 2\deg_G(u)\deg_G(v)\\
&- \deg_G(u) - \deg_G(v))\\
=& 2mEM_2(S(G)) - \left( \frac{1}{2}M_1^5(G) + \frac{3}{2}M_1^4(G) - \frac{11}{2}F(G) + \frac{7}{2}M_1(G) - 2m \right)\\
&- 2\alpha_1(G).
\end{aligned}$$

Now by Lemma 2.1,

$$\begin{aligned}
\chi_4(S(G)) =& \frac{1}{2}m(4M_2(G) - 2F(G) + 2M_1^4(G) + 4) + \frac{11}{2}F(G) - 2\alpha_1(G) - \frac{7}{2}M_1(G)\\
&- \frac{3}{2}M_1^4(G) - \frac{1}{2}M_1^5(G),
\end{aligned}$$

which is our goal. □

**Lemma 2.6.** $\chi_5(S(G)) = (8M_1(G) + 8)m^2 - (4F(G) + 10M_1(G) + 4M_2(G) + 4)m - 2(M_1(G))^2 + 2F(G) + M_1(G) + 2M_1^4(G) + 8M_2(G) + \alpha_1(G)$.

*Proof.* Again definition of subdivision graph,

$$\begin{aligned}\chi_5(S(G)) = &\sum_{uv\in E(G)} ((2m-\deg_G(u)-1)(M_2(S(G))-2\deg_G(u)^2-2\deg_G(v)\\
&-\sum_{wu\in E(G)}\deg_G(w)+\deg_G(v)-2\deg_G(v)(\deg_G(v)-1)\\
&+2(\deg_G(v)-1)^2)+(2m-\deg_G(v)-1)(M_2(S(G))\\
&-2\deg_G(v)^2-2\deg_G(u)-\sum_{vz\in E(G)}\deg_G(z)\\
&+\deg_G(u)-2\deg_G(u)(\deg_G(u)-1)+2(\deg_G(u)-1)^2))\\
=&\sum_{uv\in E(G)}(2\deg_G(u)^3+2\deg_G(v)^3-4\deg_G(u)^2m-4\deg_G(v)^2m\\
&-M_2(S(G))\deg_G(u)-M_2(S(G))\deg_G(v)+4M_2(S(G))m-2M_2(S(G))\\
&+8m-4+2\deg_G(u)^2+2\deg_G(v)^2+6\deg_G(u)\deg_G(v)-6\deg_G(u)m\\
&-6\deg_G(v)m+\deg_G(u)+\deg_G(v))-\sum_{uv\in E(G)}((2m-\deg_G(u)-1)\\
&\sum_{wu\in E(G)}\deg_G(w)-(2m-\deg_G(v)-1)\sum_{vz\in E(G)}\deg_G(z))\\
=&2M_1^4(G)-4mF(G)-M_2(S(G))M_1(G)+4M_2(S(G))m^2-2mM_2(S(G))\\
&+8m^2-4m+2F(G)+6M_2(G)-6mM_1(G)+M_1(G)\\
&-\sum_{uv\in E(G)}((2m-\deg_G(u)-1)\deg_G(v)\deg_G(u)\\
&+(2m-\deg_G(v)-1)\deg_G(u)\deg_G(v))\\
=&2M_1^4(G)-4mF(G)-M_2(S(G))M_1(G)+4M_2(S(G))m^2-2mM_2(S(G))\\
&+8m^2-4m+2F(G)+8M_2(G)-6mM_1(G)+M_1(G)+\alpha_1(G)\\
&-4mM_2(G),\end{aligned}$$

Now, by Lemma 2.1,

$$\begin{aligned}\chi_5(S(G)) =&(8M_1(G)+8)m^2-(4F(G)+10M_1(G)+4M_2(G)+4)m-2(M_1(G))^2\\
&+2F(G)+M_1(G)+2M_1^4(G)+8M_2(G)+\alpha_1(G),\end{aligned}$$

which proving the lemma. □

Let $G$ be a graph. It is easy to see that $g(S(G))\geq 6$. Therefore, by Lemma 2.1, 2.2, 2.3, 2.4, 2.5, 2.6 and Theorem 1.1, we have the following theorem.

**Theorem 2.1.** *Let $G$ be a graph with $m$ edges. Then*

$$\begin{aligned}p(S(G);5) =&\frac{1}{15}m^2(4m^3-20m^2+15m+15)+\frac{1}{12}m(8F(G)m-8M_1(G)m^2\\
&+3(M_1(G))^2+36M_1(G)m-28F(G)-24M_1(G)-6M_1^4(G)-24M_2(G))\end{aligned}$$

$$- \frac{1}{6}M_1(G)(3M_1(G) + F(G) + 6) + \alpha_1(G) + 2M_2(G) + \frac{1}{5}M_1^5(G) + M_1^4(G) + F(G).$$

We are now ready to prove our main result. For the sake of completeness, we mention here a useful result of Zhou and Gutman [18].

**Theorem 2.2.** *Let $G$ be an $n$-vertex tree. Then $c_{n-k}(G) = p(S(G); k)$, for $0 \le k \le n$.*

**Theorem 2.3.** *Let $G$ be an acyclic graph on $n$ vertices and $m$ edges. Then*

$$\begin{aligned} c_{n-5}(G) =& \frac{1}{15}m^2(4m^3 - 20m^2 + 15m + 15) + \frac{1}{12}m(8F(G)m - 8M_1(G)m^2 \\ &+ 3(M_1(G))^2 + 36M_1(G)m - 28F(G) - 24M_1(G) - 6M_1^4(G) - 24M_2(G)) \\ &- \frac{1}{6}M_1(G)(3M_1(G) + F(G) + 6) + \alpha_1(G) + 2M_2(G) + \frac{1}{5}M_1^5(G) \\ &+ M_1^4(G) + F(G). \end{aligned}$$

*Proof.* Apply Theorem 2.1 and 2.2. □

**Corollary 2.1.** *Let $T$ be a tree on $n$ vertices. Then*

$$\begin{aligned} c_{n-5}(G) =& \frac{1}{15}(n-1)^2(4n^3 - 32n^2 + 67n - 24) + \frac{1}{12}n(8nF(G) - 8n^2M_1(G) \\ &+ 3(M_1(G))^2 + 60nM_1(G) - 44F(G) - 120M_1(G) - 6M_1^4(G) - 24M_2(G)) \\ &- \frac{1}{12}M_1(G)(2F(G) + 9M_1(G) - 56) + \alpha_1(G) + \frac{1}{5}M_1^5(G) + \frac{3}{2}M_1^4(G) \\ &+ 4M_2(G) + 4F(G). \end{aligned}$$

*Proof.* The result follows from Theorem 2.3 and the fact that $m(T) = n - 1$. □

## 3. Applications

The aim of this section is to apply our results in Section 2 for computing the Laplacian coefficients $c_{n-k}(G)$, $k = 2, 3, 4, 5$, when $G$ is a certain tree. We first assume that $T(k, t)$ be a rooted tree with degree sequence $k, k, \ldots, k, 1, 1, \ldots, 1$ and $t$ is the distance between the center and any pendent vertex, Figure 1. Then,

$$\begin{aligned} c_{n-1}(T(k,t)) =& \frac{k}{k-2}(2(k-1)^t - 1)), \\ c_{n-2}(T(k,t)) =& \frac{k}{2(k-2)^2}((k-1)(k-2)^2(k-1)^{t-1} - 2((k-1)^t - 1)(k^2 - 2k(k-1)^t + k - 2)), \\ c_{n-3}(T(k,t)) =& -\frac{k}{3(k-2)^3}((k-1)(k-2)^2(k^2 - 3k(k-1)^t + 5k - 8)(k-1)^{t-1} \\ &- 2k((k-1)^t - 1)(k^2 - 2k(k-1)^t + 3k - 6)(k - (k-1)^t - 1)), \\ c_{n-4}(T(k,t)) =& \frac{k}{4(k-2)^4}((1/2)k(k-1)^2(k-2)^4((k-1)^{t-1})^2 + (4((k-1)^t)^2k^2 \\ &+ (-(14/3)k^3 - (34/3)k^2 + (76/3)k)(k-1)^t + k^4 + (29/3)k^3 - (53/3)k^2 \end{aligned}$$

$$
\begin{aligned}
&- (52/3)k + 28)(k-1)(k-2)^2(k-1)^{t-1} - 2((k-1)^t - 1)\\
&- (4/3)((k-1)^t)^3k^3 + (4k^4 - 8k^2)((k-1)^t)^2 + (-(11/3)k^5 - (10/3)k^4\\
&+ (85/3)k^3 - 20k^2 - 4k)(k-1)^t + (k^5 + (14/3)k^4 - 18k^3 + 6k^2\\
&+ 16k - 8)(k-1))),
\end{aligned}
$$

$$
\begin{aligned}
c_{n-5}(T(k,t)) =& -\frac{k}{5(k-2)^5}((5/6)(k^2 - (3/2)k(k-1)^t + (7/2)k - 8)(k-1)^2(k-2)^4k(\\
&(k-1)^{t-1})^2 + (-(10/3)((k-1)^t)^3k^3 + (25/3)(k^2 + (7/5)k - (22/5))k^2\\
&\times ((k-1)^t)^2 - (35/6)(k^4 + 5k^3 - (99/7)k^2 - (32/7)k + (116/7))k(k-1)^t\\
&+ k^6 + (95/6)k^5 - (223/6)k^4 - (341/6)k^3 + 158k^2 - (118/3)k - 48)(k-1)\\
&(k-2)^2(k-1)^{t-1} - 2((k-1)^t - 1)(k - 3/2 - (1/2)(k-1)^t)k((-(4/3)((k-1)^t)^3k^3\\
&+ (4k^4 + (8/3)k^3 - (40/3)k^2)((k-1)^t)^2 + (-(11/3)k^5\\
&- 10k^4 + (137/3)k^3 - 20k^2 - 20k)(k-1)^t + k^6 + (23/3)k^5 - 32k^4\\
&+ (310/3)k^2 - 120k + 40)),\\
c_{n-1}(T(3,t)) =& 6 \times 2^t - 6,\\
c_{n-2}(T(3,t)) =& -\frac{93}{2}2^t + 18 \times 2^{2t} + 30,\\
c_{n-3}(T(3,t)) =& 272 \times 2^t - 171 \times 2^{2t} + 36 \times 2^{3t} - 144,\\
c_{n-4}(T(3,t)) =& -\frac{5799}{4}2^t + \frac{9177}{8}2^{2t} - 405 \times 2^{3t} + 54 \times 2^{4t} + 687,\\
c_{n-5}(T(3,t)) =& \frac{74427}{10}2^t - \frac{26967}{4}2^{2t} + \frac{12267}{4}2^{3t} - 702 \times 2^{4t} + \frac{324}{5}2^{5t} - 3294,\\
c_{n-1}(T(4,t)) =& 4 \times 3^t - 4,\\
c_{n-2}(T(4,t)) =& -24 \times 3^t + 8 \times 3^{2t} + 18,\\
c_{n-3}(T(4,t)) =& \frac{392}{3}3^t - 64 \times 3^{2t} + \frac{32}{3}3^{3t} - 88,\\
c_{n-4}(T(4,t)) =& -\frac{2132}{3}3^t + \frac{1232}{3}3^{2t} - \frac{320}{3}3^{3t} + \frac{32}{3}3^{4t} + 457,\\
c_{n-5}(T(4,t)) =& \frac{19644}{5}3^t - 2480 \times 3^{2t} + \frac{2368}{3}3^{3t} - 128 \times 3^{4t} + \frac{128}{15}3^{5t} - 2484.
\end{aligned}
$$

Our second class of trees are known as Kragujevac trees. To define, we assume that $B_1, B_2, B_3, \ldots, B_k$ are branches whose structure is depicted in Figure 2. A proper Kragujevac tree is a tree possessing a central vertex of degree at least 3, to which branches of the form $B_1$ and/or $B_2$ and/or $B_3$ and/or $\ldots$ are attached [10].

Let $G_i$, for $i = 1, 2, \ldots, 7$, be the proper Kragujevac tree on $n$ vertices in Figure 3. Then

$$
\begin{aligned}
&c_{n-2}(G_1) = \frac{3}{98}n(65n - 231) - 3,\ c_{n-3}(G_1) = \frac{8}{1029}n(169n^2 - 1302n + 1127) + 60,\\
&c_{n-4}(G_1) = \frac{1}{57624}n(37349n^3 - 503594n^2 + 1625575n + 4758782) - 462,\\
&c_{n-5}(G_1) = \frac{3}{336140}n(28561n^4 - 597415n^3 + 3893785n^2 + 1016995n - 107579206) + 2868,
\end{aligned}
$$

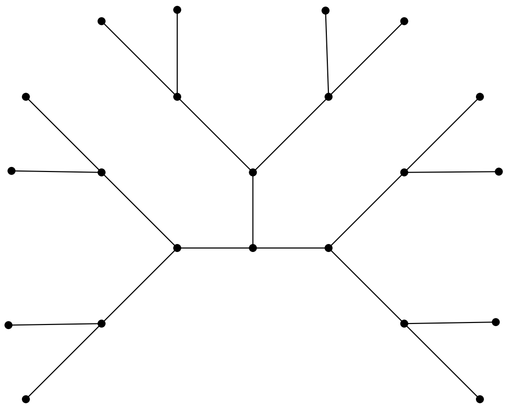

FIGURE 1. The rooted tree $T(3,3)$.

$$
\begin{aligned}
c_{n-2}(G_2) &= \frac{15}{98}(n-1)(13n-34),\ \ c_{n-3}(G_2) = \frac{2}{1029}(n-1)(676n^2-4649n+8481),\\
c_{n-4}(G_2) &= \frac{1}{57624}(n-1)(37349n^3-478413n^2+2146954n-3432552),\\
c_{n-5}(G_2) &= \frac{1}{336140}(n-1)(85683n^4-1750502n^3+13991793n^2-52528222n+79270320),\\
c_{n-2}(G_3) &= \frac{1}{98}(15n-16)(13n-33),\\
c_{n-3}(G_3) &= \frac{1}{1029}n(1352n^2-10611n+26563)-\frac{6480}{343},\\
c_{n-4}(G_3) &= \frac{1}{57624}n(37349n^3-513734n^2+2635015n-5871574)+\frac{184469}{2401},\\
c_{n-5}(G_3) &= \frac{1}{1008420}n(257049n^4-5486585n^3+47226105n^2-204551395n+437870586)\\
&\quad -\frac{5694446}{16807},\\
c_{n-2}(G_4) &= \frac{1}{98}n(195n-701)+\frac{125}{49},\ \ c_{n-3}(G_4) = \frac{4}{1029}n(n-4)(338n-1291)+\frac{3008}{343},\\
c_{n-4}(G_4) &= \frac{1}{57624}n(37349n^3-511706n^2+2298967n-2158546)-\frac{291540}{2401},\\
c_{n-5}(G_4) &= \frac{1}{1008420}n(257049n^4-5464615n^3+43233545n^2-130350725n-74554454)\\
&\quad +\frac{15573272}{16807},\\
c_{n-2}(G_5) &= \frac{1}{98}n(195n-713)+\frac{405}{49},\\
c_{n-3}(G_5) &= \frac{2}{1029}n(676n^2-5403n+16460)-\frac{14418}{343},\\
c_{n-4}(G_5) &= \frac{1}{57624}n(37349n^3-523874n^2+3028255n-9178570)+\frac{540746}{2401},\\
c_{n-2}(G_5) &= \frac{1}{1008420}n(257049n^4-5596435n^3+52360845n^2-275208485n+846144906)
\end{aligned}
$$

$$- \frac{20524022}{16807},$$

$$c_{n-2}(G_6) = \frac{1}{98}(39n+1)(5n-18),\ c_{n-3}(G_6) = \frac{2}{1029}n(676n^2 - 5247n + 7454) + \frac{11758}{343},$$

$$c_{n-4}(G_6) = \frac{1}{57624}n(37349n^3 - 507650n^2 + 1965703n + 1289150) - \frac{709627}{2401},$$

$$c_{n-2}(G_6) = \frac{1}{1008420}n(257049n^4 - 5420675n^3 + 39172605n^2 - 60754765n - 529616214) + \frac{33001040}{16807},$$

$$c_{n-2}(G_7) = \frac{1}{98}n(195n - 709) + \frac{332}{49},\ c_{n-3}(G_7) = \frac{2}{1029}n(676n^2 - 5364n + 14831) - \frac{10096}{343},$$

$$c_{n-4}(G_7) = \frac{1}{57624}n(37349n^3 - 519818n^2 + 2830243n - 7399558) + \frac{340472}{2401},$$

$$c_{n-2}(G_7) = \frac{1}{1008420}n(257049n^4 - 5552495n^3 + 49827665n^2 - 237728905n + 619570486) - \frac{11978116}{16807}.$$

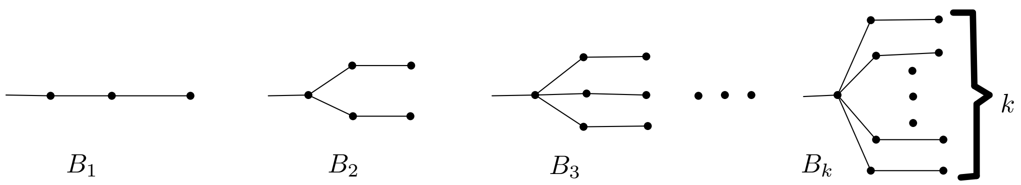


FIGURE 2. The branches of proper Kragujevac trees.

Our third class of trees are caterpillar trees. A caterpillar is a tree in which all the vertices are within distance 1 of a central path [9]. Let $T_i$, for $i = 1, 2, \dots, 5$, be the caterpillar tree on $n \geq 8$ vertices, see Figure 4. Then,

$$c_{n-2}(T_1) = \frac{1}{2}(4n-7)(n-2), \quad c_{n-3}(T_1) = \frac{1}{3}(n-2)(4n^2 - 25n + 42),$$

$$c_{n-4}(T_1) = \frac{1}{24}(n-4)(16n^3 - 168n^2 + 611n - 726),$$

$$c_{n-5}(T_1) = \frac{1}{60}(n-4)(16n^4 - 296n^3 + 2111n^2 - 6811n + 8250),$$

$$c_{n-2}(T_2) = \frac{1}{2}n(4n-15) + \frac{15}{2}, \quad c_{n-3}(T_2) = \frac{1}{3}(n-3)(4n^2 - 21n + 32),$$

$$c_{n-4}(T_2) = \frac{1}{24}n(16n^3 - 232n^2 + 1307n - 3404) + \frac{1155}{8},$$

$$c_{n-5}(T_2) = \frac{1}{60}(n-5)(16n^4 - 280n^3 + 1935n^2 - 6270n + 8079),$$

$$c_{n-2}(T_3) = 2(n-2)^2, \quad c_{n-3}(T_3) = \frac{4}{3}(n-2)(n^2 - 7n + 14),$$

$$c_{n-4}(T_3) = \frac{2}{3}n(n^3 - 16n^2 + 100n - 281) + \frac{575}{3},$$

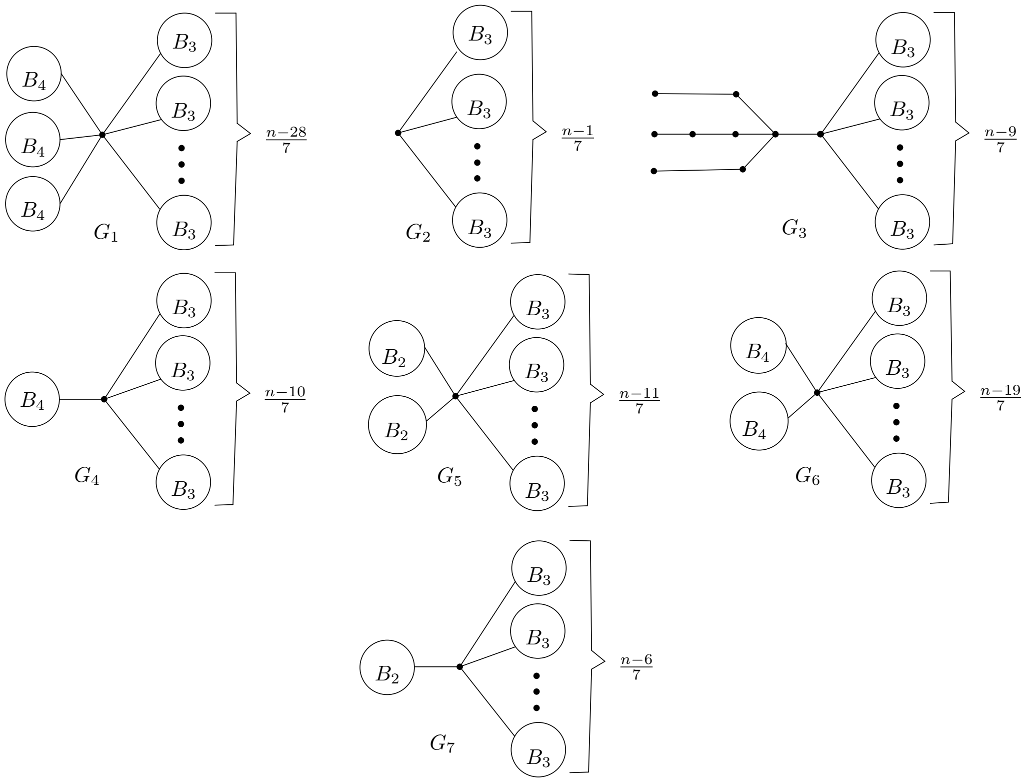


FIGURE 3. The proper Kragujevac trees that have illustrated in [10, Conjecture 3].

$$c_{n-5}(T_3) = \frac{2}{15}(n-5)(2n^4 - 40n^3 + 320n^2 - 1170n + 1573),$$
$$c_{n-2}(T_4) = 2(n-2)^2 + 1, \quad c_{n-3}(T_4) = \frac{2}{3}(n-3)(2n^2 - 12n + 23),$$
$$c_{n-4}(T_4) = \frac{2}{3}(n-4)(n^3 - 12n^2 + 55n - 93),$$
$$c_{n-5}(T_4) = \frac{4}{15}n(n^4 - 25n^3 + 265n^2 - 1480n + 4314) - 1386,$$
$$c_{n-2}(T_5) = 2(n-2)^2 + 1, \quad c_{n-3}(T_5) = \frac{2}{3}n(2n^2 - 18n + 59) - \frac{140}{3},$$
$$c_{n-4}(T_5) = \frac{2}{3}n(n^3 - 16n^2 + 103n - 315) + \frac{769}{3},$$
$$c_{n-5}(T_5) = \frac{2}{15}n(2n^4 - 50n^3 + 530n^2 - 2970n + 8773) - 1456.$$

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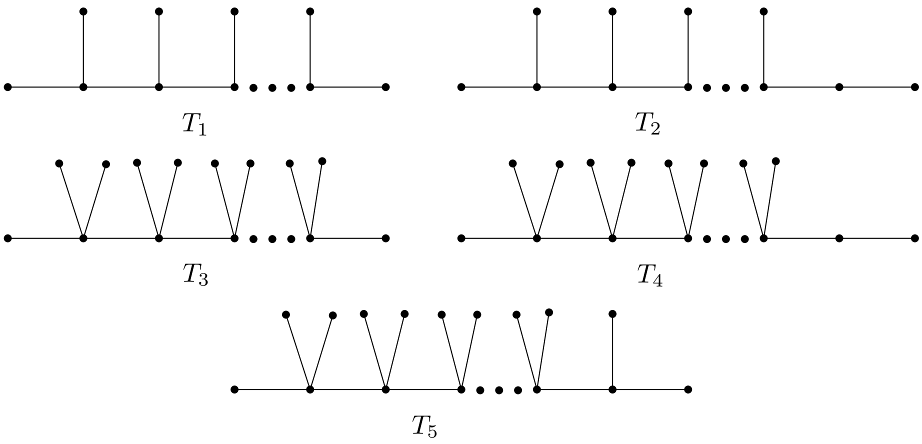


FIGURE 4. The caterpillar trees.

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