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ON KIRCHHOFF-TYPE PROBLEMS INVOLVING $p(z)$-LAPLACIAN-LIKE OPERATORS AND THE ξ-HILFER FRACTIONAL DERIVATIVE

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Abstract. We study the existence of weak solutions for a class of Kirchhoff-type boundary value problems involving a variable-exponent $p(z)$-Laplacian-like operator and the generalized ξ-Hilfer fractional derivative. An appropriate variable-exponent fractional Sobolev space is employed, and the problem is reformulated as an abstract operator equation. Using Berkovits' topological degree theory for operators of type S_+, we establish the existence of at least one weak solution under suitable assumptions on the nonlinear terms.

1. Introduction

Fractional differential equations have emerged as powerful analytical tools for modeling a wide range of complex processes exhibiting memory and hereditary properties. Unlike classical integer-order models, fractional operators are capable of capturing nonlocal dynamics such as anomalous diffusion, viscoelastic behavior, and transport phenomena in heterogeneous media that commonly arise in physics, biology, and engineering. The continuous development of fractional calculus has been driven by two complementary goals: to provide more flexible mathematical frameworks for describing phenomena that cannot be accurately represented by conventional derivatives, and to maintain the essential analytical structures of classical calculus that ensure rigorous treatment of the resulting equations. Within this evolving framework, several fractional operators have been introduced, including the classical Riemann-Liouville

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and Caputo derivatives [1], which have subsequently been generalized by incorporating an auxiliary function ξ to define the so-called ξ-fractional derivatives. A major advance in this direction was made by Sousa and Oliveira [26], who proposed the ξ-Hilfer fractional derivative for multivariable functions. This operator provides a unifying framework that elegantly bridges and extends numerous existing formulations in fractional calculus. Its behavior is governed by three parameters: the fractional order α, the interpolation parameter β, and the auxiliary function ξ. Appropriate choices of ξ enable the recovery of many classical operators as particular cases: for example, choosing $\xi(z) = z$ yields the Caputo and Riemann-Liouville derivatives as $\beta \to 1$ and $\beta \to 0$, respectively [18, 20]; setting $\xi(z) = \ln z$ produces the Hadamard derivative [3], and taking $\xi(z) = z^\rho$ with $\rho > 0$ gives the Katugampola derivative [24]. These connections demonstrate the remarkable versatility of the ξ-Hilfer framework, which now serves as a cornerstone for modern fractional modeling. Owing to this flexibility, the ξ-Hilfer operator has found widespread applications in both theoretical and applied contexts [7–10, 15, 27, 28]. However, despite this growing attention, its use in nonlocal Kirchhoff-type boundary-value problems with variable exponents remains largely unexplored. Such problems arise naturally in models describing nonhomogeneous materials, elastic strings, and electrorheological fluids, where the diffusion coefficient depends on the global energy of the system.

In 2006, Correa et al. [13] established the existence of positive solutions for the following class of p-Kirchhoff problems

$$\begin{cases} \left(-\mathfrak{M}\left(\int_\Lambda |\nabla \vartheta|^p \, dz\right)\right)^{p-1} \Delta_p \vartheta = F(z, \vartheta), & \text{in } \Lambda, \\ \vartheta = 0, & \text{on } \partial\Lambda, \end{cases}$$

and

$$\begin{cases} \left(-\mathfrak{M}\left(\int_\Lambda |\nabla \vartheta|^p \, dz\right)\right)^{p-1} \Delta_p \vartheta = F(z, \vartheta) + \omega|\vartheta|^{s-2}\vartheta, & \text{in } \Lambda, \\ \vartheta = 0, & \text{on } \partial\Lambda, \end{cases}$$

where $\Lambda \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, Δ_p is the p-Laplacian operator given by $\Delta_p \vartheta := \operatorname{div}(|\nabla\vartheta|^{p-2}\nabla\vartheta)$ with $p \in (1, N)$, $s \geq \frac{pN}{N-p}$ and $\mathfrak{M}$, F are continuous functions.

In 2009, Dai et al. [14] investigated the existence of a solution for the following class of $p(z)$-Kirchhoff-type problems

$$\begin{cases} -\mathfrak{M}\left(\int_\Lambda \frac{1}{p(z)}|\nabla \vartheta|^{p(z)} \, dz\right) \operatorname{div}\left(|\nabla\vartheta|^{p(z)-2}\nabla\vartheta\right) = F(z, \vartheta), & \text{in } \Lambda, \\ \vartheta = 0, & \text{on } \partial\Lambda, \end{cases}$$

where $\Lambda \subset \mathbb{R}^N$ is a smooth bounded domain, $p \in C(\overline{\Lambda})$ with $p^+ < N$, $\mathfrak{M}$ is continuous function and $F : \Lambda \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition.

Inspired by the aforementioned works, we investigate the existence of weak solutions for the following class of $p(z)$-Kirchhoff-like problems.

$$(1.1)\qquad \begin{cases} -\mathfrak{M}\left(\mathfrak{K}(\vartheta)\right)\Delta^{\alpha,\beta;\xi}_{L,p(z)}\vartheta = -\tau|\vartheta|^{s(z)-2}\vartheta + \omega F\left(z,\vartheta,\ {}^{H}\mathfrak{D}^{\alpha,\beta;\xi}_{0^+}\vartheta\right), & \text{in } \Lambda,\\ \vartheta = 0, & \text{on } \partial\Lambda, \end{cases}$$

where

$$\Delta^{\alpha,\beta;\xi}_{L,p(z)}\vartheta := {}^{H}\mathfrak{D}^{\alpha,\beta;\xi}_{T^-}\left(\left|{}^{H}\mathfrak{D}^{\alpha,\beta;\xi}_{0^+}\vartheta\right|^{p(z)-2}\ {}^{H}\mathfrak{D}^{\alpha,\beta;\xi}_{0^+}\vartheta + \frac{\left|{}^{H}\mathfrak{D}^{\alpha,\beta;\xi}_{0^+}\vartheta\right|^{2p(z)-2}\ {}^{H}\mathfrak{D}^{\alpha,\beta;\xi}_{0^+}\vartheta}{\sqrt{1+\left|{}^{H}\mathfrak{D}^{\alpha,\beta;\xi}_{0^+}\vartheta\right|^{2p(z)}}}\right),$$

$$\mathfrak{K}(\vartheta) = \int_\Lambda \frac{\left|{}^{H}\mathfrak{D}^{\alpha,\beta;\xi}_{0^+}\vartheta\right|^{p(z)} + \sqrt{1+\left|{}^{H}\mathfrak{D}^{\alpha,\beta;\xi}_{0^+}\vartheta\right|^{2p(z)}}}{p(z)}dz,$$

${}^{H}\mathfrak{D}^{\alpha,\beta;\xi}_{0^+}$ and ${}^{H}\mathfrak{D}^{\alpha,\beta;\xi}_{T^-}$ are the left and right ξ-Hilfer fractional partial derivatives of order $0<\alpha<1$ with $\alpha p(z)<N$ and of type $\beta\in[0,1]$, respectively. ω is a real parameter, $\Lambda=\Lambda_1\times\cdots\times\Lambda_N=[0,T]\times\cdots\times[0,T]\subset\mathbb{R}^N$, $p,s\in C(\bar{\Lambda})$ with $1<s^-\leq s(z)\leq s^+<p^-$, $\tau\in L^\infty(\Lambda)$ such that $\tau(z)>0$ a.e. in Λ and $\mathfrak{M}$, $F:\Lambda\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ are given functions satisfying certain assumptions.

The study of mathematical problems involving variable exponent growth conditions has attracted significant attention in recent years (see, for example, [5, 15, 17]). This growing interest is largely motivated by the wide range of physical applications in which such problems naturally arise. It is well known that problem (1.1) is related to the stationary version of Kirchhoff's equation

$$(1.2)\qquad \rho\frac{\partial^2\vartheta}{\partial t^2} - \left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L \frac{\partial\vartheta}{\partial z}^2 dz\right)\frac{\partial^2\vartheta}{\partial z^2} = 0,$$

introduced by Kirchhoff in 1883 [22], where E, ρ_0, ρ, h, and L are constants. This equation generalizes the classical d'Alembert wave equation by incorporating the effect of changes in the length of a vibrating string during oscillations. Kirchhoff-type boundary value problems arise in the modeling of numerous physical and biological systems in which the state variable ϑ depends on its own average, such as in models describing population density.

The main purpose of this paper is to establish the existence of weak solutions to a class of fractional boundary-value problems of Kirchhoff type involving the ξ-Hilfer fractional derivative. Our analysis relies on constructing an appropriate functional framework in a variable-exponent Sobolev space adapted to the ξ-Hilfer operator and reformulating the original problem as an abstract Hammerstein-type equation. We then apply Berkovits's topological degree theory to obtain existence results under suitable monotonicity and growth conditions on the nonlinear terms. This theory is a widely used tool in the analysis of nonlinear equations. The concept of topological degree was originally developed by Leray and Schauder [23] in their study of nonlinear problems involving compact perturbations of the identity operator in infinite-dimensional Banach spaces. Later, Browder [11] introduced a degree theory for operators of type (S_+) in reflexive Banach spaces, employing the Galerkin approximation method. Building upon this foundation, Berkovits [6] extended the Leray-Schauder degree to operators of generalized monotone type. In this framework, the compactness condition is essentially replaced by compositions of monotone operators. For additional details, see [2, 4, 16, 21]. However, to the best of our knowledge, no previous work has addressed a ξ-Hilfer fractional Kirchhoff problem with variable-exponent nonlinearities within a functional framework suitable for the application of Berkovits's topological degree theory. The combination of a nonlocal Kirchhoff term, a generalized ξ-Hilfer operator, and a variable-exponent structure presents new analytical challenges, particularly regarding coercivity, compactness, and the identification of monotone limits in nonstandard growth settings.

The structure of the paper is as follows. Section 2 introduces the main mathematical tools of the ξ-Hilfer fractional calculus, Berkovits's topological degree theory, and the functional setting used throughout the paper. Section 3 reformulates the main problem as an abstract Hammerstein-type equation and applies Berkovits's degree theory to derive the existence of solutions.

2. Preliminary Results and Notations

Let $p \in C_+(\Lambda) := \{p \in C(\Lambda) : 1 < p^- := \inf_{z\in\Lambda} p(z) \leq p^+ := \sup_{z\in\Lambda} p(z) < +\infty\}$. Then, the variable exponent Lebesgue space is given by

$$L^{p(z)}(\Lambda) = \left\{\vartheta : \Lambda \to \mathbb{R} \text{ is measurable and } \int_\Lambda |\vartheta(z)|^{p(z)} dz < +\infty\right\}.$$

Endowed with the Luxemburg norm

$$\|\vartheta\|_{p(z)} = \inf\left\{\mu > 0 : \ \sigma_{p(z)}\left(\frac{\vartheta}{\mu}\right) \leq 1\right\},$$

where

$$\sigma_{p(z)}(\vartheta) = \int_\Lambda |\vartheta(z)|^{p(z)} dz, \qquad \text{for all } \vartheta \in L^{p(z)}(\Lambda).$$

Proposition 2.1 ([15])**.** *If $\vartheta_n, \vartheta \in L^{p(z)}(\Lambda)$, then we have*

(a) $\lim_{n\to+\infty} \sigma_{p(z)}(\vartheta_n - \vartheta) = 0$ *if and only if* $\lim_{n\to+\infty} \|\vartheta_n - \vartheta\|_{p(z)} = 0$,

(b) $\sigma_{p(z)}(\vartheta) < 1$ (*resp.* $> 1, = 1$) *if and only if* $\|\vartheta\|_{p(z)} < 1$ (*resp.* $> 1, = 1$),

(c) $\|\vartheta\|_{p(z)} < 1$ *implies* $\|\vartheta\|^{p^+}_{p(z)} \leq \sigma_{p(z)}(\vartheta) \leq \|\vartheta\|^{p^-}_{p(z)}$,

(d) $\|\vartheta\|_{p(z)} > 1$ *implies* $\|\vartheta\|^{p^-}_{p(z)} \leq \sigma_{p(z)}(\vartheta) \leq \|\vartheta\|^{p^+}_{p(z)}$.

According to Proposition 2.1, we can derive the following inequalities

$$\|\vartheta\|_{p(z)} \leq \sigma_{p(z)}(\vartheta) + 1 \quad \text{ and } \quad \sigma_{p(z)}(\vartheta) \leq \|\vartheta\|^{p^-}_{p(z)} + \|\vartheta\|^{p^+}_{p(z)}.$$

Definition 2.1 ([26,27])**.** Let $a = (a_1, \dots, a_N)$, $b = (b_1, \dots, b_N)$, $\Lambda = \Lambda_1 \times \cdots \times \Lambda_N = [a_1, b_1] \times \cdots \times [a_N, b_N]$, $\xi \in C^1(\Lambda)$ with $\xi' > 0$ on Λ. Then, the right and left-sided

ξ-Riemann-Liouville fractional partial integral of order $0 < \alpha = (\alpha_1, \dots, \alpha_N) < 1$ of the function $\vartheta = (\vartheta_1, \dots, \vartheta_N) \in L^1(\Lambda)$ are given as follows

$$\mathfrak{I}_{a^+}^{\alpha;\xi}\vartheta(z) = \frac{1}{\Gamma(\alpha)}\int_{a_1}^{z_1}\int_{a_2}^{z_2}\cdots\int_{a_N}^{z_N}\Upsilon_\xi^{\alpha-1}(z,s)\vartheta(s)ds$$

and

$$\mathfrak{I}_{b^-}^{\alpha;\xi}\vartheta(z) = \frac{1}{\Gamma(\alpha)}\int_{z_1}^{b_1}\int_{z_2}^{b_2}\cdots\int_{z_N}^{b_N}\Upsilon_\xi^{\alpha-1}(s,z)\vartheta(s)ds,$$

where

$$\Gamma(\alpha) = \prod_{i=1}^N \Gamma(\alpha_i), \quad \Upsilon_\xi^\alpha(z,s) = \prod_{i=1}^N \xi'(s_i)(\xi(z_i)-\xi(s_i))^{\alpha_i}, \quad ds = \prod_{i=1}^N ds_i.$$

Definition 2.2 ([26,27])**.** Let $\xi \in C^n(\Lambda)$ with $\xi' > 0$ on Λ. Then, the right and left-sided ξ-Hilfer fractional partial derivative of the function $\vartheta = (\vartheta_1, \dots, \vartheta_N) \in C^n(\Lambda)$ of order $0 < \alpha = (\alpha_1, \dots, \alpha_N) < 1$ and type $0 \le \beta = (\beta_1, \dots, \beta_N) \le 1$ are given as

$$^H\mathfrak{D}_{a^+}^{\alpha,\beta;\xi}\vartheta(z) = \mathfrak{I}_{a^+}^{\beta(1-\alpha);\xi}\left(\frac{1}{\xi'(z)}\cdot\frac{\partial^N}{\partial z}\right)\mathfrak{I}_{a^+}^{(1-\beta)(1-\alpha);\xi}\vartheta(z)$$

and

$$^H\mathfrak{D}_{b^-}^{\alpha,\beta;\xi}\vartheta(z) = \mathfrak{I}_{b^-}^{\beta(1-\alpha);\xi}\left(\frac{1}{\xi'(z)}\cdot\frac{\partial^N}{\partial z}\right)\mathfrak{I}_{b^-}^{(1-\beta)(1-\alpha);\xi}\vartheta(z),$$

where $\partial z = \prod_{i=1}^N \partial z_i$ and $\xi'(z) = \prod_{i=1}^N \xi'(z_i)$.

Now, we define the ξ-fractional derivative space $\mathbb{H}_{p(z)}^{\alpha,\beta;\xi}(\Lambda)$ as follows

$$\mathbb{H}_{p(z)}^{\alpha,\beta;\xi}(\Lambda) = \left\{\vartheta \in L^{p(z)}(\Lambda) : \left|^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\right| \in L^{p(z)}(\Lambda)\right\},$$

endowed with the norm

$$\|\vartheta\|_{\mathbb{H}_{p(z)}^{\alpha,\beta;\xi}} = \|\vartheta\|_{p(z)} + \|\,^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\|_{p(z)}.$$

Let us define the space $\mathbb{H}_0(\Lambda)$ as the closure of $C_0^\infty(\Lambda)$ in $\mathbb{H}_{p(z)}^{\alpha,\beta;\xi}(\Lambda)$, endowed with the equivalent norm $\|\vartheta\| := \left\|^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\right\|_{p(z)}$, the spaces $\mathbb{H}_{p(z)}^{\alpha,\beta;\xi}(\Lambda)$ and $\mathbb{H}_0(\Lambda)$ are reflexive and separable Banach spaces [5, 25].

Theorem 2.1 ([25])**.** *Let $\Lambda \subset \mathbb{R}^N$ be a Lipschitz bounded domain and $\gamma \in C^0(\bar{\Lambda})$. If $\gamma : \bar{\Lambda} \to (1, +\infty)$ such that $\gamma(z) \in (1, \bar{p}(z))$, with*

$$\bar{p}(z) = \begin{cases} +\infty, & \text{if } p(z) \ge \frac{N}{\alpha}, \\ \frac{Np(z)}{N-\alpha p(z)}, & \text{if } p(z) < \frac{N}{\alpha}, \end{cases}$$

then the embedding $\mathbb{H}_0(\Lambda) \hookrightarrow L^{\gamma(z)}(\Lambda)$ is compact.

We next outline the key results and properties of the topological degree theory required for our study.

Let $\mathtt{Z}$ be a reflexive, separable real Banach space and let $\mathtt{Z}^*$ its dual space with dual pairing $\langle\cdot,\cdot\rangle$. The symbols $\rightharpoonup$ and $\to$ are used to denote weak and strong convergence, respectively.

Definition 2.3. The operator $\mathcal{M} : \Lambda \subset \mathtt{Z} \to \mathtt{Y}$, where $\mathtt{Y}$ is a real Banach space, is called

(a) compact, if $\mathcal{M}$ is continuous and for any bounded set $\mathbb{B} \subset \Lambda$, then $\mathcal{M}(\mathbb{B})$ is relatively compact,
(b) demicontinuous, if for every $(\vartheta_n) \subset \Lambda$, $\vartheta_n \to \vartheta$ implies $\mathcal{M}(\vartheta_n) \rightharpoonup \mathcal{M}(\vartheta)$,
(c) bounded, if it takes every bounded set into a bounded set.

Definition 2.4. The operator $\mathcal{M} : \Lambda \subset \mathtt{Z} \to \mathtt{Z}^*$ is called

(a) quasimonotone, if for every $(\vartheta_n) \subset \Lambda$ with $\vartheta_n \rightharpoonup \vartheta$, it follows that $\limsup_{n\to+\infty}\langle \mathcal{M}\vartheta_n, \vartheta_n - \vartheta\rangle \geq 0$,
(b) of type (S_+), if for every $(\vartheta_n) \subset \Lambda$ with $\vartheta_n \rightharpoonup \vartheta$ and $\limsup_{n\to+\infty}\langle \mathcal{M}\vartheta_n, \vartheta_n - \vartheta\rangle \leq 0$, it follows that $\vartheta_n \to \vartheta$.

Definition 2.5. Let $\mathcal{M} : \Lambda_1 \subset \mathtt{Z} \to \mathtt{Z}^*$ be a bounded operator with $\Lambda \subset \Lambda_1$. For any $\mathcal{P} : \Lambda \subset \mathtt{Z} \to \mathtt{Z}$, we say that

(a) $\mathcal{P}$ satisfies the condition $(S_+)_{\mathcal{M}}$, if for every $(\vartheta_n) \subset \Lambda$ with $\vartheta_n \rightharpoonup \vartheta$, $\chi_n := \mathcal{M}\vartheta_n \rightharpoonup \chi$ and $\limsup_{n\to+\infty}\langle \mathcal{P}\vartheta_n, \chi_n - \chi\rangle \leq 0$, we have $\vartheta_n \to \vartheta$,
(b) $\mathcal{P}$ has the $(QM)_{\mathcal{M}}$ property, if for every sequence $(\vartheta_n) \subset \Lambda$ with $\vartheta_n \rightharpoonup \vartheta, \chi_n := \mathcal{M}\vartheta_n \rightharpoonup \chi$, we have $\limsup_{n\to+\infty}\langle \mathcal{P}\vartheta_n, \chi - \chi_n\rangle \geq 0$.

Let $\Lambda \subset \mathtt{Z}$, we define the following classes of operators by

(a) $\mathtt{E}_1(\Lambda) := \{\mathcal{P} : \Lambda \to \mathtt{Z}^* : \mathcal{P}$ is of type (S_+), bounded and demicontinuous$\}$,
(b) $\mathtt{E}_{\mathcal{M}}(\Lambda) := \{\mathcal{P} : \Lambda \to \mathtt{Z} : \mathcal{P}$ is of type $(S_+)_{\mathcal{M}}$ and demicontinuous$\}$,
(c) $\mathtt{E}(\mathtt{Z}) := \left\{\mathcal{M} \in \mathtt{E}_{\mathcal{M}}(\overline{\Lambda}) : \Lambda \in \mathtt{O},\ \mathcal{M} \in \mathtt{E}_1(\overline{\Lambda})\right\}$,

where $\mathtt{O}$ is the family of all bounded open subsets of $\mathtt{Z}$.

Lemma 2.1 ([21])**.** *Let $\Lambda \subset \mathtt{Z}$ be a open bounded set, $\mathcal{M} \in \mathtt{E}_1(\overline{\Lambda})$ be continuous and $\mathcal{P} : D_{\mathcal{P}} \subset \mathtt{Z}^* \to \mathtt{Z}$ be demicontinuous such that $\mathcal{M}(\bar{\Lambda}) \subset D_{\mathcal{P}}$. Then, the following hold.*

(a) *If $\mathcal{P}$ is of type (S_+), then $\mathcal{P}o\mathcal{M} \in \mathtt{E}_{\mathcal{M}}(\overline{\Lambda})$.*
(b) *If $\mathcal{P}$ is quasimonotone, then $\mathtt{I} + \mathcal{P}o\mathcal{M} \in \mathtt{E}_{\mathcal{M}}(\overline{\Lambda})$, where $\mathtt{I}$ is the identity operator.*

Definition 2.6. Let $\Lambda \subset \mathtt{Z}$ be a bounded open set, $\mathcal{M} \in \mathtt{E}_1(\overline{\Lambda})$ be continuous. Suppose that $\mathcal{M}, \mathcal{P} \in \mathtt{E}_{\mathcal{M}}(\overline{\Lambda})$. The affine homotopy $\Theta : [0,1] \times \bar{\Lambda} \to \mathtt{Z}$ is given by

$$\Theta(\varepsilon, \vartheta) = (1-\varepsilon)\mathcal{M}\vartheta + \varepsilon\mathcal{P}\vartheta, \qquad \text{for } (\varepsilon, \vartheta) \in [0,1] \times \bar{\Lambda}.$$

Remark 2.1 ([6]). The homotopy above is of type $(S_+)_{\mathcal{M}}$.

Theorem 2.2 ([21])**.** *There exists a unique degree function*

$$\mathcal{D}\mathsf{eg} : \mathbb{B} = \left\{ (\mathcal{P}, \Lambda, \varpi) : \Lambda \in \mathsf{O},\ \mathcal{M} \in \mathtt{E}_1(\overline{\Lambda}),\ \mathcal{P} \in \mathtt{E}_{\mathcal{M}}(\overline{\Lambda}),\ \varpi \notin \mathcal{P}(\partial\Lambda) \right\} \to \mathbb{Z},$$

that satisfies the following.

(a) *If* $\Theta : [0,1]\times\bar{\Lambda} \to \mathtt{Z}$ *is an admissible affine homotopy with a common continuous essential inner map and* $\varpi : [0,1] \to \mathtt{Z}$ *is a continuous path in* $\mathtt{Z}$ *such that* $\varpi(\varepsilon) \notin \Theta(\varepsilon, \partial\Lambda)$ *for every* $\varepsilon \in [0,1]$*, then* $\mathcal{D}\mathsf{eg}(\Theta(\varepsilon,\cdot), \Lambda, \varpi(\varepsilon))$ *is constant for every* $\varepsilon \in [0,1]$.

(b) *Let* $\varpi \in \Lambda$*, we have* $\mathcal{D}\mathsf{eg}(\mathsf{I}, \Lambda, \varpi) = 1$.

(c) *If* $\mathcal{D}\mathsf{eg}(\mathcal{P}, \Lambda, \varpi) \neq 0$*, then the equation* $\mathcal{P}\vartheta = \varpi$ *possesses a solution in* Λ.

3. Existence Results

This section is devoted to establishing the existence of weak solutions to problem (1.1).

Definition 3.1. A function $\vartheta \in \mathbb{H}_0(\Lambda)$ is called a weak solution of (1.1) if

$$\begin{aligned}
&\mathfrak{M}\left(\mathfrak{K}(\vartheta)\right) \int_\Lambda \left(\left|{}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\right|^{p(z)-2}\ {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta + \frac{\left|{}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\right|^{2p(z)-2}\ {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta}{\sqrt{1+\left|{}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\right|^{2p(z)}}} \right) \\
&\quad \times {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\chi + |\vartheta|^{p(z)-2}\vartheta\chi dz \\
&= \int_\Lambda \left(-\tau(z)|\vartheta|^{p(z)-2}\vartheta + \omega\mathcal{F}\left(z, \vartheta, {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\right)\right)\chi dz,
\end{aligned}$$

for all $\chi \in \mathbb{H}_0(\Lambda)$.

In the following, the functions $\mathfrak{M}$ and $\mathcal{F}$ are assumed to satisfy the following hypotheses.

($\mathcal{A}_1$) $\mathfrak{M} : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, increasing function such that

$$m_0 t^{r-1} \leq \mathfrak{M}(t) \leq m_1 t^{r-1},$$

where m_0 and m_1 are two positive constants with $m_0 < m_1$ and r is constant with $1 \leq r < p^- \leq p(z) \leq p^+ < +\infty$ for all $t \in \mathbb{R}^+$.

($\mathcal{A}_2$) $\mathcal{F}$ is a Carathéodory function.

($\mathcal{A}_3$) There exist a function $\zeta \in L^{p'(z)}(\Lambda)$ with $p'(z) = \frac{p(z)}{p(z)-1}$ and a constant $\kappa > 0$ such that

$$|\mathcal{F}(z,\vartheta,\Psi)| \leq \kappa\left(\zeta(z) + |\vartheta|^{q(z)-1} + |\Psi|^{q(z)-1}\right), \quad 1 < q^- \leq q(z) \leq q^+ < p^-,$$

for a.e. $z \in \Lambda$ and every $(\vartheta, \Psi) \in \mathbb{R} \times \mathbb{R}^N$.

In what follows, we define the operators $\mathfrak{L}$, $\mathfrak{S} : \mathbb{H}_0(\Lambda) \to (\mathbb{H}_0(\Lambda))^*$ by

$$\langle \mathfrak{L}\vartheta, \chi\rangle = \mathfrak{M}(\mathfrak{K}(\vartheta)) \int_\Lambda \left(\left|{}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\right|^{p(z)-2}\ {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta + \frac{\left|{}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\right|^{2p(z)-2}\ {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta}{\sqrt{1+\left|{}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\right|^{2p(z)}}} \right)$$

$$\times^{H}\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\chi+|\vartheta|^{p(z)-2}\vartheta\chi dz$$

and

$$\langle\mathfrak{S}\vartheta,\chi\rangle=\int_{\Lambda}\tau|\vartheta|^{s(z)-2}\vartheta\chi-\omega F\left(z,\vartheta,\ {}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta\right)\chi\,dz,$$

for all $\chi\in\mathbb{H}_0(\Lambda)$.

Moreover, we consider the following functional

$$\mathfrak{J}(\vartheta)=\widehat{\mathfrak{M}}\left(\mathfrak{K}(\vartheta)\right),\quad\text{where }\widehat{\mathfrak{M}}(t)=\int_0^t\mathfrak{M}(s)ds.$$

Lemma 3.1. *The operator $\mathfrak{L}$ is a bounded, continuous, strictly monotone, of type (S_+) and homeomorphism.*

Proof. (a) The operator $\mathfrak{J}$ is a continuously Gâteaux differentiable function whose Gâteaux derivative at the point $\vartheta\in\mathbb{H}_0(\Lambda)$ is the functional from $\mathbb{H}_0(\Lambda)$ to the dual space $(\mathbb{H}_0(\Lambda))^*$, such that

$$\begin{aligned}\langle\mathfrak{J}'\vartheta,\chi\rangle=&\mathfrak{M}\left(\mathfrak{K}(\vartheta)\right)\int_{\Lambda}\left(\left|{}^{H}\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\right|^{p(z)-2}\ {}^{H}\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta+\frac{\left|{}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta\right|^{2p(z)-2}\ {}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta}{\sqrt{1+\left|{}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta\right|^{2p(z)}}}\right)\\&\times^{H}\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\chi dz\\=&\langle\mathfrak{L}\vartheta,\chi\rangle.\end{aligned}$$

Thus, the operator $\mathfrak{L}$ is continuous.

(b) Let $\vartheta,\chi\in\mathbb{H}_0(\Lambda)$, then by using Hölder's inequality and the involved embeddings, we have

$$\begin{aligned}|\langle\mathfrak{L}\vartheta,\chi\rangle|\leq& m_1\left(\int_{\Lambda}\frac{2\left|{}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta\right|^{p(z)}+1}{p^-}dz\right)^{r-1}\int_{\Lambda}\left(2\left|{}^{H}\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\right|^{p(z)-1}\right)\ {}^{H}\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\chi dz\\\leq& c\left(1+\|\vartheta\|^{\theta}\right)^{(r-1)}\left(\int_{\Lambda}\left|{}^{H}\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta^{(p(z)-1)}\right|^{p'(z)}dz\right)^{\frac{1}{\nu}}\|\chi\|\\\leq& c\left(1+\|\vartheta\|^{\theta(r-1)+\frac{\theta}{\nu}}\right)\|\chi\|,\end{aligned}$$

where c is constant,

$$\theta=\begin{cases}p^+,&\text{if }\|\vartheta\|\geq1,\\p^-,&\text{if }\|\vartheta\|\leq1,\end{cases}\qquad\text{and}\qquad\nu=\begin{cases}p'^-,&\text{if }\|\vartheta\|\geq1,\\p'^+,&\text{if }\|\vartheta\|\leq1.\end{cases}$$

Therefore, $\mathfrak{L}$ is bounded.

(c) Let $\vartheta_1,\vartheta_2\in\mathbb{H}_0(\Lambda)$ such that $\vartheta_1\neq\vartheta_2$, by using the following well-known inequality [19].

$$\left(|\vartheta_1|^{\gamma-2}\vartheta_1-|\vartheta_2|^{\gamma-2}\vartheta_2\right)(\vartheta_1-\vartheta_2)>0,\qquad\text{for every }\gamma\geq1.\tag{3.1}$$

Then, we can deduce that $\langle\mathfrak{K}'\vartheta_1-\mathfrak{K}'\vartheta_2,\vartheta_1-\vartheta_2\rangle>0$. So, $\mathfrak{K}'$ is strictly monotone. Then by [29, Proposition 2.1] we infer that $\mathfrak{K}$ is strictly convex. Moreover, since $\mathfrak{M}$

is increasing, then $\widehat{\mathfrak{M}}$ is convex in $\mathbb{R}^+$. Thus, for every $\vartheta_1, \vartheta_2 \in \mathbb{H}_0(\Lambda)$ with $\vartheta_1 \neq \vartheta_2$, and every $t_1, t_2 \in (0,1)$ with $t_1 + t_2 = 1$, we have

$$\widehat{\mathfrak{M}}(\mathfrak{K}(t_1\vartheta_1 + t_2\vartheta_2)) < \widehat{\mathfrak{M}}(t_1\mathfrak{K}(\vartheta_1) + t_2\mathfrak{K}(\vartheta_2)) \leq t_1\widehat{\mathfrak{M}}(\mathfrak{K}(\vartheta_1)) + t_2\widehat{\mathfrak{M}}(\mathfrak{K}(\vartheta_2)).$$

This means that $\mathfrak{J}$ is strictly convex, then $\mathfrak{L}$ is strictly monotone in $\mathbb{H}_0(\Lambda)$.

(d) Let $\vartheta \in \mathbb{H}_0(\Lambda)$, then by using $(\mathcal{A}_1)$ we have

$$\begin{aligned}
\langle \mathfrak{L}(\vartheta), \vartheta \rangle =& \mathfrak{M}(\mathfrak{K}(\vartheta)) \int_\Lambda \left(\left| {}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta \right|^{p(z)} + \frac{\left| {}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta \right|^{2p(z)}}{\sqrt{1 + \left| {}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta \right|^{2p(z)}}} \right) dz \\
&\geq m_0 \left(\frac{1}{p^+} \int_\Lambda \left(\left| {}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta \right|^{p(z)} \right) dz \right)^{r-1} \left(\int_\Lambda \left| {}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta \right|^{p(z)} dz \right) \\
&\geq m_0 \left(\frac{1}{p^+} \right)^{r-1} \left(\int_\Lambda \left| {}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta \right|^{p(z)} dz \right)^r.
\end{aligned}$$

Thus,

$$\frac{\langle \mathfrak{L}(\vartheta), \vartheta \rangle}{\|\vartheta\|} \to +\infty, \quad \text{as } \|\vartheta\| \to +\infty.$$

Hence, $\mathfrak{L}$ is coercive.

(e) Let $(\vartheta_n)_n \subset \mathbb{H}_0(\Lambda)$ such that

$$\begin{cases} \vartheta_n \rightharpoonup \vartheta, \quad \text{in } \mathbb{H}_0(\Lambda), \\ \limsup\limits_{n\to+\infty} \langle \mathfrak{L}\vartheta_n, \vartheta_n - \vartheta \rangle \leq 0. \end{cases}$$

Since $\vartheta_n \rightharpoonup \vartheta$ in $\mathbb{H}_0(\Lambda)$, then $(\vartheta_n)_n$ is a bounded sequence in $\mathbb{H}_0(\Lambda)$, then there exist a subsequence still denoted by $(\vartheta_n)_n$ such that $\vartheta_n \to \vartheta$ in $\mathbb{H}_0(\Lambda)$, and under the strict monotonicity of $\mathfrak{L}$, we get

$$\lim_{n\to+\infty} \langle \mathfrak{L}\vartheta_n, \ \vartheta_n - \vartheta \rangle = 0.$$

Thus,

$$\begin{aligned}
&\lim_{n\to+\infty} \mathfrak{M}\left(\mathfrak{K}(\vartheta_n)\right) \int_\Lambda \left(\left| {}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta_n \right|^{p(z)-2} + \frac{\left| {}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta_n \right|^{2p(z)-2} {}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta_n}{\sqrt{1 + \left| {}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta_n \right|^{2p(z)}}} \right) \\
&\times {}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta_n(\vartheta_n - \vartheta)\, dz = 0. \qquad (3.2)
\end{aligned}$$

Given that $\mathfrak{K}(\vartheta)$ is bounded, then we infer that $\mathfrak{K}(\vartheta_n)$ is bounded. So since $\mathfrak{M}$ is continuous, up to a subsequence, there is $\ell > 0$ such that

$$\mathfrak{M}\left(\mathfrak{K}(\vartheta_n)\right) \to \mathfrak{M}(\ell) \geq m_0\ell^{r-1}, \quad \text{as } n \to +\infty. \qquad (3.3)$$

From (3.2) and (3.3), we can deduce that

$$\lim_{n\to+\infty}\int_\Lambda\left(\left|{}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta_n\right|^{p(z)-2}+\frac{\left|{}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta_n\right|^{2p(z)-2}\ {}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta_n}{\sqrt{1+\left|{}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta_n\right|^{2p(z)}}}\right)$$

$$\times{}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\xi}\vartheta_n(\vartheta_n-\vartheta)\,dz=0. \tag{3.4}$$

Since, the functional (3.4) is of type (S_+) [15, Proposition 2.7], then we have

$$\vartheta_n\to\vartheta,\quad \text{in } \mathbb{H}_0(\Lambda).$$

Thus, $\mathfrak{L}$ is of type (S_+).

(f) Given that $\mathfrak{L}$ is strictly monotone in $\mathbb{H}_0(\Lambda)$, thus $\mathfrak{L}$ is an injection and by the Minty Browder theorem [29], $\mathfrak{L}$ is a surjection. Therefore $\mathfrak{L}$ possesses an inverse $\mathfrak{L}^{-1}:(\mathbb{H}_0(\Lambda))^*\to\mathbb{H}_0(\Lambda)$. Thus, the continuity of $\mathfrak{L}^{-1}$ suffices to conclude that $\mathfrak{L}$ is a homeomorphism. If $\digamma_n,\digamma\in(\mathbb{H}_0(\Lambda))^*$, $\digamma_n\to\digamma$, let $\vartheta_n=\mathfrak{L}^{-1}(\digamma_n)$, $\vartheta=\mathfrak{L}^{-1}(\digamma)$, it follows $\mathfrak{L}(\vartheta_n)=\digamma_n$, $\mathfrak{L}(\vartheta)=\digamma$. Thus, $(\vartheta_n)_n$ is bounded in $\mathbb{H}_0(\Lambda)$. Without loss of generality, we suppose that $\vartheta_n\rightharpoonup\vartheta_0$.

Since $\digamma_n\to\digamma$, thus

$$\lim_{n\to+\infty}\langle\mathfrak{L}(\vartheta_n)-\mathfrak{L}(\vartheta_0),\vartheta_n-\vartheta_0\rangle=\lim_{n\to+\infty}\langle\digamma_n-\digamma,\vartheta_n-\vartheta_0\rangle=0.$$

Given that $\mathfrak{L}$ is of type (S_+) and $\vartheta_n\to\vartheta_0$, we deduce that $\vartheta_n\to\vartheta$.

Thus, $\mathfrak{L}^{-1}$ is continuous. □

Lemma 3.2. *The operator $\mathfrak{G}$ is compact.*

Proof. We define the following operator $\Phi_1:\mathbb{H}_0(\Lambda)\to L^{p'(z)}(\Lambda)$ by

$$\Phi_1\vartheta(z)=\tau|\vartheta(z)|^{s(z)-2}\vartheta(z),\quad \text{for } \vartheta\in\mathbb{H}_0(\Lambda).$$

Then, it is clear that Φ_1 is continuous.

Now, let us demonstrate that Φ_1 is bounded on $\mathbb{H}_0(\Lambda)$. Let $\vartheta\in\mathbb{H}_0(\Lambda)$. By employing the compact embedding $\mathbb{H}_0(\Lambda)\hookrightarrow L^{p(z)}(\Lambda)$ and the Poincaré inequality, we obtain

$$\begin{aligned}\|\Phi_1\vartheta\|_{p'(z)}&\le\sigma_{p'(z)}(\Phi_1\vartheta)+1\le\int_\Lambda\left|\tau|\vartheta|^{s(z)-2}\vartheta\right|^{p'(z)}dz+1\\&\le\|\tau\|_{L^\infty(\Lambda)}^{p'+}\int_\Lambda|\vartheta|^{\gamma_1(z)}dz+1\le\|\tau\|_{L^\infty(\Lambda)}^{p'+}\sigma_{\gamma_1(z)}(\vartheta)+1\\&\le\|\tau\|_{L^\infty(\Lambda)}^{p'+}\left(\|\vartheta\|_{\gamma_1(z)}^{\gamma+}+\|\vartheta\|_{\gamma_1(z)}^{\gamma-}\right)+1,\end{aligned}$$

where $\gamma_1(z)=p'(z)(s(z)-1)<p(z)$.

By employing the continuous embedding $L^{p(z)}(\Lambda)\hookrightarrow L^{\gamma_1(z)}(\Lambda)$ along with the Poincaré inequality, it follows that

$$\|\Phi_1\vartheta\|_{p'(z)}\le c\left(\|\vartheta\|^{\gamma+}+\|\vartheta\|^{\gamma-}\right)+1,\quad c=const.$$

Thus, Φ_1 is bounded on $\mathbb{H}_0(\Lambda)$.

Now, we introduce the operator $\Phi_2 : \mathbb{H}_0(\Lambda) \to L^{p'(z)}(\Lambda)$ by

$$\Phi_2\vartheta(z) = -\omega F(z, \vartheta, {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta), \quad \text{for } \vartheta \in \mathbb{H}_0(\Lambda).$$

First, we demonstrate that Φ_2 is bounded. Let $\vartheta \in \mathbb{H}_0(\Lambda)$. Then by using $(\mathcal{A}_3)$ we have

$$\begin{aligned}
\|\Phi_2\vartheta\|_{p'(z)} \leq& \sigma_{p'(z)}(\Phi_2\vartheta) + 1\\
\leq& \int_\Lambda |\omega|^{p'(z)} \, |F(z, \vartheta(z), {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta(z))|^{p'(z)} + 1\\
\leq& (|\omega|^{p'+} + |\omega|^{p'-}) \int_\Lambda \kappa \left|\left(\zeta(z) + |\vartheta(z)|^{q(z)-1} + |{}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta(z)|^{q(z)-1}\right)\right|^{p'(z)} dz\\
&+ 1\\
\leq& c(|\omega|^{p'+} + |\omega|^{p'-}) \int_\Lambda \left(|\zeta(z)|^{p'(z)} + |\vartheta(z)|^{\gamma_2(z)} + |{}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta(z)|^{\gamma_2(z)}\right) dz\\
&+ 1\\
\leq& c(|\omega|^{p'+} + |\omega|^{p'-}) \left(\sigma_{p'(z)}(\zeta) + \sigma_{\gamma_2(z)}(\vartheta) + \sigma_{\gamma_2(z)}({}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta)\right)\\
&+ 1\\
\leq& c\left(\|\zeta\|_{p'(z)}^{p'+} + \|\vartheta\|_{\gamma_2(z)}^{\gamma_2-} + \|\vartheta\|_{\gamma_2(z)}^{\gamma_2+} + \|{}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\|_{\gamma_2(z)}^{\gamma_2-} + \|{}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\|_{\gamma_2(z)}^{\gamma_2+}\right)\\
&+ 1, \quad c = conts.,
\end{aligned}$$

where $\gamma_2(z) = p'(z)(q(z) - 1) < p(z)$.

By the continuous embedding $L^{p(z)}(\Lambda) \hookrightarrow L^{\gamma_2(z)}(\Lambda)$ along with the Poincaré inequality, it follows that

$$\|\Phi_2\vartheta\|_{p'(z)} \leq c\left(\|\zeta\|_{p'(z)}^{p'+} + \|\vartheta\|^{\gamma_2-} + \|\vartheta\|^{\gamma_2+}\right) + 1, \quad c = const.$$

Consequently, the operator Φ_2 is bounded on $\mathbb{H}_0(\Lambda)$.

Now, let us demonstrate that Φ_2 is continuous. For any $\vartheta_n \to \vartheta$ in $\mathbb{H}_0(\Lambda)$, it follows that $\vartheta_n \to \vartheta$ in $L^{p(z)}(\Lambda)$ and ${}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta_n \to {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta$ in $(L^{p(z)}(\Lambda))^N$. Thus, there exist (ϑ_k) a subsequence of (ϑ_n) and $(\rho, \varrho) \in L^{p(z)}(\Lambda) \times (L^{p(z)}(\Lambda))^N$ a measurable functions that satisfy

$$\begin{aligned}
&\vartheta_k(z) \to \vartheta(z) \quad \text{and} \quad {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta_k(z) \to {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta(z),\\
&|\vartheta_k(z)| \leq \rho(z) \quad \text{and} \quad \left|{}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta_k(z)\right| \leq |\varrho(z)|,
\end{aligned}$$

for every $k \in \mathbb{N}$.

By $(\mathcal{A}_2)$, we have

$$F\left(z, \vartheta_k(z), {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta_k(z)\right) \to F\left(z, \vartheta(z), {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta(z)\right), \quad \text{a.e. } z \in \Lambda. \tag{3.5}$$

Thanks to $(\mathcal{A}_3)$ we have

$$\left|F\left(z, \vartheta_k(z), {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta_k(z)\right)\right| \leq \kappa\left(\zeta(z) + |\rho(z)|^{p(z)-1} + |\varrho(z)|^{p(z)-1}\right),$$

for every $k \in \mathbb{N}$ and for a.e. $z \in \Lambda$.

Note that
$$\zeta(z) + |\rho(z)|^{p(z)-1} + |\varrho(z)|^{p(z)-1} \in L^{p'(z)}(\Lambda),$$
and from (3.5), we deduce that
$$\int_\Lambda \left| F(z, \vartheta_k(z), {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta_k(z)) - F(z, \vartheta(z), {}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta(z)) \right|^{p'(z)} dz \to 0.$$
By means of the dominated convergence theorem, we find that
$$\Phi_2\vartheta_k \to \Phi_2\vartheta, \quad \text{in } L^{p'(z)}(\Lambda).$$
Hence, the entire sequence $(\Phi_2\vartheta_n)_n$ converges to $\Phi_2\vartheta$ in $L^{p'(z)}(\Lambda)$, then Φ_2 is continuous.

Given that the embedding $\mathsf{I} : \mathbb{H}_0(\Lambda) \to L^{p(z)}(\Lambda)$ is compact [2], then the adjoint operator $\mathsf{I}^* : L^{p'(z)}(\Lambda) \to (\mathbb{H}_0(\Lambda))^*$ is also compact. Thus, the composition $\mathsf{I}^* \circ \Phi_2 : \mathbb{H}_0(\Lambda) \to (\mathbb{H}_0(\Lambda))^*$ and $\mathsf{I}^* \circ \Phi_1 : \mathbb{H}_0(\Lambda) \to (\mathbb{H}_0(\Lambda))^*$ are compact. Therefore, $\mathfrak{S} = \mathsf{I}^* \circ \Phi_2 + \mathsf{I}^* \circ \Phi_1$ is compact. □

Theorem 3.1. *Suppose that* $(\mathcal{A}_1)$-$(\mathcal{A}_3)$ *hold. Then, the problem* (1.1) *has at least a weak solution in* $\mathbb{H}_0(\Lambda)$.

Proof. Let $\vartheta \in \mathbb{H}_0(\Lambda)$, then ϑ is a weak solution of (1.1) if and only if
$$\mathfrak{L}\vartheta = -\mathfrak{S}\vartheta. \tag{3.6}$$
By applying Lemma 3.1, we can conclude that $\mathcal{W} := \mathfrak{L}^{-1} : (\mathbb{H}_0(\Lambda))^* \to \mathbb{H}_0(\Lambda)$ is of type (S_+), bounded and continuous. Thus by Lemma 3.2, $\mathfrak{S}$ is bounded, continuous and quasimonotone. Thus, the equation (3.6) is equivalent to
$$\vartheta = \mathcal{W}u \quad \text{and} \quad u + \mathfrak{S}o\mathcal{W}u = 0. \tag{3.7}$$
Now, we can apply the Berkovits topological degree to solve the equation (3.7). To this purpose, we consider the set $\mathbb{B}$ defined by
$$\mathbb{B} := \left\{u \in (\mathbb{H}_0(\Lambda))^* \mid u + \varepsilon\mathfrak{S}o\mathcal{W}u = 0 \text{ for some } \varepsilon \in [0,1]\right\}.$$
Let $u \in \mathbb{B}$ and $\vartheta := \mathcal{W}u$. Then, $\|\mathcal{W}u\| = \|{}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\|_{p(z)}$.

If $\|{}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\|_{p(z)} \le 1$, it follows that $\|\mathcal{W}u\|$ is bounded.

If $\|{}^H\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\|_{p(z)} > 1$, then, by applying $(\mathcal{A}_3)$ with the Young inequality and the Hölder inequality, we get
$$\begin{aligned}
\|\mathcal{W}u\|^{p-} \le& \langle \mathfrak{L}\vartheta, \vartheta\rangle = \langle u, \mathcal{W}u\rangle = -\varepsilon\langle \mathfrak{S}o\mathcal{W}u, \mathcal{W}u\rangle \\
=&\varepsilon \int_\Lambda \left(-\tau|\vartheta|^{s(z)-2}\vartheta + \omega F\left(z, \vartheta(z), \mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta(z)\right)\right)\vartheta dz \\
&\le c\left(\int_\Lambda |\vartheta|^{s(z)}dz + \int_\Lambda |\zeta(z)\vartheta(z)|dz + \int_\Lambda |\vartheta|^{q(z)}dz \right.\\
&\left. + \int_\Lambda \left|\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\right|^{q(z)-1}|\vartheta|dz\right) \\
\le& c\left(\sigma_{s(z)}(\vartheta) + \int_\Lambda |\zeta(z)\vartheta(z)|dz + \sigma_{q(z)}(\vartheta) + \int_\Lambda \left|\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\right|^{q(z)-1}|\vartheta|dz\right)
\end{aligned}$$

$$\begin{aligned}
\leq & c\Big(\|\vartheta\|_{s(z)}^{s-} + \|\vartheta\|_{s(z)}^{s+} + \|\zeta\|_{p'(z)}\|\vartheta\|_{p(z)} + \|\vartheta\|_{q(z)}^{q-} + \|\vartheta\|_{q(z)}^{q+} \\
& + \frac{1}{q-}\sigma_{q(z)}(\vartheta) + \frac{1}{q'-}\sigma_{q(z)}\left(\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\right)\Big) \\
\leq & c\left(\|\vartheta\|_{s(z)}^{s-} + \|\vartheta\|_{s(z)}^{s+} + \|\vartheta\|_{p(z)} + \|\vartheta\|_{q(z)}^{q-} + \|\vartheta\|_{q(z)}^{q+} + \|\mathfrak{D}_{0+}^{\alpha,\beta;\xi}\vartheta\|_{q(z)}^{q+}\right),
\end{aligned}$$

where $c = const.$ By the continuous embedding $L^{p(z)}(\Lambda) \hookrightarrow L^{q(z)}(\Lambda)$ and $L^{p(z)}(\Lambda) \hookrightarrow L^{s(z)}(\Lambda)$, it follows

$$\|\mathcal{W}u\|^{p-} \leq c(\|\mathcal{W}u\|^{s+} + \|\mathcal{W}u\| + \|\mathcal{W}u\|^{q+}), \quad c = const.$$

Thus, $\{\mathcal{W}u \mid u \in \mathbb{B}\}$ is bounded.

Given that $\mathfrak{S}$ is bounded, it follows from (3.7) that the set $\mathbb{B}$ is bounded in $(\mathbb{H}_0(\Lambda))^*$. Therefore, there exists $\mathsf{R} > 0$ such that $\|u\|_{(\mathbb{H}_0(\Lambda))^*} < \mathsf{R}$, for every $u \in \mathbb{B}$. Thus,

$$u + \varepsilon\mathfrak{S}o\mathcal{W}u \neq 0, \quad \text{for every } u \in \partial\mathbb{B}_{\mathsf{R}}(0) \text{ and } \varepsilon \in [0,1].$$

Using Lemma 2.1, we obtain $\mathsf{I} + \mathfrak{S}o\mathcal{W} \in \mathsf{E}_{\mathcal{W}}\left(\overline{\mathbb{B}_{\mathsf{R}}(0)}\right)$ and $\mathsf{I} = \mathfrak{L}o\mathcal{W} \in \mathsf{E}_{\mathcal{W}}(\overline{\mathbb{B}_{\mathsf{R}}(0)})$. Now, we define the following homotopy $\Theta : [0,1] \times \overline{\mathbb{B}_{\mathsf{R}}(0)} \to (\mathbb{H}_0(\Lambda))^*$ given by

$$\Theta(\varepsilon, u) := u + \varepsilon\mathfrak{S}o\mathcal{W}u, \quad \text{for } (\varepsilon, u) \in [0,1] \times \overline{\mathbb{B}_{\mathsf{R}}(0)}.$$

By Theorem 2.2, we get $\mathcal{D}\mathsf{eg}\,(\mathsf{I} + \mathfrak{S}o\mathcal{W}, \mathbb{B}_{\mathsf{R}}(0), 0) = \mathcal{D}\mathsf{eg}\,(\mathsf{I}, \mathbb{B}_{\mathsf{R}}(0), 0) = 1$. Thus, there exists $u \in \mathbb{B}_{\mathsf{R}}(0)$ such that $u + \mathfrak{S}o\mathcal{W}u = 0$. It follows that $\vartheta = \mathcal{W}u$ is a weak solution of the main problem (1.1). □

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