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FRACTIONAL POWER SERIES METHODS FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this study, we develop a novel Modified Generalized Fractional Power Series (MGFPS) method for solving ordinary and fractional differential equations within the Caputo fractional derivative framework. This approach extends and refines classical and fractional power series techniques, overcoming their inherent limitations through dynamic exponent adjustment based on the equation characteristics. The proposed framework is supported by rigorous analytical foundations ensuring enhanced accuracy and flexibility in solutions. Applications to linear fractional-order equations demonstrate superior precision and computational efficiency compared to existing techniques, providing a robust tool bridging classical and fractional calculus.

1. Introduction

The power series method was first introduced by Newton in his work [7] for solving differential equations. The extension of the power series method to the so-called fractional calculus is presented in the work [3]. Specifically, the authors proposed solutions of fractional differential equations expressed in power series of the form

$$\sum_{n=0}^{+\infty} \gamma_n (t-t_0)^{n\nu} = \gamma_0 + \gamma_1 (t-t_0)^{\nu} + \gamma_2 (t-t_0)^{2\nu} + \cdots ,$$

where $0 \leq m-1 < \nu \leq m$ and $t \geq t_0$, which is referred to as an FPS centered at t_0, with t as a variable and γ_n as constants known as the coefficients of the series.

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In the same spirit, many authors have generalized the form of the last series, including the so-called generalized fractional power series [2] of the following form:

(1.1)
$$\sum_{n\geq 0, m\geq 0}^{+\infty} \gamma_{(n,m)} t^{n\nu+m} = \gamma_{(0,0)} + \gamma_{(0,1)} t + \gamma_{(1,0)} t^{\nu} + \gamma_{(0,2)} t^2 + \gamma_{(1,1)} t^{\nu+1} + \gamma_{(2,0)} t^{2\nu} + \cdots ,$$

where $n, m \in \mathbb{N}$, $t \geq 0$ and $\gamma_{(n,m)}$ are the series coefficients. Comprehensive treatments of fractional power series theory and applications can be found in [1–6].

We note, however, that the previous forms of power series are not suitable for a large class of concrete problems, as several examples in the book [8] show. For example, the following fractional differential equation cannot simply be solved using the latest approaches:

$$\begin{aligned} {}^{C}D_0^{\nu} y(t) &= \rho t^{\mu} y(t), \quad \rho \neq 0, \\ y^{(k)}(0) &= c_k, \quad c_k \in \mathbb{R}, \quad k = 0, 1, \ldots, j-1, \end{aligned} \tag{1.2}$$

where $j - 1 < \nu \leq j$, $j \in \mathbb{N}^*$ and $\mu \in \mathbb{R}^+$, with $\mu \notin \mathbb{N}$.

Motivated by this remark, we propose the following form of the power series:

$$y(t) = \sum_{n_0=0}^{+\infty} \sum_{n_1=0}^{+\infty} \cdots \sum_{n_m=0}^{+\infty} \gamma(n_0, \ldots, n_m) t^{\sum_{k=0}^{m} n_k \nu_k}, \quad \nu_k \in \mathbb{R}^+,$$

where $n_0, \ldots, n_m \in \mathbb{N}$, $t \geq 0$ and $\gamma(n_0, \ldots, n_m)$ are the series coefficients.

This form will be used to find solutions for several examples of fractional differential equations, in particular for the significant problem in (1.2).

The rest of this work is organized as follows. In Section 2, we give an overview of the main definitions and properties of fractional derivatives. In Section 3, we give further details and explanations of our method and examine some important examples known in the literature to shed light on our present method. In the last section, we provide an overview of the main results and advances presented in this paper.

2. Definitions and Properties

In this section, we present the essential definitions and properties that will be referenced in the subsequent discussions [8].

Definition 2.1. The Riemann-Liouville integral of order ν for $\nu \in \mathbb{R}^+$ and $t \geq 0$ is given by

$$(\mathcal{J}_{0^+}^{\nu} y)(t) = \frac{1}{\Gamma(\nu)} \int_0^t \frac{y(\xi)}{(t-\xi)^{1-\nu}} \, d\xi.$$

The Euler gamma function $\Gamma(\cdot)$ is defined as:

$$\Gamma(\epsilon) = \int_0^{+\infty} t^{\epsilon-1} e^{-t} \, dt, \quad \mathrm{Re}(\epsilon) > 0.$$

Definition 2.2. For $n-1<\nu\leq n$ and $t\geq 0$, the Riemann-Liouville derivative of y of order ν is defined by

$$^{RL}D^{\nu}_{a^{+}}y(t)=\frac{1}{\Gamma(n-\nu)}\cdot\frac{d^{n}}{dt^{n}}\int_{a}^{t}\frac{y(\xi)}{(t-\xi)^{\nu-n+1}}d\xi.$$

Definition 2.3. For $n-1<\nu\leq n$ and $t\geq 0$, the Caputo derivative of y of order ν is

$$^{C}D^{\nu}_{a}y(t)=\frac{1}{\Gamma(n-\nu)}\int_{a}^{t}\frac{y^{(n)}(\xi)}{(t-\xi)^{\nu-n+1}}d\xi. \tag{2.1}$$

Property 2.1. If $\nu\in\mathbb{R}^{+}$ and $\mu>0$, then

$$\left(^{C}D^{\nu}_{0}t^{\mu-1}\right)(\xi)=\frac{\Gamma(\mu)}{\Gamma(\mu-\nu)}\xi^{\mu-\nu-1}.$$

In particular, if $\mu=1$ and $\nu\in\mathbb{R}^{+}$, the Caputo fractional derivatives of a constant are generally zero:

$$\left(^{C}D^{\nu}_{0}1\right)(\xi)=0.$$

Moreover, for $j=0,1,\dots,[\nu]$,

$$\left(^{C}D^{\nu}_{0}t^{j}\right)(\xi)=0.$$

Definition 2.4. For $\nu\in\mathbb{R}^{+}$ and $\xi\in\mathbb{R}$, the Mittag-Leffler function is defined as

$$E_{\nu}(\xi)=\sum_{n=0}^{+\infty}\frac{\xi^{n}}{\Gamma(\nu n+1)}.$$

Definition 2.5. For $\nu,\mu\in\mathbb{R}^{+}$ and $\xi\in\mathbb{R}$, the generalized Mittag-Leffler function is defined as

$$E_{(\nu,\mu)}(\xi)=\sum_{n=0}^{+\infty}\frac{\xi^{n}}{\Gamma(\nu n+\mu)}.$$

Definition 2.6. For ν,μ and $\gamma\in\mathbb{R}^{+}$ and $\xi\in\mathbb{C}$, the Mittag-Leffler function of three indices is defined as

$$E_{(\nu,\mu,\gamma)}(\xi)=1+\sum_{k=1}^{+\infty}\left[\prod_{i=0}^{k-1}\frac{\Gamma[\nu(i\mu+\gamma)+1]}{\Gamma[\nu(i\mu+\gamma+1)+1]}\right]\xi^{k}.$$

Definition 2.7. If $\nu,\mu\in\mathbb{R}^{+}$ and $\xi\in\mathbb{R}$, the Wright function is defined as

$$W_{(\nu,\mu)}(\xi):=\sum_{n=0}^{+\infty}\frac{\xi^{n}}{n!\Gamma(\nu n+\mu)}.$$

3. Methods, Theory and Calculations

In this section, we present a new general form of generalized fractional power series designed to address both ordinary and fractional differential equations within the Caputo framework.

Definition 3.1. The new general form of fractional power series, defined by

$$y(t) = \sum_{n_0=0}^{+\infty} \sum_{n_1=0}^{+\infty} \cdots \sum_{n_m=0}^{+\infty} \gamma(n_0, \dots, n_m) t^{\sum_{k=0}^{m} n_k \nu_k}, \quad \nu_k \in \mathbb{R}^+, \tag{3.1}$$

where $m \in \mathbb{N}$ and $t \geq 0$ serves as an indeterminate variable, while $\gamma(n_0, \dots, n_m)$ represents the coefficients of the series.

For simplicity, we assume that the center of the generalized fractional power series is at zero, which is always possible by performing a linear transformation on the variable, namely $(t - t_0) \mapsto t$.

Definition 3.2. If $\nu_k \in \mathbb{R}^+$, then a Cauchy product fractional power series for a real function $y(t)$ is an infinite series of the form

$$\begin{aligned} y(t) &= \sum_{n_0=0}^{+\infty} \sum_{n_1=0}^{+\infty} \cdots \sum_{n_m=0}^{+\infty} \gamma(n_0, \dots, n_m) t^{\sum_{k=0}^{m} n_k \nu_k} \\ &= \left(\sum_{n_0=0}^{+\infty} a_{n_0} t^{n_0 \nu_0} \right) \left(\sum_{n_1=0}^{+\infty} a_{n_1} t^{n_1 \nu_1} \right) \cdots \left(\sum_{n_k=0}^{+\infty} a_{n_m} t^{n_m \nu_m} \right), \end{aligned}$$

where $\gamma(n_0, \dots, n_m) = a_{n_0} a_{n_1} \cdots a_{n_m}$ for all $n_0, n_1, \dots, n_m$.

Proposition 3.1 ([2])**.** *If $\sum_{i=0}^{+\infty} a_i t^{i\nu}$ converges for some $t = c > 0$, then it converges absolutely for $t \in (0, c)$.*

Corollary 3.1. *If $\sum_{k=0}^{+\infty} d_k t^k$ converges for some $t = d > 0$, then it converges absolutely for $t \in (0, d)$.*

To derive a fractional power series solution characterized by $m+1$ fractional indices $\nu_k \in \mathbb{R}^+ \setminus \mathbb{N}$ for each $k \in \{0, \dots, m\}$, the method determines the number of indices in the exponent based on the varying orders of fractional derivatives. This approach also accommodates cases where terms in the equation involve powers of t raised to the exponent ν_k . Consequently, for a fractional derivative of order ν_k, the term $n_k \nu_k$ appears in the exponent, as well as when any term is multiplied by t^{ν_k} . In the case of classical derivatives, we include $n_k \nu_k$ in the exponent, where $\nu_k = 1$.

To demonstrate the application of the fundamental method and highlight its key properties, we present several examples. Specifically, we will use this method to solve the following fractional differential equation.

Example 3.1. The initial value problem for the fractional differential equation is defined by

$$\begin{aligned} {}^{C}D_0^{\nu} y(t) + \lambda y(t) &= 0, \\ y(0) &= y_0, \end{aligned}$$

where $0 < \nu \leq 1$, has a solution given by

$$y(t) = y_0 E_{\nu}\left(-\lambda t^{\nu}\right).$$

Consider the following expression for a fractional power series:

$$y(t) = \sum_{n\geq 0}^{+\infty} \gamma_n t^{n\nu}.$$

The selection of $i\nu$ as the exponent is based on the fractional derivative ${}^{C}D_0^{\nu}$. The fractional derivative can be applied term by term to represent

$${}^{C}D_0^{\nu} y(t) = \sum_{n\geq 1}^{+\infty} \gamma_n \frac{\Gamma(n\nu+1)}{\Gamma((n-1)\nu+1)} t^{(n-1)\nu}.$$

By re-indexing the summation, we obtain the balance equations

$$\sum_{n\geq 1}^{+\infty} \gamma_n \frac{\Gamma(n\nu+1)}{\Gamma((n-1)\nu+1)} t^{(n-1)\nu} + \lambda \sum_{n\geq 1}^{+\infty} \gamma_{n-1} t^{(n-1)\nu} = 0.$$

We can immediately deduce that

$$\gamma_n = -\lambda \frac{\Gamma((n-1)\nu+1)}{\Gamma(n\nu+1)} \gamma_{n-1}, \quad n \geq 1.$$

Equivalently we can write

$$\gamma_n = \gamma_0 \frac{(-\lambda)^n}{\Gamma(n\nu+1)}, \quad \text{with } \gamma_0 = y(0).$$

We can now express the solution as

$$y(t) = y_0 \sum_{n=0}^{+\infty} \frac{(-\lambda)^n t^{n\nu}}{\Gamma(n\nu+1)} = y_0 E_{\nu}\left(-\lambda t^{\nu}\right). \tag{3.2}$$

The plot shows the behavior of the function $y(t)$ for different values of ν. The parameter $\lambda = 1$ and the coefficient $y_0 = 1$. The curves are plotted for $\nu \in \{0.6, 0.7, 0.8, 0.9, 1\}$, presenting the following relaxation behavior (see Figure 1).

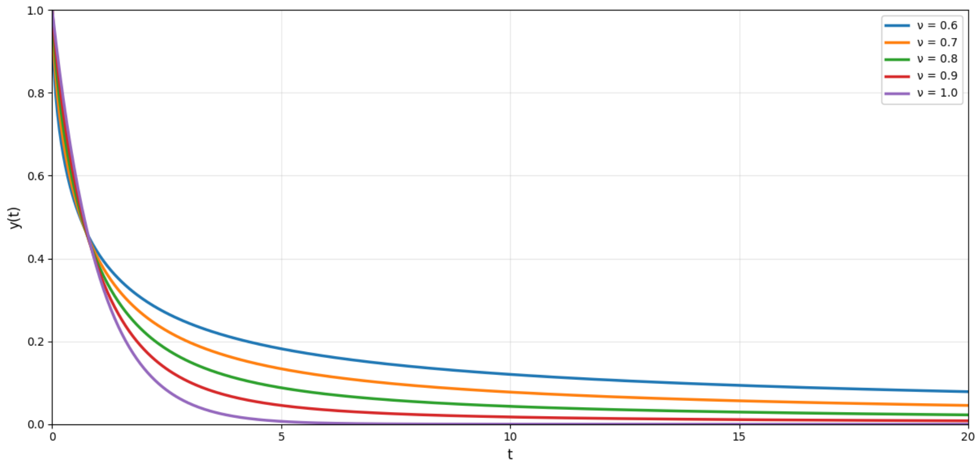

FIGURE 1. Plot of $y(t)$ for different values of $0 < \nu \le 1$ in the time range $0 \le t \le 20$

Let us move on to another example. This one concerns the fractional oscillation equation, a subject that has been thoroughly examined by various authors (see [13,16]) using the Laplace transform for its solution. Here, we will address it using a generalized fractional power series.

Lemma 3.1 ([8])**.** *Let $n-1 < \nu \le n$, $n \in \mathbb{N}^*$. The equality $\left({}^cD_0^\nu y\right)(t) = 0$ is valid if and only if*

$$y(t) = \sum_{j=0}^{n-1} c_j t^j,$$

where $c_j \in \mathbb{R}$.

Example 3.2. The initial value problem associated with the fractional differential equation is defined as follows:

$$\begin{aligned} {}^CD_0^\nu y(t) + \lambda y(t) &= 0, \\ y(0) &= c_0, \\ y'(0) &= c_1, \end{aligned}$$

where $1 < \nu \le 2$ and $c_0, c_1 \in \mathbb{R}$. The solution to this equation can be expressed as $y(t) = c_0 E_{(\nu,1)}\left(-\lambda t^\nu\right) + c_1 t E_{(\nu,2)}\left(-\lambda t^\nu\right)$.

Proof. The generalized fractional power series representation of $y(t)$ is expressed as an infinite series:

$$y(t) = \sum_{n\ge 0, m\ge 0}^{+\infty} \gamma_{(n,m)} t^{n\nu+m}. \tag{3.3}$$

The selection of $n\nu$ as the exponent in (3.3) is dictated by the fractional derivative ${}^{C}D_0^{\nu}$, while the integer m associated with the initial value stems from the classical derivative. Applying the fractional derivative term by term, by using Lemma 3.1 we obtain:

$$
\begin{aligned}
{}^{C}D_0^{\nu}y(t) = &\sum_{m\geq 2}^{+\infty}\gamma_{(0,m)}\frac{\Gamma(m+1)}{\Gamma(-\nu+m+1)}t^{-\nu+m} \\
&+\sum_{n\geq 1, m\geq 0}^{+\infty}\gamma_{(n,m)}\frac{\Gamma(n\nu+m+1)}{\Gamma((n-1)\nu+m+1)}t^{(n-1)\nu+m}.
\end{aligned}
$$

We have also

$$
\lambda y(t) = \lambda\sum_{n\geq 1, m\geq 0}^{+\infty}\gamma_{(n-1,m)}t^{(n-1)\nu+m}.
$$

The balance equations are obtained by re-indexing the summation

$$
\begin{aligned}
&\sum_{m\geq 2}^{+\infty}\gamma_{(0,m)}\frac{\Gamma(m+1)}{\Gamma(-\nu+m+1)}t^{-\nu+m}+\sum_{n\geq 1, m\geq 0}^{+\infty}\gamma_{(n,m)}\frac{\Gamma(n\nu+m+1)}{\Gamma((n-1)\nu+m+1)}t^{(n-1)\nu+m} \\
&-\lambda\sum_{n\geq 1, m\geq 0}^{+\infty}\gamma_{(n-1,m)}t^{(n-1)\nu+m}=0.
\end{aligned}
$$

We can immediately deduce that

$$
\gamma_{(0,m)} = 0, \quad m\geq 2, \tag{3.4}
$$

$$
\gamma_{(n,m)} = -\lambda\frac{\Gamma((n-1)\nu+m+1)}{\Gamma(n\nu+m+1)}\gamma_{(n-1,m)}, \quad n\geq 1, m\geq 0. \tag{3.5}
$$

It can be deduced from (3.4) and through successive applications of (3.5) that $\gamma_{(n,m)} = 0$ for all $n\geq 0$ and $m\geq 2$. As a result, the only non-zero coefficients are those where $m<2$, yielding a solution of the form:

$$
y(t) = \sum_{m=0}^{1}\sum_{n=0}^{+\infty}\gamma_{(n,m)}t^{n\nu+m} = \sum_{m=0}^{1}t^m\sum_{n=0}^{+\infty}\gamma_{(n,m)}t^{n\nu}.
$$

Then, from (3.5), we have

$$
\gamma_{(n,m)} = -\lambda\frac{\Gamma((n-1)\nu+m+1)}{\Gamma(n\nu+m+1)}\gamma_{(n-1,m)}, \quad n\geq 1, m\leq 1.
$$

Equivalently, we can write

$$
\gamma_{(n,m)} = (-1)^n\frac{\lambda^n\Gamma(m+1)}{\Gamma(n\nu+m+1)}\gamma_{(0,m)}, \quad n\geq 1, m\leq 1, \tag{3.6}
$$

where $\gamma_{(0,m)} = \frac{y^{(m)}(0)}{m!}$ and from (3.6) we have

$$
\gamma_{(n,m)} = (-1)^n\frac{\lambda^n}{\Gamma(n\nu+m+1)}y^{(m)}(0), \quad n\geq 1, m\leq 1.
$$

The solution can now be expressed as:

$$y(t) = c_0 E_{(\nu,1)}\left(-\lambda t^{\nu}\right) + c_1 t E_{(\nu,2)}\left(-\lambda t^{\nu}\right). \qquad \square$$

The plot shows the behavior of the function $y(t)$ for different values of α. The parameter $\lambda = 1$, and the coefficients c_0 and c_1 are chosen as 1 and 0.25, respectively. The curves are plotted for $\nu \in \{1.5, 1.6, 1.7, 1.8, 1.9, 2\}$, presenting the following oscillation behavior (see Figure 2).

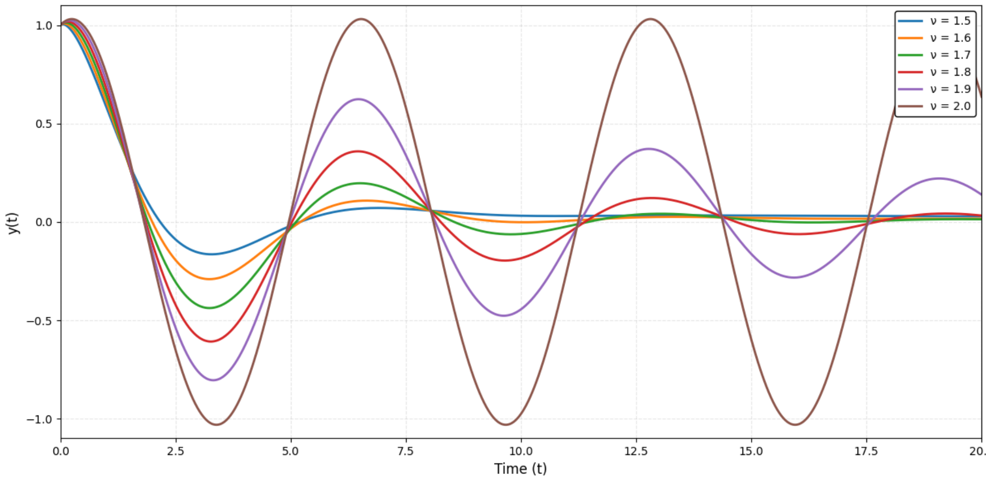

FIGURE 2. Plot of $y(t)$ for different values of alpha $1 < \nu \leq 2$ in the time range $0 \leq t \leq 20$

One of the contributions of this paper lies in the utilization of a generalized fractional power series to address a specific class of classical differential equations. To illustrate this approach, we consider the following equation.

Proposition 3.2. *Let $l \in \mathbb{N}$ and $\mu \in \mathbb{R}^+$. Consider the initial value problem defined by the differential equation*

$$y^{(l)}(t) = \rho t^{\mu} y(t), \quad 0 \leq t < +\infty, \rho \in \mathbb{R}^+, \tag{3.7}$$

$$y^{(k-1)}(0) = c_k, \quad k = 1, 2, \ldots, l, \tag{3.8}$$

where $c_k \in \mathbb{R}$. The solution to this problem can be written as:

$$y(t) = \sum_{k=1}^{l} \frac{c_k}{(k-1)!} t^{k-1} E_{\left(l, 1+\frac{\mu}{l}, \frac{\mu+k-1}{l}\right)}\left(\rho t^{\mu+l}\right).$$

Proof. Consider the expansion of the fractional power series for $y(t)$:

$$y(t) = \sum_{n \geq 0, m \geq 0}^{+\infty} \gamma_{(n,m)} t^{n\mu+m}. \tag{3.9}$$

The selection of $n\mu$ and m in (3.9) is influenced by the monomial t^{μ} in (3.9) and the classical derivative as presented in (3.8) then we have:

$$t^{\mu}y(t) = \sum_{n\geq 0, m\geq 0}^{+\infty} \gamma_{(n,m)} t^{(n+1)\mu+m} = \sum_{n\geq 1, m\geq l}^{+\infty} \gamma_{(n-1,m-l)} t^{n\mu+m-l}. \tag{3.10}$$

Additionally, we have

$$y^{(l)}(t) = \sum_{n\geq 0, m\geq 0}^{+\infty} \gamma_{(n,m)} (n\mu+m)(n\mu+m-1)\cdots(n\mu-l+1) t^{n\mu+m-l}.$$

$$\begin{aligned} y^{(l)}(t) = &\sum_{n\geq 1, m\geq l}^{+\infty} \gamma_{(n,m)} \frac{\Gamma(n\mu+m+1)}{\Gamma(n\mu+m-l+1)} t^{n\mu+m-l} + \sum_{m\geq l}^{+\infty} \gamma_{(0,m)} \frac{\Gamma(m+1)}{\Gamma(m-l+1)} t^{m-l} \\ &+ \sum_{m=0}^{l-1} \sum_{n\geq 1}^{+\infty} \gamma_{(n,m)} \frac{\Gamma(n\mu+m+1)}{\Gamma(n\mu+m-l+1)} t^{n\mu+m-l}. \end{aligned} \tag{3.11}$$

We then substitute (3.10) and (3.11) into (3.7) and adjust the summation indices to obtain the balance equations:

$$\sum_{n\geq 0, m\geq 0}^{+\infty} \gamma_{(n,m)} \frac{\Gamma(n\mu+m+1)}{\Gamma(n\mu+m-l+1)} t^{n\mu+m-l} - \rho \sum_{n\geq 1, m\geq l}^{+\infty} \gamma_{(n-1,m-l)} t^{n\mu+m-l} = 0.$$

It can be concluded that

$$\gamma_{(0,m)} = 0, \quad m \geq l, \tag{3.12}$$

$$\gamma_{(n,m)} = 0, \quad n \geq 1, m \leq l-1 \tag{3.13}$$

$$\gamma_{(n,m)} = \rho \frac{\Gamma(n\mu+m-l+1)}{\Gamma(n\mu+m+1)} \gamma_{(n-1,m-l)}, \quad n \geq 1, m \geq l. \tag{3.14}$$

It can be deduced from (3.12) and the recursive applications of (3.14) that $\gamma_{(n,m)} = 0$ for all $n \geq 1$ and $m > nl + l - 1$. Similarly, from (3.13) in conjunction with (3.14) it follows that $\gamma_{(n,m)} = 0$ for all $m \geq 1$ and $m < nl$. Therefore, the only non-zero coefficients that remain are those for which $nl \leq m \leq nl + l - 1$. Consequently, we can express the solution in the following form:

$$y(t) = \sum_{k=0}^{l-1} \sum_{n=0}^{+\infty} \gamma_{(n,nl+k)} t^{n\mu+nl+k} = \sum_{k=0}^{l-1} t^k \sum_{n=0}^{+\infty} \gamma_{(n,nl+k)} t^{n(\mu+l)},$$

where $\phi(n,k) = \gamma_{(n,nl+k)}$ is derived from the recurrence relation (3.14). Then,

$$\phi(n,k) = \rho \frac{\Gamma(n\mu+nl+k-l+1)}{\Gamma(n\mu+nl+k+1)} \phi(n-1,k), \quad n \geq 1, k \leq l-1.$$

This can be represented in closed form, resulting in

$$\phi(n,k) = \prod_{j=0}^{n-1} \rho \frac{\Gamma((j+1)(\mu+l)+k-l+1)}{\Gamma((j+1)(\mu+l)+k+1)} \phi(0,k), \quad n \geq 1, k \leq l-1.$$

Alternatively, this can be represented as

$$\phi(n,k)=\prod_{j=0}^{n-1}\rho\frac{\Gamma(l(j(\frac{\mu}{l}+1)+\frac{\mu}{l}+\frac{k}{l})+1)}{\Gamma(l(j(\frac{\mu}{l}+1)+\frac{\mu}{l}+1+\frac{k}{l})+1)}\phi(0,k),\quad n\geq 1, k\leq l-1,$$

where $\phi(0,k)=\gamma_{(0,k)}=\frac{y^{(k)}(0)}{k!}$. We can now express the solution in the form:

(3.15)

$$y(t)=\sum_{k=0}^{l-1}\frac{y^{(k)}(0)}{k!}t^kE_{(l,1+\frac{\mu}{l},\frac{\mu+k}{l})}\left(\rho t^{\mu+l}\right)=\sum_{k=1}^{l}\frac{c_k}{(k-1)!}t^{k-1}E_{(l,1+\frac{\mu}{l},\frac{\mu+k-1}{l})}\left(\rho t^{\mu+l}\right).$$

□

To further demonstrate our method and highlight the limitations of the generalized fractional power series in solving such equations, let us consider another example. In this study, we will apply our modified power series approach to solve it.

Proposition 3.3. *Let $\mu\in\mathbb{R}^+$. The fractional differential equation, defined as an initial value problem, is given by*

$$\begin{aligned}{}^{C}D_0^{\nu}y(t)&=\rho t^{\mu}y(t),\quad 0\leq t<+\infty,\rho\in\mathbb{R}^+,\\ y(0)&=y_0,\end{aligned}\tag{3.16}$$

where $0<\nu<1$. This problem admits a modified generalized power series solution expressed as $y(t)=y_0E_{(\nu,1+\frac{\mu}{\nu},\frac{\mu}{\nu})}\left(\rho t^{\nu+\mu}\right)$.

Proof. Consider the following modified generalized power series:

$$y(t)=\sum_{n\geq 0,m\geq 0}^{+\infty}\gamma_{(n,m)}t^{n\nu+m\mu}.$$

The selection of the exponent $n\nu+m\mu$ is determined by the monomial t^{μ} and the fractional derivative ${}^{C}D_0^{\nu}$. Applying the fractional derivative term by term, we obtain the following expression:

$${}^{C}D_0^{\nu}y(t)=\sum_{\substack{n+m\geq 1\\ n\geq 0,m\geq 0}}^{+\infty}\gamma_{(n,m)}\frac{\Gamma(n\nu+m\mu+1)}{\Gamma((n-1)\nu+m\mu+1)}t^{(n-1)\nu+m\mu}.\tag{3.17}$$

We also have an alternative expression

$$t^{\mu}y(t)=\sum_{n\geq 0,m\geq 0}^{+\infty}\gamma_{(n,m)}t^{n\nu+(m+1)\mu}=\sum_{n\geq 1,m\geq 1}^{+\infty}\gamma_{(n-1,m-1)}t^{(n-1)\nu+m\mu}.\tag{3.18}$$

Next, we substitute (3.17) and (3.18) into (3.16) and adjust the summation indices to derive the balance equations

$$\sum_{\substack{n+m\geq 1\\ n\geq 0,m\geq 0}}^{+\infty}\gamma_{(n,m)}\frac{\Gamma(n\nu+m\mu+1)}{\Gamma((n-1)\nu+m\mu+1)}t^{(n-1)\nu+m\mu}-\rho\sum_{n\geq 1,m\geq 1}^{+\infty}\gamma_{(n-1,m-1)}t^{(n-1)\nu+m\mu}=0.$$

It follows directly that

$$\gamma_{(0,m)} = 0, \quad m \geq 1, \tag{3.19}$$

$$\gamma_{(n,0)} = 0, \quad n \geq 1, \tag{3.20}$$

$$\gamma_{(n,m)} = \lambda \frac{\Gamma((n-1)\nu + m\mu + 1)}{\Gamma(n\nu + m\mu + 1)} \gamma_{(n-1,m-1)}, \quad n \geq 1, m \geq 1. \tag{3.21}$$

From (3.19) and by recursively applying (3.21), it can be established that $\gamma_{(n,m)} = 0$ for $n \geq 1$ and $m > n$. Similarly, from (3.20) in conjunction with (3.21), we conclude that $\gamma_{(n,m)} = 0$ for $m \geq 1$ and $m < n$. Consequently, the only non-zero coefficients correspond to the case where $m = n$, resulting in the solution of the form:

$$y(t) = \sum_{n=0}^{+\infty} \gamma(n,n) t^{n\nu + n\mu} = \sum_{n=0}^{+\infty} \gamma(n,n) t^{n(\nu+\mu)},$$

where $\psi(n) = \gamma_{(n,n)}$ is obtained from the recurrence relation (3.21) by setting $m = n$, leading to a solution of the form

$$\psi(n) = \rho \frac{\Gamma((n-1)\nu + n\mu + 1)}{\Gamma(n\nu + n\mu + 1)} \psi(n-1), \quad n \geq 1.$$

This can be expressed as

$$\psi(n) = \prod_{j=0}^{n-1} \rho \frac{\Gamma(j(\nu+\mu) + \mu + 1)}{\Gamma(j(\nu+\mu) + \nu + \mu + 1)} \psi(0), \quad n \geq 1.$$

Thus, we can deduce that

$$\psi(n) = \prod_{j=0}^{n-1} \rho \frac{\Gamma(\nu(j(\frac{\mu}{\nu}+1) + \frac{\mu}{\nu}) + 1)}{\Gamma(\nu(j(\frac{\mu}{\nu}+1) + \frac{\mu}{\nu} + 1) + 1)} \psi(0), \quad n \geq 1,$$

where $\psi(0) = \gamma_{(0,0)} = y(0)$. The solution can be expressed in the following expression:

$$y(t) = y_0 E_{(\nu, 1+\frac{\mu}{\nu}, \frac{\mu}{\nu})} \left(\rho t^{\nu+\mu} \right). \qquad \square$$

The plot shows the behavior of the function $y(t)$ for different values of ν. The parameter $\rho = 1$, and the coefficient $y_0 = 1$. The curves are plotted for $\mu \in \{0, -0.3, -0.5, -0.6\}$ and $\nu = 1$, presenting the following relaxation behavior (see Figure 3).

To enhance the applicability of our method in various situations, we propose a generalized equation that extends equations (3.7) and (3.16) as follows.

Theorem 3.1. *Let $j - 1 < \nu \leq j$, where $j \in \mathbb{N}^*$ and $\mu \in \mathbb{R}^+$. Consider the initial value problem given by*

$$^{C}D_0^{\nu} y(t) = \rho t^{\mu} y(t), \quad \rho \neq 0, \tag{3.22}$$

$$y^{(k)}(0) = c_k, \quad c_k \in \mathbb{R}, k = 0, 1, \ldots, j-1. \tag{3.23}$$

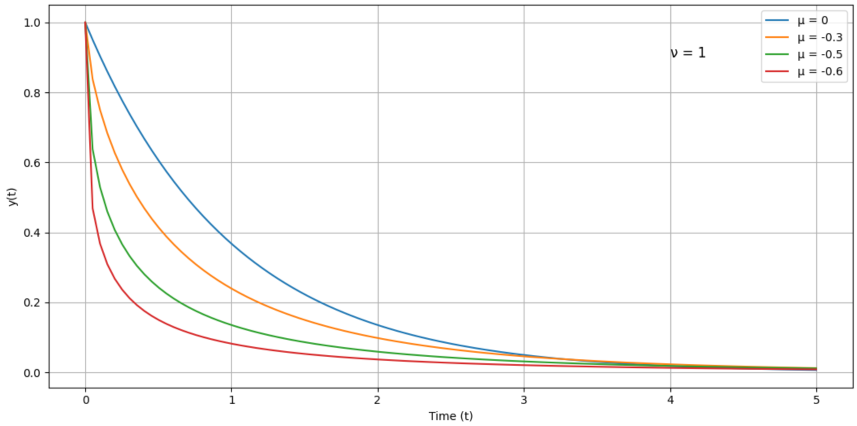

FIGURE 3. Plot of $y(t)$ for different values of mu in the time range $0 \leq t \leq 5$

Therefore, the solution to this problem is expressed as:

$$y(t) = \sum_{k=0}^{j-1} \frac{c_k}{k!} t^k E_{\left(\nu, 1+\frac{\mu}{\nu}, \frac{\mu+k}{\nu}\right)} \left(\rho t^{\mu+\nu}\right).$$

Proof. We assume that the solution to this equation takes the form:

$$y(t) = \sum_{n,m,k \geqslant 0}^{+\infty} \gamma_{(n,m,k)} t^{n\nu + m\mu + k}. \tag{3.24}$$

The selection of $n\nu + m\mu + k$ in (3.24) raised to a power is contingent upon the monomial t^μ and the fractional derivative ${}^C D_0^\nu$ in the equation (3.22). Moreover, the integer k see (3.23) for the initial value is determined by the classical derivative. We can apply the fractional derivative term by term to represent:

$$\begin{aligned} {}^C D_0^\nu \left(\sum_{n,m,k \geq 0}^{+\infty} \gamma_{(n,m,k)} t^{n\nu + m\mu + k} \right) &= \sum_{k \geq j}^{+\infty} \gamma_{(0,0,k)} \frac{\Gamma(k+1)}{\Gamma(-\nu + k + 1)} t^{-\nu + k} \\ &+ \sum_{\substack{n,m,k \geq 0 \\ n+m \geq 1}}^{+\infty} \gamma_{(n,m,k)} \frac{\Gamma(n\nu + m\mu + k)}{\Gamma((n-1)\nu + m\mu + k)} t^{(n-1)\nu + m\mu + k}. \end{aligned} \tag{3.25}$$

We additionally have

$$t^\mu y(t) = \sum_{n,m,k \geq 0}^{+\infty} \gamma_{(n,m,k)} t^{n\nu + (m+1)\mu + k} = \sum_{k \geq 0}^{+\infty} \sum_{n \geq 1, m \geq 1}^{+\infty} \gamma_{(n-1,m-1,k)} t^{(n-1)\nu + m\mu + k}. \tag{3.26}$$

We subsequently substitute equations (3.25) and (3.26) into (3.22) and adjust the summation indices, leading us to the balance equations:

$$\sum_{k\geq j}^{+\infty}\gamma_{(0,0,k)}\frac{\Gamma(k+1)}{\Gamma(-\nu+k+1)}t^{-\nu+k}+\sum_{\substack{n,m,k\geq 0\\ n+m\geq 1}}^{+\infty}\gamma_{(n,m,k)}\frac{\Gamma(n\nu+m\mu+k)}{\Gamma((n-1)\nu+m\mu+k)}t^{(n-1)\nu+m\mu+k}$$

(3.27)

$$-\rho\sum_{k\geq 0}^{+\infty}\sum_{n\geq 1,m\geq 1}^{+\infty}\gamma_{(n-1,m-1,k)}t^{(n-1)\nu+m\mu+k}=0.$$

Immediate deduction yields

(3.28) $$\gamma_{(0,0,k)}=0,\quad k\geq j,$$

(3.29) $$\gamma_{(0,m,k)}=0,\quad m\geq 1,k\geq 0,$$

(3.30) $$\gamma_{(n,0,k)}=0,\quad n\geq 1,k\geq 0,$$

(3.31) $$\gamma_{(n,m,k)}=\rho\frac{\Gamma((n-1)\nu+m\mu+k+1)}{\Gamma(n\nu+m\mu+k+1)}\gamma_{(n-1,m-1,k)},\quad n\geq 1,m\geq 1,k\geq 0.$$

From (3.28) and the iterative application of (3.31), it can be deduced that $\gamma_{(n,m,k)}=0$ for all $n\geq 0$, $m\geq 0$ and $k\geq j$. Therefore, the only non-zero coefficients correspond to the case where $k<j$.

From (3.29) and recursively from (3.31), $\gamma_{(n,m,k)}=0$ for $n\geq 1$, $k\geq 0$ and $m>n$. Similarly, from (3.30) and (3.31), $\gamma_{(n,m,k)}=0$ for $m\geq 1$, $k\geq 0$ and $m<n$. As a result, the only non-zero coefficients occur when $m=n$, leading to a solution of the form:

$$y(t)=\sum_{k=0}^{j-1}\sum_{n\geq 0}^{+\infty}\gamma_{(n,n,k)}t^{n\nu+n\mu+k}=\sum_{k=0}^{j-1}t^k\sum_{n\geq 0}^{+\infty}\gamma_{(n,n,k)}t^{n(\nu+\mu)},$$

where $\varphi(n,k)=\gamma_{(n,n,k)}$ are found from the recurrence relation (3.31), with $m=n$, and we then have

$$\varphi(n,k)=\rho\frac{\Gamma((n-1)\nu+n\mu+k+1)}{\Gamma(n\nu+n\mu+k+1)}\varphi(n-1,k),\quad n\geq 1,k\leq j-1.$$

This can be written as

$$\varphi(n,k)=\prod_{i=0}^{n-1}\rho\frac{\Gamma(i(\nu+\mu)+\mu+k+1)}{\Gamma(i(\nu+\mu)+\nu+\mu+k+1)}\varphi(0,k),\quad n\geq 1,k\leq j-1.$$

Equivalently, we deduce that

$$\varphi(n,k)=\prod_{i=0}^{n-1}\rho\frac{\Gamma(\nu(i(\frac{\mu}{\nu}+1)+\frac{\mu+k}{\nu})+1)}{\Gamma(\nu(i(\frac{\mu}{\nu}+1)+\frac{\mu+k}{\nu}+1)+1)}\varphi(0,k),\quad n\geq 1,k\leq j-1.$$

Given that $\varphi(0,k)=\gamma_{(0,0,k)}=\frac{y^{(k)}(0)}{k!}$ the series solution can now be expressed as:

$$y(t)=\sum_{k=0}^{j-1}\frac{c_k}{k!}t^kE_{\left(\nu,1+\frac{\mu}{\nu},\frac{\mu+k}{\nu}\right)}\left(\rho t^{\mu+\nu}\right).\qquad\square$$

Let us consider another equation to enhance the applicability of our methodology for multiple derivatives. This examples, presented by Francesco Mainardi [14] in the context of Wright functions applied to fractional differential equations. We now present the following examples.

Example 3.3. Let $0 < \mu < 1$ and $\nu \in \mathbb{R}^+$, with $\nu \notin \mathbb{N}$. The fractional differential equation is formulated as follows:

$$ {}^{C}D_0^{\mu}\left(t^{\nu}\frac{dy(t)}{dt}\right) = t^{\nu-1}y(t), \tag{3.32}$$

with the initial condition $y(0) = y_0$. The solution to this equation is given by

$$y(t) = \Gamma(\nu)y_0 W_{(\mu,\nu)}\left(\frac{t^{\mu}}{\mu}\right).$$

Proof. We hypothesize that the solution to this equation has the form

$$y(t) = \sum_{n,m,k\geqslant 0}^{+\infty} \gamma_{(n,m,k)} t^{n\mu+m\nu+k}. \tag{3.33}$$

The decision to raise $n\mu+m\nu+k$ to a power in (3.33) is influenced by the fractional derivative ${}^{C}D_0^{\mu}$ and the monomial t^{ν}, whereas the integer k is determined by the classical derivative. One can easily observe that

$$t^{\nu-1}y(t) = \sum_{n,m,k\geqslant 0}^{+\infty} \gamma_{(n,m,k)} t^{n\mu+(m+1)\nu+k-1}.$$

On the other hand, it is known that:

$$t^{\nu}\frac{dy(t)}{dt} = \sum_{\substack{n,m,k\geqslant 0\\ n+m+k\geqslant 1}}^{+\infty} \gamma_{(n,m,k)}\,(n\mu+m\nu+k)\,t^{n\mu+(m+1)\nu+k-1}.$$

Then, one has

$$\begin{aligned} {}^{C}D_0^{\mu}\left(t^{\nu}\frac{dy(t)}{dt}\right) &= \sum_{\substack{n,m,k\geqslant 0\\ n+m+k\geqslant 1}}^{+\infty} \gamma_{(n,m,k)}\,(n\mu+m\nu+k) \\ &\quad \times \frac{\Gamma(n\mu+(m+1)\nu+k)}{\Gamma((n-1)\mu+(m+1)\nu+k)} t^{(n-1)\mu+(m+1)\nu+k-1}. \end{aligned}$$

By modifying the summation indices, we derive the balance equations

$$\begin{aligned} &\sum_{\substack{n,m,k\geqslant 0\\ n+m+k\geqslant 1}}^{+\infty} \gamma_{(n,m,k)}\,(n\mu+m\nu+k)\,\frac{\Gamma(n\mu+(m+1)\nu+k)}{\Gamma((n-1)\mu+(m+1)\nu+k)} t^{(n-1)\mu+(m+1)\nu+k-1} \\ &- \sum_{n\geq 1 m,k\geqslant 0}^{+\infty} \gamma_{(n-1,m,k)} t^{(n-1)\mu+(m+1)\nu+k-1} = 0. \end{aligned}$$

It can be readily concluded that

(3.34)
$$\gamma_{(0,m,k)} = 0, \quad m + k \geq 1,$$

(3.35)
$$\gamma_{(n,m,k)} = \frac{\Gamma((n-1)\mu + (m+1)\nu + k)}{(n\mu + m\nu + k)\Gamma(n\mu + (m+1)\nu + k)}\gamma_{(n-1,m,k)}, \quad n \geq 1, m \geq 0, k \geq 0.$$

From equations (3.34) and (3.35), we find that $\gamma_{(n,m,k)} = 0$ for all $n \geq 0$ and $m+k \geq 1$. The only term that remains in the series for which $m + k$ is zero is when $m + k = 0$, leading to a simplification of the recurrence relation to

$$\gamma_{(n,0,0)} = \frac{1}{n\mu} \cdot \frac{\Gamma((n-1)\mu + \nu)}{\Gamma(n\mu + \nu)}\gamma_{(n-1,0,0)}.$$

We subsequently obtain

$$\gamma_{(n,0,0)} = \frac{1}{n!\mu^n} \cdot \frac{\Gamma(\nu)}{\Gamma(n\mu + \nu)}\gamma_{(0,0,0)}.$$

The series solution can now be written as

$$y(t) = y_0\Gamma(\nu)W_{(\mu,\nu)}\left(\frac{t^\mu}{\mu}\right). \qquad \square$$

Remark 3.1. Through direct calculations, we find that

$$\begin{aligned} {}^{C}D_0^\mu \left(t^\nu \frac{d}{dt}\right) W_{\mu,\nu}\left(\frac{t^\mu}{\mu}\right) &= {}^{C}D_0^\mu \sum_{k=1}^{+\infty} \frac{t^{k\mu+\nu-1}}{(k-1)!\mu^{k-1}\Gamma(k\mu+\nu)} \\ &= \sum_{k=1}^{+\infty} \frac{t^{k\mu+\nu-1-\mu}}{(k-1)!\mu^{k-1}\Gamma(k\mu+\nu-\mu)} = t^{\nu-1}W_{(\mu,\nu)}\left(\frac{t^\mu}{\mu}\right). \end{aligned}$$

Conclusion and Comments

In this paper, we have explored the application of generalized fractional power series methods to solve linear fractional order differential equations with both constant and variable coefficients. We have identified the limitations of traditional methods such as the Laplace transform, particularly when dealing with inhomogeneous equations and variable coefficients. To address these challenges, we developed a new method that extends the power series method to fractional differential equations, offering a more comprehensive and effective approach.

Our findings demonstrate that this new method not only generalizes the classical and fractional power series method but also provides a robust solution framework for equations involving fractional derivatives. Through various examples, we illustrated the efficacy and versatility of our approach in solving complex differential equations. However, applying our method to solve a special type of recurrence relation with more than two indices presents significant challenges because there is no general procedure for solving such recurrence relations. This implies that when an equation contains more than two terms, it becomes more challenging to apply our algorithm, except in some particular cases.

Overall, the contributions of this paper lie in extending the power series method to a broader class of differential equations, thereby opening new avenues for research and application in various scientific domains.

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