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EVALUATION SUBGROUPS OF A MAP BETWEEN RATIONAL FINITE H-SPACES

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Abstract. We use the theory of Sullivan minimal models and derivation to compute the evaluation subgroups and moreover the relative evaluation subgroups of a map $f : X \to Y$ between rational finite H-spaces. As a consequence, we show that the G-sequence is exact if f induces a zero map on rational homotopy groups.

1. Introduction

In this paper all spaces are assumed to be simply connected CW-complex and are of finite type over $\mathbb{Q}$, that is, have finite dimensional rational cohomology in each degree.

An important problem in homotopy theory is the computation of the Gottlieb groups. It is often difficult to describe these groups fully and the best that can be hoped for is some partial information about them. As is well known, the homotopy theory of rational spaces, i.e., spaces whose homotopy groups are vector spaces over $\mathbb{Q}$, is equivalent to the homotopy theory of minimal commutative differential graded algebras over $\mathbb{Q}$. More precisely, there is an equivalence between the homotopy category of rational spaces and the homotopy category of minimal cdgas. However, it is known that the category of continuous map between rational spaces is equivalent to the category of morphism between corresponding models.

Now, let us recall the definition of Gottlieb groups. Given a based space X, the n-th Gottlieb group or the n-th evaluation subgroups of X is the subgroup of $\pi_n(X)$ consisting of homotopy classes of map $h : \mathbb{S}^n \to X$ such that the wedge

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$(h \vee Id_X) : \mathbb{S}^n \vee X \to X$ extends to a map $H : \mathbb{S}^n \times X \to X$. Alternately, it is known that

$$G_n(X) = \mathrm{Im}\left(\mathrm{ev}_\sharp : \pi_n(\mathrm{aut}_1(X)) \to \pi_n(X)\right),$$

where $\mathrm{aut}_1(X)$ is the set of self-homotopy equivalences of X which are homotopic to the identity map. These groups were discovered and studied by Gottlieb in the early 1960's (the interested reader may consult [4]). They have led to many interesting results in algebraic topology, especially, in theory of fibrations.

The generalization of Gottlieb groups was initiated by Woo and Kim [10]. Let $f : X \to Y$ be a based map of spaces, then the n-th evaluation subgroups of f, also called the n-th generalized Gottlieb group is given by:

$$G_n(Y, X; f) = \mathrm{Im}\left(\mathrm{ev}_\sharp : \pi_n(map(X, Y; f)) \to \pi_n(Y)\right).$$

Here $\mathrm{map}(X, Y; f)$ means that the space of all maps from X to Y that are homotopic to f. Note that

$$G_n(X, X; Id_X) = G_n(X),$$

and in general we have $G_n(Y) \subset G_n(Y, X; f)$.

The Gottlieb groups and the generalized Gottlieb groups play a profound role in the homotopy theory of fibrations. But, until now, there are not many explicit computations of $G_*(X)$ and $G_*(Y, X; f)$. Since a map of spaces does not necessarily induce a corresponding homomorphism of Gottlieb groups, Woo and Lee was introduced the n-th relative evaluation subgroups $G_n^{rel}(Y, X; f)$, also called the n-th relative Gottlieb group [11]. The authors showed also that these groups fit in the following G-sequence

$$\cdots \to G_{n+1}^{rel}(Y, X; f) \to G_n(X) \to G_n(Y, X; f) \to G_n^{rel}(Y, X; f) \to \cdots \tag{1.1}$$

The computation of rational relative evaluation subgroups have been receiving a growing attention and have become a popular subject of study with a lot of progresses (see [3, 6, 12, 13] for instance). Our goal in this paper is to compute these subgroups in some new cases. So, by using Sullivan minimal models and other invariants in rational homotopy theory we compute the relative Gottlieb groups of a map between rational finite H-spaces. These spaces form a very well-studied and interesting class of spaces which appear abundantly in geometry and topology. They include products of rational spheres.

The paper is organized as follows. In Section 2, we introduce our notation and recall some background of rational homotopy theory, namely Sullivan minimal models, derivations and mapping cone. Next, we use them to recall the algebraic version of all terms involved in the G-sequence (1.1). Section 3 is devoted to our results and their proofs.

2. Preliminaries

We will work with $\mathbb{Q}$ as ground field and our principal tools are Sullivan minimal models. A detailed description of these and the standard tools of rational homotopy theory can be found in [1]. For our purposes, we recall the following.

A commutative differential graded algebra (abbreviated cdga) (A, d) consists of graded vector spaces $A = \oplus_{i\geq 0} A^i$ with a multiplication $A^i \otimes A^j \to A^{i+j}$ satisfying $ab = (-1)^{ij}\, ba$ and a map $d : A^i \to A^{i+1}$, such that $d^2 = 0$ and $d\,(ab) = d\,(a)\,b + (-1)^i\, ad\,(b)$ for all $a \in A^i$ and $b \in A^j$. A cdga (A, d) is called simply connected if $H^0(A, d) = \mathbb{Q}$ and $H^1(A, d) = 0$. Denote by

$$V^{even} = \oplus_{i\geq 1} V^{2i} \quad \text{and} \quad V^{odd} = \oplus_{i\geq 1} V^{2i+1}.$$

We say that a simply connected cdga (A, d) is a Sullivan minimal algebra if it is of the form

$$(\Lambda V, d) = (S(V^{even}) \otimes E(V^{odd}), d),$$

such that $dV \subset \Lambda^{\geq 2} V$.

For a simply connected CW-complex X of finite type, there is a cdga $A_{PL}(X)$ verifying

$$H^*(A_{PL}(X)) \cong H^*(X; \mathbb{Q}).$$

Then the Sullivan minimal model of X is defined to be the Sullivan minimal model of $A_{PL}(X)$ which is unique up to isomorphism [1]. Moreover, the rational homotopy type of X is completely determined by its Sullivan minimal model $(\Lambda V, d)$, that is, $Hom(V, \mathbb{Q}) \cong \pi_*(X) \otimes \mathbb{Q}$ as graded vector spaces.

Now, we go to generalize this situation to any map $f : X \to Y$ of simply connected spaces. A Sullivan model of f is a morphism of cdga's

$$\phi : (\Lambda W, d_W) \to (\Lambda V, d_V)\,,$$

where $(\Lambda V, d_V)$ and $(\Lambda W, d_W)$ are the Sullivan minimal models of X and Y, respectively.

Next, consider $\phi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ a morphism of cdga's. A ϕ-derivation θ of degree n is a linear map

$$\theta : (\Lambda W)^m \to (\Lambda V)^{m-n}$$

verifying $\theta\,(xy) = \theta\,(x)\,\phi\,(y) + (-1)^{n|x|}\,\phi\,(x)\,\theta\,(y)$ for x and y are in W. In the following, we denote by $Der_n\,(\Lambda W, \Lambda V; \phi)$ the vector space of ϕ-derivations of degree n and when $n = 1$, we require additionally that all derivations are cycles, that is,

$$d_V \circ \theta = -\theta \circ d_W.$$

There is a differential

$$\delta : Der_n\,(\Lambda W, \Lambda V; \phi) \to Der_{n-1}\,(\Lambda W, \Lambda V; \phi)\,,$$

given by

$$\delta\,(\theta) = d_V \circ \theta - (-1)^n\,\theta \circ d_W.$$

Moreover, let

$$Der_*\,(\Lambda W, \Lambda V; \phi) = \oplus_n Der_n\,(\Lambda W, \Lambda V; \phi)\,,$$

and in particular

$$Der_*\,(\Lambda V, \Lambda V; Id_{\Lambda V}) = Der_*\,(\Lambda V)\,.$$

Further, it is easy to see that there is an isomorphism of graded vector spaces

$$Der_*(\Lambda W, \Lambda V; \phi) \cong Hom_*(W, \Lambda V).$$

Thus, we denote by $(w, v) \in Der_{|w|-|v|}(\Lambda W, \Lambda V; \phi)$ the unique ϕ-derivation sending an element $w \in W$ to $v \in \Lambda V$ and the other generators to zero, and in particular $(w, 1) = w^*$ for an element $w \in W$.

In the remainder of this section, we give a description in rational homotopy theory of all terms involved in the G-sequence (1.1) (see [5]). For this, let us consider the augmentation map $\varepsilon : \Lambda V \to \mathbb{Q}$ which induces $\varepsilon_* : Der_*(\Lambda W, \Lambda V; \phi) \to Der_*(\Lambda W, \mathbb{Q}; \varepsilon)$. Thus, the n-th evaluation subgroups of ϕ is given by

$$G_n(\Lambda W, \Lambda V; \phi) = \operatorname{Im}\{H_n(\varepsilon_*) : H_n(Der(\Lambda W, \Lambda V; \phi)) \to Hom_n(W, \mathbb{Q})\}, \quad \text{for } n \geq 2.$$

So, an element w^* in $Hom_n(W, \mathbb{Q})$ is in $G_n(\Lambda W, \Lambda V; \phi)$ if and only if w^* extends to a cycle-derivation θ of $Der_n(\Lambda W, \Lambda V; \phi)$.

A special case of the preceding that is of interest to us is the one in which $\Lambda W \cong \Lambda V$ and $\phi = Id_{\Lambda V}$. In this case, we get:

$$G_n(\Lambda V) = \operatorname{Im}\{H_n(\varepsilon_*) : H_n(Der(\Lambda V)) \to Hom_n(V, \mathbb{Q})\}, \quad \text{for } n \geq 2,$$

which is called the n-th Gottlieb group of $(\Lambda V, d_V)$.

The following result is due to S.B. Smith which shows that the rational evaluation subgroups of a map are completely determined only by the graded vector space of derivations (see [9]).

Theorem 2.1. *Suppose $\phi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ is a Sullivan model of a map $f : X \to Y$ between simply connected CW-complexes such that X is finite. Then*

$$G_n(Y, X; f) \otimes \mathbb{Q} \;\cong\; G_n(\Lambda W, \Lambda V; \phi), \quad n \geq 2.$$

Next, let us remind the notion of mapping cone which is very useful to characterize the rational relative evaluation subgroups in terms of derivations.

Suppose $\phi : (A, d_A) \to (B, d_B)$ is a map of differential graded vector spaces. We define differential graded vector spaces $(Rel_*(\phi), D)$, called the mapping cone of ϕ, as follows

$$Rel_n(\phi) = A_{n-1} \oplus B_n.$$

The differential

$$D : Rel_n(\phi) \to Rel_{n-1}(\phi)$$

is given by

$$D(a, b) = (-d_A(a), \phi(a) + d_B(b)).$$

Furthermore, we define the following maps

$$J : B_n \to Rel_n(\phi) \quad \text{and} \quad P : Rel_n(\phi) \to A_{n-1}$$

by

$$J(b) = (0, b) \quad \text{and} \quad P(a, b) = a, \quad \text{for } (a, b) \in A \times B.$$

Next, suppose given a differential graded algebra map $\phi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ of a map $f : X \to Y$. Then, we can constructed a map of chain complexes

$$\phi^* : Der_* (\Lambda V) \to Der_* (\Lambda W, \Lambda V; \phi), \quad \text{given by } \phi^*(\theta) = \phi \circ \theta.$$

The consideration above induces the following commutative diagram

$$\begin{array}{ccccc} Der_* (\Lambda V) & \xrightarrow{\phi^*} & Der_* (\Lambda W, \Lambda V; \phi) & \xrightarrow{J} & Rel_*(\phi^*) \\ \downarrow \varepsilon_* & & \downarrow \varepsilon_* & & \downarrow (\varepsilon_*, \varepsilon_*) \\ Der_* (\Lambda V, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{\phi^*}} & Der_* (\Lambda W, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{J}} & Rel_*(\widehat{\phi^*}) \end{array}.$$

In this diagram, ε denotes the augmentation of either ΛV or ΛW. So, the n-th relative evaluation subgroups of ϕ is defined as follows

$$G_n^{rel} (\Lambda W, \Lambda V; \phi) = \mathrm{Im} \left\{ H_n (\varepsilon_*, \varepsilon_*) : H_n(Rel(\phi^*)) \to H_n(Rel(\widehat{\phi^*})) \right\}, \quad \text{for } n \geq 2.$$

We finish this section by some notations and conventions. The cohomology of a cdga (A, d) is denoted $H^* (A, d)$ or just $H^* (A)$ and let $[x] \in H^* (A, d)$ stand for the cohomolgy class of the cocycle $x \in A$. In the sequel, all spaces appearing in this paper are assumed to be *rational simply connected CW-complexes*, i.e., all spaces satisfy $X = X_{\mathbb{Q}}$.

3. Relative Evaluation Subgroups of a Map Between Rational Finite H-Spaces

Our aim in this section is devoted to compute in terms of Sullivan minimal models and derivations the rational evaluation subgroups and moreover the rational relative evaluation subgroups of a map between finite H-spaces.

3.1. **Evaluation subgroups of a map between rational finite H-spaces.** In the first place, we are interested in determining the evaluation subgroups of a rational H-space.

Proposition 3.1. *Suppose X is a rational H-space. Then $G_* (X) \cong \pi_* (X)$.*

Proof. It is easy, but for the sake of completeness we write a proof. It is well known that a rational H-space X has the rational homotopy type of a product of Eilenberg-MacLane spaces, i.e., $X \simeq_{\mathbb{Q}} \prod_i K(\mathbb{Q}, n_i)$ (see [8, Corollary 1]). It follows that

$$G_* (X) \cong G_* \left(\prod_i K(\mathbb{Q}, n_i) \right) \cong \oplus_i G_* (K(\mathbb{Q}, n_i)) \cong \oplus_i \pi_* (K(\mathbb{Q}, n_i)) \cong \pi_*(X),$$

as required. □

We note that Proposition 3.1 can also be showed by using the Sullivan minimal model of a rational H-space.

Now let's move on to evaluation subgroups of a map between rational finite H-spaces.

Proposition 3.2. *Let $f : X \to Y$ be a map between rational finite H-spaces, then*

$$G_*(Y, X; f) \cong \pi_*(Y).$$

Proof. From [1, Example 3, p. 143] we know that the Sullivan minimal model of Y is given by

$$(\Lambda W, 0) = (\Lambda(y_1, y_2, \ldots, y_p), 0)$$

and the Sullivan minimal model of X is of the form

$$(\Lambda V, 0) = (\Lambda(x_1, x_2, \ldots, x_q), 0),$$

where $p = \dim \pi_*(Y)$ and $q = \dim \pi_*(X)$. Now, denote by $(y_j, 1)$ the derivation θ_j in $Der_{|y_j|}(\Lambda W, \Lambda V; \phi)$ for $1 \leq j \leq p$. Since the differential on ΛV and ΛW are trivial, it follows that the differential δ on $Der_*(\Lambda W, \Lambda V; \phi)$ is trivial. It is therefore automatic that θ_j is closed and not a boundary. Further, we consider

$$\varepsilon_* : Der_*(\Lambda W, \Lambda V; \phi) \to Hom(W, \mathbb{Q}),$$

which is given by $\varepsilon_*(\theta_j) = y_j^*$ for $1 \leq j \leq p$. Hence, combining the preceding we obtain

$$G_*(\Lambda W, \Lambda V; \phi) = \langle y_1^*, y_2^*, \ldots, y_p^* \rangle \cong Hom(W, \mathbb{Q}),$$

which completes the proof. ☐

3.2. **Relative evaluation subgroups of a map between rational finite H-spaces.** In this subsection, we will present our main result concerning the relative evaluation subgroups of a map between rational finite H-spaces. First, we offer one example to illustrate the general idea and then give a summary result.

Example 3.1. Suppose $f : X \to Y$ is a map of rational H-spaces which its Sullivan model

$$\phi : (\Lambda W, 0) = (\Lambda(x_3, y_4, z_8), 0) \to (\Lambda V, 0) = (\Lambda(u_3, v_5, w_9), 0)$$

is given on generators by $\phi(x) = u$, $\phi(y) = 0$ and $\phi(z) = uv$. In both Sullivan minimal models, subscripts denote degrees. We compute $G_*^{rel}(\Lambda W, \Lambda V; \phi)$ as follows.

Let us consider

$$\phi^* : Der_*(\Lambda V) \to Der_*(\Lambda W, \Lambda V; \phi),$$

which is given by $\phi^*(u^*) = x^* + (z, v)$, $\phi^*(v^*) = (z, u)$ and $\phi^*(w^*) = 0$. Thus, we have immediately

$$D(u^*, 0) = (0, x^* + (z, v)), \quad D(v^*, 0) = (0, (z, u)) \quad \text{and} \quad D(w^*, 0) = 0.$$

Further, it is easy to see that

$$D(0, x^*) = D(0, y^*) = D(0, z^*) = 0.$$

An easy argument shows that the elements $[(w^*, 0)]$, $[(0, y^*)]$ and $[(0, z^*)]$ are nonzero in $H_*(Rel(\phi^*))$. Next, denote by $\varepsilon_*(a^*) = \widehat{a^*}$ for an element a in W or V. Since, we have $H_*(\varepsilon_*,\varepsilon_*)([(w^*,0)]) = \left[\left(\widehat{w^*},0\right)\right]$, $H_*(\varepsilon_*,\varepsilon_*)([(0,y^*)]) = \left[\left(0,\widehat{y^*}\right)\right]$ and also $H_*(\varepsilon_*,\varepsilon_*)([(0,z^*)]) = \left[\left(0,\widehat{z^*}\right)\right]$ are nonzero in $H_*\left(Rel\left(\widehat{\phi^*}\right)\right)$. It follows that

$$G_*^{rel}(\Lambda W,\Lambda V;\phi) = \left\langle \left[\left(\widehat{w^*},0\right)\right], \left[\left(0,\widehat{y^*}\right)\right], \left[\left(0,\widehat{z^*}\right)\right] \right\rangle.$$

The following discussion will fix our notation. Suppose $f : X \to Y$ is a map between rational finite H-spaces and denote by $\phi : (\Lambda W, 0) \to (\Lambda V, 0)$ the Sullivan model of f. Let $\{y_1,\dots,y_r,y_{r+1},\dots,y_s,y_{s+1},\dots,y_p\}$ be a basis for W and let us consider $\{x_1,\dots,x_r,x_{r+1},\dots,x_t,x_{t+1},\dots,x_q\}$ be a basis for V. By using part (a) of ([2], Proposition 2.2) and a change of KS-basis, we can write $\phi(y_{i_0}) = x_{i_0}$ for $1 \le i_0 \le r$, $\phi(y_{i_1}) \in \Lambda^{\ge 2}(x_1,\dots,x_t)$ -contains only decomposable elements- for $r+1 \le i_1 \le s$ and $\phi(y_{i_2}) = 0$ for $s+1 \le i_2 \le p$. Hence, it is easy to see that ϕ is well defined.

Now, we may extend the argument explained in Example 3.1 to give our main result in this section.

Theorem 3.1. *With the same notation as above, let $f : X \to Y$ be a map between rational finite H-spaces and $\phi : (\Lambda W, 0) \to (\Lambda V, 0)$ its Sullivan model. Then*

$$G_*^{rel}(\Lambda W,\Lambda V;\phi) = \left\langle \left[\left(\widehat{x_{j_2}^*},0\right)\right], \left[\left(0,\widehat{y_{i_1}^*}\right)\right], \left[\left(0,\widehat{y_{i_2}^*}\right)\right] \right\rangle,$$

for $t+1 \le j_2 \le q$, $r+1 \le i_1 \le s$ and $s+1 \le i_2 \le p$.

Proof. First, as recalled above denote by

$$\phi : (\Lambda(y_1,\dots,y_p),0) \to (\Lambda(x_1,\dots,x_q),0)$$

the Sullivan model of f in which is given by $\phi(y_{i_0}) = x_{i_0}$ for $1 \le i_0 \le r$, $\phi(y_{i_1}) \in \Lambda^{\ge 2}(x_1,\dots,x_t)$ for $r+1 \le i_1 \le s$ and $\phi(y_{i_2}) = 0$ for $s+1 \le i_2 \le p$. Thus, it induces the following map

$$\phi^* : Der_*(\Lambda V) \to Der_*(\Lambda W,\Lambda V;\phi),$$

where $\phi^*\left(x_{i_0}^*\right) = y_{i_0}^* + (y_{i_1}, v_{i_1})$ such that $v_{i_1} = x_{i_0}^*(\phi(y_{i_1}))$, $\phi^*\left(x_{j_1}^*\right) = (y_{i_1}, w_{i_1})$ such that $w_{i_1} = x_{j_1}^*(\phi(y_{i_1}))$ for $r+1 \le j_1 \le t$ and finally, $\phi^*\left(x_{j_2}^*\right) = 0$ for $t+1 \le j_2 \le q$. Here, since $\phi(y_{i_1})$ is decomposable, we note that the elements v_{i_1} and w_{i_1} are in $\Lambda^+(x_1,\dots,x_t)$. Further, an easy computation gives that

$$D\left(x_{i_0}^*,0\right) = \left(0, y_{i_0}^* + (y_{i_1}, v_{i_1})\right), \quad D\left(x_{j_1}^*,0\right) = (0,(y_{i_1},w_{i_1})) \quad \text{and} \quad D\left(x_{j_2}^*,0\right) = 0.$$

Hence, for $r+1 \le i_1 \le s$ and $s+1 \le i_2 \le p$, we have

$$D\left(0,y_{i_1}^*\right) = D\left(0,y_{i_2}^*\right) = 0.$$

Next, an easy argument by contradiction shows that $\left(x_{j_2}^*,0\right)$, $\left(0,y_{i_1}^*\right)$ and $\left(0,y_{i_2}^*\right)$ are not D-boundaries. This means that $\left[\left(x_{j_2}^*,0\right)\right]$, $\left[\left(0,y_{i_1}^*\right)\right]$ and $\left[\left(0,y_{i_2}^*\right)\right]$ are non null in $H_*(Rel(\phi^*))$. Otherwise, we see that

$$Rel_*\left(\widehat{\phi^*}\right) = Der_{*-1}(\Lambda V,\mathbb{Q};\varepsilon) \oplus Der_*(\Lambda W,\mathbb{Q};\varepsilon).$$

Moreover, we recall that for $a \in V$

$$\varepsilon_* : Der_*(\Lambda V) \to Der_*(\Lambda V, \mathbb{Q}; \varepsilon), \quad \varepsilon_*(a^*) = \widehat{a^*},$$

and let also for $b \in W$

$$\varepsilon_* : Der_*(\Lambda W, \Lambda V; \phi) \to Der_*(\Lambda W, \mathbb{Q}; \varepsilon), \quad \varepsilon_*(b^*) = \widehat{b^*}$$

Next to determine $H_*\left(Rel\left(\widehat{\phi^*}\right)\right)$, we need to compute the differential $\widehat{D}$ in $\mathrm{Rel}_*\left(\widehat{\phi^*}\right)$. For this, let

$$\widehat{\phi^*} : Der_*(\Lambda V, \mathbb{Q}; \varepsilon) \to Der_*(\Lambda W, \mathbb{Q}; \varepsilon),$$

which is defined as follows: $\widehat{\phi^*}\left(\widehat{x^*_{i_0}}\right) = \widehat{y^*_{i_0}}$ and $\widehat{\phi^*}\left(\widehat{x^*_{j_1}}\right) = \widehat{\phi^*}\left(\widehat{x^*_{j_2}}\right) = 0$. Hence, in a similar fashion as above, we prove that the elements $\left(\widehat{x^*_{j_2}}, 0\right)$, $\left(0, \widehat{y^*_{i_1}}\right)$ and $\left(0, \widehat{y^*_{i_2}}\right)$ are cycles which are not $\widehat{D}$-boundaries. Then, by summarizing all the above we get

$$H_*(\varepsilon_*, \varepsilon_*)\left(\left[\left(x^*_{j_2}, 0\right)\right]\right) = \left[\left(\widehat{x^*_{j_2}}, 0\right)\right], \quad \text{for } t+1 \leq j_2 \leq q,$$

$$H_*(\varepsilon_*, \varepsilon_*)\left(\left[\left(0, y^*_{i_1}\right)\right]\right) = \left[\left(0, \widehat{y^*_{i_1}}\right)\right], \quad \text{for } r+1 \leq i_1 \leq s,$$

and also

$$H_*(\varepsilon_*, \varepsilon_*)\left(\left[\left(0, y^*_{i_2}\right)\right]\right) = \left[\left(0, \widehat{y^*_{i_2}}\right)\right], \quad \text{for } s+1 \leq i_2 \leq p.$$

In summary, we have proved that

$$G^{rel}_*(\Lambda W, \Lambda V; \phi) = \left\langle \left[\left(\widehat{x^*_{j_2}}, 0\right)\right], \left[\left(0, \widehat{y^*_{i_1}}\right)\right], \left[\left(0, \widehat{y^*_{i_2}}\right)\right] \right\rangle,$$

for $t+1 \leq j_2 \leq q$, $r+1 \leq i_1 \leq s$ and $s+1 \leq i_2 \leq p$. □

Proposition 3.3. *Given a map $f : X \to Y$ of rational finite H-spaces. Suppose that f induces an injective morphism on rational homotopy groups, then*

$$G^{rel}_*(Y, X; f) \cong \pi_*(Y) \diagup \pi_*(X).$$

Proof. Denote by $(\Lambda W, 0)$ and $(\Lambda V, 0)$ the Sullivan minimal models respectively of Y and X. Let $\{x_1, \ldots, x_q\}$ be a homogeneous basis for V. As f induces an injective morphism on rational homotopy groups, we may choose $y_{q+1}, \ldots, y_p$ such that $\{x_1, \ldots, x_q, y_{q+1}, \ldots, y_p\}$ is a homogeneous basis for W. Further, let us denote by

$$\phi : (\Lambda(x_1, \ldots, x_q, y_{q+1}, \ldots, y_p), 0) \to (\Lambda(x_1, \ldots, x_q), 0)$$

the Sullivan model of f which is defined as follows: $\phi(x_i) = x_i$ for $1 \leq i \leq q$ and $\phi(y_j) = 0$ for $q+1 \leq j \leq p$. Of course, we have $\phi^*(x^*_i) = x^*_i$ for $1 \leq i \leq q$. Then by using a similar argument given in the proof of Theorem 3.1, we obtain that

$$\begin{aligned} G^{rel}_*(\Lambda W, \Lambda V; \phi) &= \left\langle \left[(0, \widehat{y_j{}^*})\right] \text{ for } q+1 \leq j \leq p \right\rangle \\ &\cong \pi_*(Y) \diagup \pi_*(X), \quad \text{as graded vector spaces.} \end{aligned}$$ □

Various conditions are known under which the G-sequence of a map is exact [5, 7]. However, in general there is not information about the exactness of the G-sequence.

Corollary 3.1. *Let $f : X \to Y$ be a map between rational finite H-spaces in which f induces an injective morphism on rational homotopy groups. Then the G-sequence of f splits into short exact sequence*

$$0 \to G_* (X) \to G_* (Y, X; f) \to G_*^{rel} (Y, X; f) \to 0.$$

Proof. It follows directly from Proposition 3.1, Proposition 3.2 and Proposition 3.3 together with the G-sequence (1.1). □

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References

[1] Y. Félix, S. Halperin and J.-C. Thomas, *Rational Homotopy theory*, Graduate Texts in Mathematics **205**, Springer, 2001.

[2] Y. Félix and G. Lupton, *Evaluation maps in rational homotopy*, Topology **46** (2007), 493–506. https://doi.org/10.1016/j.top.2007.03.005

[3] J.-B. Gatsinzi, *Rational homotopy type of mapping spaces between complex projective spaces and their evaluation subgroups*, Commun. Korean Math. Soc. **37**(1) (2022), 259—267. https://doi.org/10.4134/CKMS.c200431

[4] D. H. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. **87**(4) (1965), 840–856. https://doi.org/10.2307/2373248

[5] G. Lupton and S. B. Smith, *Rationalized evaluation subgroups of a map I: Sullivan models, derivations and G-sequences*, J. Pure Appl. Algebra. **209**(1) (2007), 159–171. https://doi.org/10.1016/j.jpaa.2006.05.018

[6] P. A. Otieno, J.-B. Gatsinzi and V. O. Otieno, *Relative Gottlieb groups of the Plücker embedding of some complex Grassmanians*, Commun. Korean Math. Soc. **35**(1) (2020), 279–285. https://doi.org/10.4134/CKMS.c180453

[7] J. Z. Pan and M. H. Woo, *Exactness of G-sequence and monomorphisms*. Topology Appl. **109**(3) (2001), 315–320. https://doi.org/10.1016/S0166-8641(99)00178-9

[8] H. Scheerer, *On Rationalized H- and co-H-Spaces. With an Appendix on decomposable H- and co-H-Spaces*, Manuscripta Math. **51** (1985), 63–87. https://doi.org/10.1007/BF01168347

[9] S. B. Smith, *Rational evaluation subgroups*, Math. Z. **221** (1996), 387–400. https://doi.org/10.1007/BF02622121

[10] M. H. Woo and J.-R. Kim, *Certain subgroups of homotopy groups*, J. Korean Math Soc. **21** (1984), 109–120.

[11] M. H. Woo and K. Y. Lee, *On the relative evaluation subgroups of a CW-pair*, J. Korean Math. Soc. **25**(1) (1988), 149–160.

[12] A. Zaim, *Relative Gottlieb groups of mapping spaces and their rational cohomology*, Proceedings of the International Geometry Center **15**(1) (2022), 1–15. https://doi.org/10.15673/tmgc.v15i1.2196

[13] A. Zaim, *Evaluation subgroups of mapping spaces over Grassmann manifolds*, Kyungpook Math. J. **63**(1) (2023), 131–139. https://doi.org/10.5666/KMJ.2022.63.1.131

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