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PRODUCT CORDIAL LABELING OF DOUBLE WHEEL AND DOUBLE FAN RELATED GRAPHS

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Abstract. I prove that Double wheel, path union of finite copies of Double wheel are product cordial graphs. Further I prove that the graph obtained by joining two copies of double wheel by a path of arbitrary length is product cordial graph. I also prove that $DW_n \oplus K_{1,n}$ and $DF_n \oplus K_{1,n}$ are product cordial graphs.

1. Introduction

I consider simple, finite, undirected graph $G = (V, E)$. In this paper P_n denotes path with n vertices. For all other terminology and notations I follow Harary [1]. First I will provide some definitions useful for the present work.

Definition 1.1. Let G be a graph and G_1,G_2,...,G_n, $n \geq 2$ be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1}, for $i = 1, 2, \dots, n-1$, is called *path union of G*.

Definition 1.2. If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*. Detailed survey on graph labeling is given and updated by Gallian [2].

Definition 1.3. Let $G = (V, E)$ be a graph. A mapping $f : V(G) \to \{0, 1\}$ is called *binary vertex labeling* of G and $f(v)$ is called *label* of the vertex v of G under f.

Definition 1.4. A binary vertex labeling $f : V(G) \to \{0, 1\}$ of graph G with induced edge labeling $f^* : E(G) \to \{0, 1\}$ defined by $f^*(uv) = f(u)f(v)$ is called a *product cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(0)$, $v_f(1)$ denote the number of vertices of G having labels 0, 1 respectively under f and $e_f(0)$, $e_f(1)$

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denote the number of edges of G having labels 0, 1 by respectively under f^*. A graph G is *product cordial* if it admits product cordial labeling.

Definition 1.5. A *Ring sum* $G_1 \oplus G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \oplus G_2 = (V_1 \cup V_2, (E_1 \cup AE_2) - (E_1 \cap E_2))$.

Definition 1.6. A *double wheel* graph DW_n of size n can be composed of $2C_n + K_1$. It consists of two cycles of size n where vertices of two cycles are all connected to a central vertex.

Definition 1.7. The *double fan* DF_n consists of two fan graph that have a common path. In other words $DF_n = P_n + \overline{K_2}$.

The concept of product cordial labeling was introduced by R. Ponraj, M. Sivakumar, M. Sundaram [5]. A. H. Rokad and G. V. Ghodasara [4] proved that graph $C_n \oplus K_{1,n}$, $G \oplus K_{1,n}$, where G is cycle with one chord, $G \oplus K_{1,n}$, where G is cycle with twin chord, $G \oplus K_{1,n}$, where G is cycle with triangle are product cordial graphs. Vaidya and Barasara [6] proved that shell admits product cordial labeling. They also proved that cycle C_n, split graph $\mathrm{Spl}(P_6)$ and total graph $T(P_n)$ are product cordial graphs. Vaidya and Dani [7] proved that the graph obtained by joining apex vertices of two stars is product cordial. They also proved similar results for shell and wheel. In [8], same authors proved that path union of k copies of cycle C_n, the graph obtained by joining two copies of cycle C_n by path P_k, the path union of k copies of $D_2(C_n)$ are product cordial graphs.

2. Main Results

Theorem 2.1. *Double wheel DW_n is a product cordial graph.*

Proof. Let u be the apex vertex, $u_1, u_2, \ldots, u_n$ be the vertices of inner cycle and $v_1, v_2, \ldots, v_n$ be the vertices of outer cycle of DW_n. Then $|V(DW_n)| = 2n + 1$ and $|E(DW_n)| = 4n$.

I define labeling function $f : V(G) \to \{0, 1\}$ as follow. For all i

$$f(u) = 1, \quad f(u_i) = 1, \quad f(v_i) = 0.$$

In view of above labeling pattern I have $v_f(0) = v_f(1) - 1 = n$ and $e_f(0) = e_f(1) = 2n$. Thus I have $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$.

Hence, DW_n is product cordial graph. □

Example 2.1. The product cordial labeling of Double wheel DW_7 is shown in Figure 1.

Theorem 2.2. *The path union of k copies of Double wheel DW_n admits product cordial labeling.*

Proof. Let G be the path union of k copies $G_1, G_2, \ldots, G_k$ of Double wheel DW_n. Let $\{u_{i0}, u_{i1}, \ldots, u_{in}, u'_{i1}, u'_{i2}, \ldots, u'_{in}\}$ denote the vertices of G. Where u_{i0} is apex vertex,

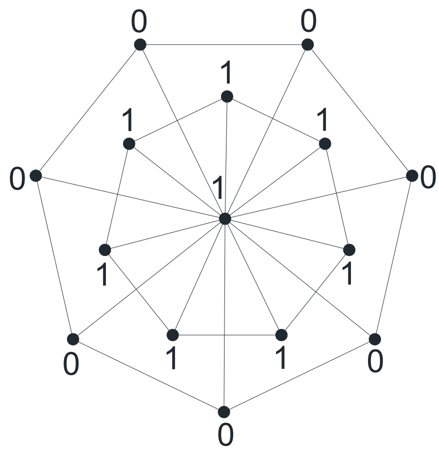

FIGURE 1. Product cordial labeling of DW_7

$\{u_{i1}, \ldots, u_{in}\}$ are internal vertices and $\{u'_{i1}, u'_{i2}, \ldots, u'_{in}\}$ are external vertices. Let $e_i = u_{i0}u_{(i+1)0}$ be the edge joining G_i and G_{i+1}.

I define labeling function $f : V(G) \to \{0, 1\}$ as follow.

Case 1: k is odd.

$$f(u_{ij}) = \begin{cases} 1, & 1 \le i \le \dfrac{k-1}{2}, 0 \le j \le n, \\ 0, & \dfrac{k+3}{2} \le i \le k, 0 \le j \le n, \end{cases}$$

$$f(u_{ij})' = \begin{cases} 1, & 1 \le i \le \dfrac{k-1}{2}, 1 \le j \le n, \\ 0, & \dfrac{k+3}{2} \le i \le k, 1 \le j \le n, \end{cases}$$

$$\begin{aligned} f(u_{(\frac{k+1}{2})j}) &= 1, \quad 0 \le j \le n, \\ f(u_{(\frac{k+1}{2})j})' &= 0, \quad 1 \le j \le n. \end{aligned}$$

Case 2: k is even.

$$f(u_{ij}) = \begin{cases} 1, & 1 \le i \le \dfrac{k}{2}, 0 \le j \le n, \\ 0, & \dfrac{k}{2} + 1 \le i \le k, 0 \le j \le n, \end{cases}$$

$$f(u_{ij})' = \begin{cases} 1, & 1 \le i \le \dfrac{k}{2}, 1 \le j \le n, \\ 0, & \dfrac{k}{2} + 1 \le i \le k, 1 \le j \le n. \end{cases}$$

The graph G under consideration satisfies the conditions $|v_f(1) - v_f(0)| \le 1$ and $|e_f(1) - e_f(0)| \le 1$ in each case.

Hence, the graph G is product cordial graph. $\square$

Example 2.2. The product cordial labeling of the path union of 3-copies of Double wheel DW_7 is shown in Figure 2. It is the case related to k is odd.

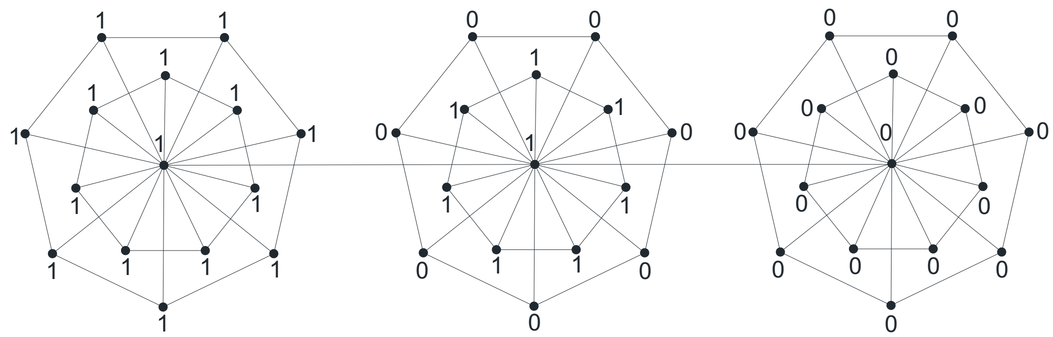

FIGURE 2. Product cordial labeling of the path union of 3-copies of DW_7

Theorem 2.3. *The graph obtaining by joining two copies of Double wheel DW_n by path of arbitrary length admits product cordial labeling.*

Proof. Let G be the graph obtained by joining two copies of closed helm CH_n by path P_k of $k-1$ length. Let $\{u_0, u_1, \ldots, u_n, u'_1, u'_2, \ldots, u'_n\}$ denote the consecutive vertices of first copy of Double wheel DW_n and let $\{w_0, w_1, \ldots, w_n, w'_1, w'_2, \ldots, w'_n\}$ denote the consecutive vertices of second copy of closed helm CH_n. Let $\{v_1, v_2, \ldots, v_k\}$ denote the vertices of the path P_k with $u_0 = v_1$ and $v_k = w_0$. First I label the vertices of first copy of CH_n by label 1 and label the vertices of second copy of CH_n by label 0. At this stage the vertex condition and the edge condition of product cordial labeling are satisfied. Now the remaining task is to label the vertices of path P_k for which I define labeling function $f : V(G) \to \{0, 1\}$ as follow.

Case 1: k is even.

In this case I label the vertices as:

$$f(v_i) = \begin{cases} 1, & 1 \le i \le \dfrac{k}{2}, \\ 0, & \dfrac{k}{2} + 1 \le i \le k. \end{cases}$$

It can easily seen that the vertex condition and edge conditions of product cordial labeling are satisfied in this case.

Case 2: k is odd.

In this case I label the vertices as:

$$f(v_i) = \begin{cases} 1, & 1 \le i \le \dfrac{k+1}{2}, \\ 0, & \dfrac{k+3}{2} \le i \le k. \end{cases}$$

It can easily seen that the vertex condition and edge conditions of product cordial labeling are satisfied in this case.

Hence the graph G is product cordial graph. □

Example 2.3. Product cordial labeling of the graph obtained by joining two copies of Double wheel DW_9 by path P_3 is shown in Figure 3. It is the case related to k is odd.

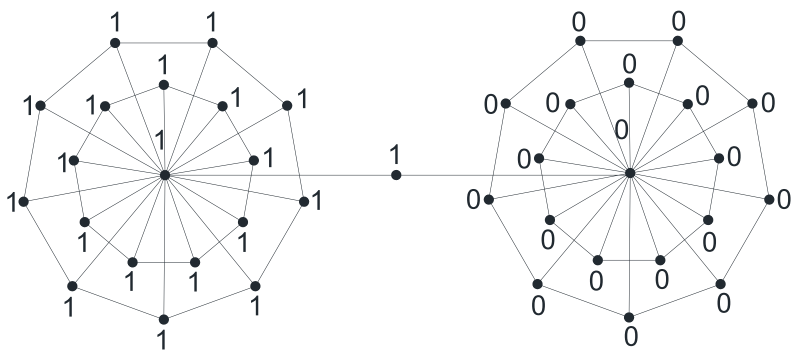

FIGURE 3. Product cordial labeling of the graph obtained by joining two copies of DW_9 by path P_3

Theorem 2.4. *The graph $DW_n \oplus K_{1,n}$ is product cordial graph.*

Proof. Let $V(G) = V_1 \cup V_2$, where $V_1 = \{u, u_1, u_2, \ldots, u_n\}$ be the vertex set of DW_n, where u is the apex vertex, $u_1, u_2, \ldots, u_n$ inner vertex set, $v_1, v_2, \ldots, v_n$ are outer vertex set of DW_n respectively and $V_2 = \{w, w_1, w_2, \ldots, w_n\}$ be the vertex set of $K_{1,n}$, where $w_1, w_2, \ldots, w_n$ are pendant vertices and $w = v_1$.

Note that $|V(DW_n \oplus K_{1,n})| = 3n + 1$ and $|E(DW_n \oplus K_{1,n})| = 5n$.

I define labeling function $f : V(DW_n \oplus K_{1,n}) \to \{0, 1\}$, as follows:

$$f(u) =1, \quad f(u_i) = 0, \quad f(v_i) = 1,$$

$$f(w_i) = \begin{cases} 0, & 1 \leq i \leq \left\lceil \dfrac{k}{2} \right\rceil, \\ 1, & \left\lceil \dfrac{k}{2} \right\rceil + 1 \leq i \leq k. \end{cases}$$

Thus I have $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$.

Hence the graph $DW_n \oplus K_{1,n}$ is product cordial graph. □

Example 2.4. The product cordial labeling of $DW_7 \oplus K_{1,7}$ is shown in Figure 4.

Theorem 2.5. *The graph $DF_n \oplus K_{1,n}$ is product cordial graph.*

Proof. Let $V(DF_n \oplus K_{1,n}) = V_1 \cup V_2$, where $V_1 = \{u, w, u_1, u_2, \ldots, u_n\}$ be the vertex set of DF_n and $V_2 = \{v = w, v_1, v_2, \ldots, v_n\}$ be the vertex set of $K_{1,n}$. Here $v_1, v_2, \ldots, v_n$ are pendant vertices and v be the apex vertex of $K_{1,n}$. Also $|V(DF_n \oplus K_{1,n})| = 2n+2$, $|E(DF_n \oplus K_{1,n})| = 4n - 1$.

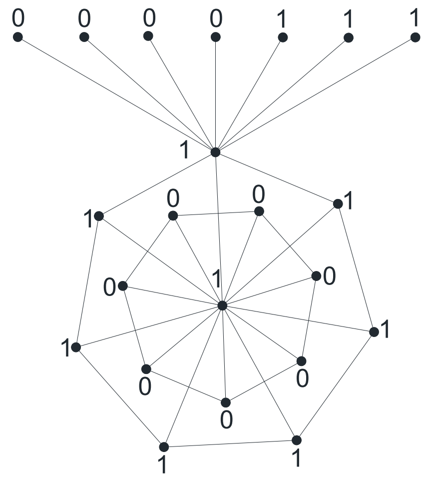

FIGURE 4. Product cordial labeling of $DW_7 \oplus K_{1,7}$

I define labeling function $f : V(DF_n \oplus K_{1,n}) \to \{0, 1\}\}$, as follows. For all $1 \le i \le n$,

$$f(u) = 1, \quad f(w) = 0, \quad f(u_i) = 1, \quad f(v_i) = 0.$$

In view of above defined labeling pattern I have $v_f(0) = v_f(1) = n+1$ and $e_f(0) = 2n$ and $e_f(1) = 2n - 1$. Therefore $|e_f(0) - e_f(1)| \le 1$.

Thus, the graph $DF_n \oplus K_{1,n}$ is a product cordial graph. □

Example 2.5. The product cordial labeling of $DF_n \oplus K_{1,n}$ is shown in Figure 5.

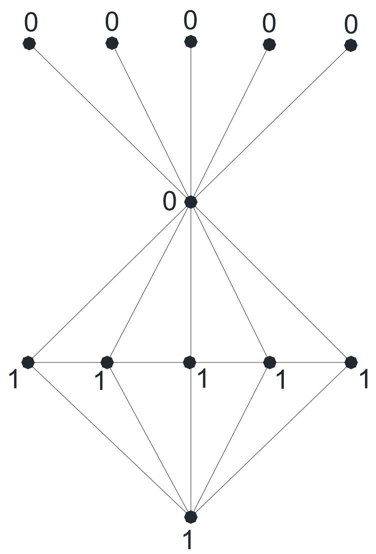

FIGURE 5. Product cordial labeling of $DF_n \oplus K_{1,n}$

3. Conclusion

In this paper I investigated five new product cordial graphs. The results proved in this paper are novel. Example are provided at the end of each theorem for better understanding of the labeling pattern defined in each theorem. To explore some new product cordial graphs is an open problem.

References

[1] F. Harary, *Graph Theory*, Addision-Wesley, Reading, Massachusetts, 1969.

[2] J. A. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin. **19** (2012), 1–260.

[3] I. Cahit, *Cordial graphs: a weaker version of graceful and harmonic graphs*, Ars Combin. **23** (1987), 201–207.

[4] A. H. Rokad and G. V. Ghodasara, *Product cordial labeling of some cycle related graphs*, International Journal of Innovative Research in Science, Engineering and Technology **4**(11) (2015), 11590–11594.

[5] R. Ponraj, M. Sundaram and M. Sivakumar, *k-product cordial labeling of graphs*, International Journal of Contemporary Mathematical Sciences **7**(15) (2012), 733–742.

[6] S. K. Vaidya and C. M. Barasara, *Further results on product cordial graphs*, Int. J. Math. Soft. Comput. **2**(2) (2012), 67–74.

[7] S. K. Vaidya and N. A. Dani, *Some new product cordial graphs*, Journal of Applied Computer Science & Mathematics **4**(8) (2010), 62–65.

[8] S. K. Vaidya and K. K. Kanani, *Some cycle related product cordial graphs*, International Journal of Algorithms, Computing and Mathematics **3**(1) (2010), 109–116.

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