APPLICATIONS OF FRACTIONAL DERIVATIVE ON A DIFFERENTIAL SUBORDINATIONS AND SUPERORDINATORS FOR ANALYTIC FUNCTIONS ASSOCIATED WITH DIFFERENTIAL OPERATOR

ABBAS KAREEM WANAS[1] AND MASLINA DARUS[2]

ABSTRACT. The purpose of this paper is to derive subordination and superordination results involving fractional derivative of differential operator for analytic functions in the open unit disk. These results are applied to obtain sandwich results. Our results extend corresponding previously known results.

1. Introduction and Preliminaries

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of analytic functions in the open unit disk $U = \{z \in C : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \quad a \in \mathbb{C}.$$

Also, let A be the subclass of $\mathcal{H}$ consisting of functions of the form:

$$(1.1) \qquad f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let $f, g \in \mathcal{H}$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$, $z \in U$. It is well known that, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$. Let $p, h \in \mathcal{H}$ and

$\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If p and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second-order differential superordination

$$(1.2) \qquad h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z),$$

then p is called a solution of the differential superordination (1.2). An analytic function q is called a subordinate of (1.2), if $q \prec p$ for all p satisfying (1.2). An univalent subordinat $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinants q of (1.2) is called the best subordinant.

Miller and Mocanu [6] obtained conditions on the functions h, q and ψ for which the following implication holds:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

Ali et al. [1] have used the results of Bulboacă [3] to obtain sufficient conditions for certain normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

Also, Tuneski [16] obtain sufficient condition for starlikeness of $f \in A$ in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Shanmugam et al. [14], Goyal et al. [4], Wanas [17, 18] and Attiya and Yassen [2] have obtained sandwich results for certain classes of analytic functions.

Definition 1.1 ([9]). For $f \in A$ the operator $I_{\lambda_1,\lambda_2,\ell,d}^{n,m}$ is defined by $I_{\lambda_1,\lambda_2,\ell,d}^{n,m} : A \to A$,

$$I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z) = \mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}(z) * R^n f(z), \quad z \in U,$$

where

$$\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{m}(z) = z + \sum_{k=2}^{\infty} \left[\frac{\ell(1 + (\lambda_1 + \lambda_2)(k-1)) + d}{\ell(1 + \lambda_2(k-1)) + d} \right]^m z^k,$$

and $R^n f(z)$ denotes the Ruscheweyh derivative operator [10] given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} C(n, k) a_k z^k,$$

where $C(n, k) = \frac{\Gamma(k+n)}{\Gamma(n+1)\Gamma(k)}$, $n, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda_2 \geq \lambda_1 \geq 0$, $\ell \geq 0$ and $\ell + d > 0$.

If f given by (1.1), then we easily find that

$$I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+n)}{\Gamma(n+1)\Gamma(k)} \left[\frac{\ell(1 + (\lambda_1 + \lambda_2)(k-1)) + d}{\ell(1 + \lambda_2(k-1)) + d} \right]^m a_k z^k.$$

Definition 1.2 ([15]). The fractional derivative of order δ, $0 \leq \delta < 1$, of a function f is defined by

$$D_z^\delta f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\delta} \, dt,$$

where the function f is analytic in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z - t)^{-\delta}$ is removed by requiring $\log(z - t)$ to be real, when $\mathrm{Re}(z - t) > 0$.

From Definition 1.1 and Definition 1.2, we have

$$(1.3) \qquad D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z) = \frac{1}{\Gamma(2 - \delta)} z^{1-\delta} + \sum_{k=2}^{\infty} \frac{k\Gamma(n + k)}{\Gamma(k - \delta + 1)\Gamma(n + 1)}$$
$$\times \left[\frac{\ell(1 + (\lambda_1 + \lambda_2)(k - 1)) + d}{\ell(1 + \lambda_2(k - 1)) + d} \right]^m a_k z^{k-\delta}.$$

It follows from (1.3) that

(1.4)
$$\ell\lambda_1 z \left(D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z) \right)' = [\ell(1 + (\lambda_2(k - 1)) + d] D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)$$
$$- [\ell(1 + (\lambda_2(k - 1) - (1 - \delta)\lambda_1) + d] D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z).$$

In order to prove our results, we make use of the following known results.

Definition 1.3 ([5]). Denote by Q the set of all functions f that are analytic and injective on $\bar{U} \backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.

Lemma 1.1 ([5]). *Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(1) $Q(z)$ is starlike univalent in U;
(2) $\mathrm{Re}\left\{ \frac{zh'(z)}{Q(z)} \right\} > 0$ for $z \in U$.

If p is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$ and

$$(1.5) \qquad \theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p \prec q$ and q is the best dominant of (1.5).

Lemma 1.2 ([6]). *Let q be a convex univalent function in U and let $\alpha \in C, \beta \in C \backslash \{0\}$, with*

$$\mathrm{Re}\left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max\left\{ 0, -\mathrm{Re}\left(\frac{\alpha}{\beta} \right) \right\}.$$

If p is analytic in U and

$$(1.6) \qquad \alpha p(z) + \beta z p'(z) \prec \alpha q(z) + \beta z q'(z),$$

then $p \prec q$ and q is the best dominant of (1.6).

Lemma 1.3 ([6]). *Let q be a convex univalent function in U and let $\beta \in \mathbb{C}$. Further assume that $\mathrm{Re}(\beta) > 0$. If $p \in \mathcal{H}[q(0), 1] \cap Q$ and $p(z) + \beta z p'(z)$ is univalent in U, then*

$$(1.7) \qquad\qquad q(z) + \beta z q'(z) \prec p(z) + \beta z p'(z),$$

which implies that $q \prec p$ and q is the best subordinant of (1.7).

Lemma 1.4 ([3]). *Let q be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

(1) $\mathrm{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ *for $z \in U$;*
(2) $Q(z) = z q'(z) \phi(q(z))$ *is starlike univalent in U.*

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subset D, \theta(p(z)) + z p'(z) \phi(p(z))$ is univalent in U, and

$$(1.8) \qquad \theta(q(z)) + z q'(z) \phi(q(z)) \prec \theta(p(z)) + z p'(z) \phi(p(z)),$$

then $q \prec p$ and q is the best subordinant of (1.8).

2. Subordination Results

Theorem 2.1. *Let q be convex univalent in U with $q(0) = 1$, $\sigma \in \mathbb{C} \setminus \{0\}$, $\gamma > 0$ and suppose that q satisfies*

$$(2.1) \qquad\qquad \mathrm{Re}\left\{1 + \frac{z q''(z)}{q'(z)}\right\} > \max\left\{0, -\mathrm{Re}\left(\frac{(1-\delta)\gamma}{\sigma}\right)\right\}.$$

If $f \in A$ satisfies the subordination

$$(2.2) \quad \left(1 - \frac{\sigma\left[\ell(1 + (\lambda_2(k-1)) + d\right]}{\ell \lambda_1(1-\delta)}\right)\left(\frac{\Gamma(2-\delta)\, D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^\gamma$$

$$+ \frac{\sigma\left[\ell(1 + (\lambda_2(k-1)) + d\right]}{\ell \lambda_1(1-\delta)}\left(\frac{\Gamma(2-\delta)\, D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^\gamma\left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)$$

$$\prec q(z) + \frac{\sigma}{(1-\delta)\gamma} z q'(z),$$

then

$$(2.3) \qquad\qquad \left(\frac{\Gamma(2-\delta)\, D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^\gamma \prec q(z)$$

and q is the best dominant of (2.2).

Proof. Define the function p by

$$(2.4) \qquad\qquad p(z) = \left(\frac{\Gamma(2-\delta)\, D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^\gamma, \quad z \in U.$$

Then the function p is analytic in U and $p(0) = 1$. Differentiating (2.4) logarithmically with respect to z, we have

$$\frac{zp'(z)}{p(z)} = \gamma \left(\frac{z \left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z) \right)'}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)} - (1 - \delta) \right).$$

Now, in view of (1.4), we obtain

$$\frac{zp'(z)}{p(z)} = \frac{\gamma \left[\ell(1 + (\lambda_2(k-1))) + d \right]}{\ell \lambda_1} \left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)} - 1 \right).$$

Therefore,

$$\frac{zp'(z)}{(1-\delta)\gamma} = \frac{\ell(1 + (\lambda_2(k-1))) + d}{\ell \lambda_1 (1 - \delta)} \left(\frac{\Gamma(2-\delta)\, D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}} \right)^\gamma$$
$$\times \left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)} - 1 \right).$$

It follows from (2.2) that

$$p(z) + \frac{\sigma}{(1-\delta)\gamma} zp'(z) \prec q(z) + \frac{\sigma}{(1-\delta)\gamma} zq'(z).$$

Thus, an application of Lemma 1.2, with $\alpha = 1$ and $\beta = \frac{\sigma}{(1-\delta)\gamma}$, we obtain (2.3). □

Theorem 2.2. *Let $\eta_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, $\gamma > 0$, $t \in \mathbb{C} \setminus \{0\}$ and q be convex univalent in U with $q(0) = 1$, $q(z) \neq 0$, $z \in U$, and assume that q satisfies*

$$(2.5) \qquad \mathrm{Re} \left\{ 1 + \frac{\eta_2}{t} q(z) + \frac{2\eta_3}{t} q^2(z) + \frac{3\eta_4}{t} q^3(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0.$$

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. If $f \in A$ satisfies
(2.6)
$$\Psi(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z) \prec \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t \frac{zq'(z)}{q(z)},$$

where
(2.7)

$$\Psi(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z)$$
$$= \eta_1 + \eta_2 \left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)} \right)^\gamma + \eta_3 \left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)} \right)^{2\gamma} + \eta_4 \left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)} \right)^{3\gamma}$$
$$+ \frac{\gamma t \left[\ell(1 + (\lambda_2(k-1))) + d \right]}{\ell \lambda_1} \left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+2} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)} - \frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)} \right),$$

then

$$\left(\frac{D_z^{\delta}I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1}f(z)}{D_z^{\delta}I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}\right)^{\gamma} \prec q(z)$$

and q is the best dominant of (2.6).

Proof. Define the function p by

(2.8) $$p(z) = \left(\frac{D_z^{\delta}I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1}f(z)}{D_z^{\delta}I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}\right)^{\gamma}, \quad z \in U.$$

Then the function p is analytic in U and $p(0) = 1$.

By a straightforward computation and using (1.4), we have
(2.9)
$$\eta_1 + \eta_2 p(z) + \eta_3 p^2(z) + \eta_4 p^3(z) + t\frac{zp'(z)}{p(z)} = \Psi\left(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right),$$

where $\Psi\left(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ is given by (2.7). From (2.6) and (2.9), we obtain

$$\eta_1 + \eta_2 p(z) + \eta_3 p^2(z) + \eta_4 p^3(z) + t\frac{zp'(z)}{p(z)} \prec \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)}.$$

By setting

$$\theta(w) = \eta_1 + \eta_2 w + \eta_3 w^2 + \eta_4 w^3 \quad \text{and} \quad \phi(w) = \frac{t}{w}, \quad w \neq 0,$$

we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = t\frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U,

$$\text{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \text{Re}\left\{1 + \frac{\eta_2}{t}q(z) + \frac{2\eta_3}{t}q^2(z) + \frac{3\eta_4}{t}q^3(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0.$$

Thus, by Lemma 1.1, we get $p(z) \prec q(z)$. By using (2.8), we obtain the desired result. $\qquad\square$

Theorem 2.3. *Let $\eta_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, $t \in V \setminus \{0\}$ and q be convex univalent in U with $q(0) = 1$, $q(z) \neq 0$, $z \in U$, and assume that q satisfies* (2.5). *Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. If $f \in A$ satisfies*
(2.10)

$$\Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right) \prec \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)},$$

where

(2.11)

$$\Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$$

$$=\eta_1 + \eta_2 \frac{z^{1-\delta} D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{\Gamma(2-\delta)\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)\right)^2} + \eta_3 \left(\frac{1}{\Gamma(2-\delta)}\right)^2 \frac{z^{2(1-\delta)}\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)\right)^2}{\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)\right)^4}$$

$$+\eta_4 \left(\frac{1}{\Gamma(2-\delta)}\right)^3 \frac{z^{3(1-\delta)}\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)\right)^3}{\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)\right)^6} + \frac{t\left[\ell(1+(\lambda_2(k-1)))+d\right]}{\ell\lambda_1}\times$$

$$\times \left(1 + \frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+2} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)} - \frac{2D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right),$$

then

$$\frac{z^{1-\delta} D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{\Gamma(2-\delta)\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)\right)^2} \prec q(z)$$

and q is the best dominant of (2.10).

Proof. Define the function p by

(2.12)
$$p(z) = \frac{z^{1-\delta} D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{\Gamma(2-\delta)\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)\right)^2}, \quad z \in U.$$

Then the function p is analytic in U and $p(0) = 1$.

We note that

(2.13)
$$\eta_1 + \eta_2 p(z) + \eta_3 p^2(z) + \eta_4 p^3(z) + t\frac{zp'(z)}{p(z)} = \Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right),$$

where $\Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ is given by (2.11). From (2.10) and (2.13), we obtain

$$\eta_1 + \eta_2 p(z) + \eta_3 p^2(z) + \eta_4 p^3(z) + t\frac{zp'(z)}{p(z)} \prec \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)}.$$

The remaining part of the proof Theorem 2.3 is similar to that of Theorem 2.2 and hence we omit it. □

3. Superordination Results

Theorem 3.1. *Let q be convex univalent in U with $q(0) = 1$, $\gamma > 0$ and* $\mathrm{Re}\left\{\sigma\right\} > 0$. *Let $f \in A$ satisfies*

$$\left(\frac{\Gamma(2-\delta)\, D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^\gamma \in \mathcal{H}\left[q(0), 1\right] \cap Q$$

and

$$\left(1 - \frac{\sigma\left[\ell(1 + (\lambda_2(k-1))) + d\right]}{\ell\lambda_1(1-\delta)}\right)\left(\frac{\Gamma(2-\delta)\, D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}{z^{1-\delta}}\right)^\gamma$$

$$+ \frac{\sigma\left[\ell(1 + (\lambda_2(k-1))) + d\right]}{\ell\lambda_1(1-\delta)}\left(\frac{\Gamma(2-\delta)\, D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}{z^{1-\delta}}\right)^\gamma\left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1}f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}\right)$$

be univalent in U. If

$$(3.1)\quad q(z) + \frac{\sigma}{(1-\delta)\gamma}zq'(z)$$

$$\prec \left(1 - \frac{\sigma\left[\ell(1 + (\lambda_2(k-1))) + d\right]}{\ell\lambda_1(1-\delta)}\right)\left(\frac{\Gamma(2-\delta)\, D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}{z^{1-\delta}}\right)^\gamma$$

$$+ \frac{\sigma\left[\ell(1 + (\lambda_2(k-1))) + d\right]}{\ell\lambda_1(1-\delta)}\left(\frac{\Gamma(2-\delta)\, D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}{z^{1-\delta}}\right)^\gamma\left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1}f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}\right),$$

then

$$(3.2)\qquad\qquad q(z) \prec \left(\frac{\Gamma(2-\delta)\, D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}{z^{1-\delta}}\right)^\gamma$$

and q is the best subordinant of (3.1).

Proof. Define the function p by

$$(3.3)\qquad\qquad p(z) = \left(\frac{\Gamma(2-\delta)\, D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}{z^{1-\delta}}\right)^\gamma, \quad z \in U.$$

Then the function p is analytic in U and $p(0) = 1$. Differentiating (3.3) logarithmically with respect to z, we get

$$\frac{zp'(z)}{p(z)} = \gamma\left(\frac{z\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)\right)'}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)} - (1-\delta)\right).$$

After some computations and using (1.4), we find that

$$(3.4)\quad \left(1 - \frac{\sigma\left[\ell(1 + (\lambda_2(k-1))) + d\right]}{\ell\lambda_1(1-\delta)}\right)\left(\frac{\Gamma(2-\delta)\, D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}{z^{1-\delta}}\right)^\gamma$$

$$+ \frac{\sigma\left[\ell(1 + (\lambda_2(k-1))) + d\right]}{\ell\lambda_1(1-\delta)}\left(\frac{\Gamma(2-\delta)\, D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}{z^{1-\delta}}\right)^\gamma\left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1}f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}\right)$$

$$= p(z) + \frac{\sigma}{(1-\delta)\gamma}zp'(z).$$

From (3.1) and (3.4), we have

$$q(z) + \frac{\sigma}{(1-\delta)\gamma}zq'(z) \prec p(z) + \frac{\sigma}{(1-\delta)\gamma}zp'(z).$$

Thus, an application of Lemma 1.3, with $\alpha = 1$ and $\beta = \frac{\sigma}{(1-\delta)\gamma}$, we obtain the results. $\square$

Theorem 3.2. *Let $\eta_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, $\gamma > 0$, $t \in \mathbb{C} \setminus \{0\}$ and q be convex univalent in U with $q(0) = 1$, $q(z) \neq 0$, $z \in U$, and assume that q satisfies*

$$(3.5) \qquad \operatorname{Re}\left\{\frac{\eta_2}{t}q(z) + \frac{2\eta_3}{t}q^2(z) + \frac{3\eta_4}{t}q^3(z)\right\} > 0.$$

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let $f \in A$ satisfies

$$\left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)^\gamma \in H\left[q(0), 1\right] \cap Q$$

and $\quad \Psi\left(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right) \quad$ *is* $\quad$ *univalent* $\quad$ *in* $\quad U$, $\quad$ *where* $\Psi\left(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ *is given by (2.7). If*
(3.6)

$$\eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)} \prec \Psi\left(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right),$$

then

$$q(z) \prec \left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)^\gamma$$

and q is the best subordinant of (3.6).

Proof. Define the function p by

$$(3.7) \qquad\qquad p(z) = \left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)^\gamma, \quad z \in U.$$

Then the function p is analytic in U and $p(0) = 1$.

By some computation, we have
(3.8)

$$\Psi\left(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right) = \eta_1 + \eta_2 p(z) + \eta_3 p^2(z) + \eta_4 p^3(z) + t\frac{zp'(z)}{p(z)},$$

where $\Psi\left(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ is given by (2.7). From (3.6) and (3.8), we obtain

$$\eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)} \prec \eta_1 + \eta_2 p(z) + \eta_3 p^2(z) + \eta_4 p^3(z) + t\frac{zp'(z)}{p(z)}.$$

By setting $\theta(w) = \eta_1 + \eta_2 w + \eta_3 w^2 + \eta_4 w^3$ and $\phi(w) = \frac{t}{w}$, $w \neq 0$, we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = t\frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U,

$$\text{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = \text{Re}\left\{\frac{\eta_2}{t}q(z) + \frac{2\eta_3}{t}q^2(z) + \frac{3\eta_4}{t}q^3(z)\right\} > 0.$$

Thus, by Lemma 1.4, we get $q(z) \prec p(z)$. By using (3.7), we obtain the desired result. □

Theorem 3.3. *Let $\eta_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, $t \in \mathbb{C} \setminus \{0\}$ and q be convex univalent in U with $q(0) = 1$, $q(z) \neq 0$, $z \in U$, and assume that q satisfies (3.5). Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let $f \in A$ satisfies*

$$\frac{z^{1-\delta}D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1}f(z)}{\Gamma(2-\delta)\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)\right)^2} \in H\left[q(0), 1\right] \cap Q$$

and $\Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ is univalent in U, where $\Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ is given by (2.11). If
(3.9)

$$\eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)} \prec \Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right),$$

then

$$q(z) \prec \frac{z^{1-\delta}D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1}f(z)}{\Gamma(2-\delta)\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)\right)^2}$$

and q is the best subordinant of (3.9).

Proof. Define the function p by

(3.10)
$$p(z) = \frac{z^{1-\delta}D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1}f(z)}{\Gamma(2-\delta)\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)\right)^2}, \quad z \in U.$$

Then the function p is analytic in U and $p(0) = 1$.

We note that
(3.11)

$$\Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right) = \eta_1 + \eta_2 p(z) + \eta_3 p^2(z) + \eta_4 p^3(z) + t\frac{zp'(z)}{p(z)},$$

where $\Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ is given by (2.11). From (3.9) and (3.11), we obtain

$$\eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)} \prec \eta_1 + \eta_2 p(z) + \eta_3 p^2(z) + \eta_4 p^3(z) + t\frac{zp'(z)}{p(z)}.$$

The remaining part of the proof Theorem 3.3 is similar to that of Theorem 3.2 and hence we omit it. □

4. Sandwich Results

Combining results of differential subordinations and superordinations, we state the following "sandwich results".

Theorem 4.1. *Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$. Suppose q_2 satisfies (2.1), $\gamma > 0$ and $\mathrm{Re}\{\sigma\} > 0$. Let $f \in A$ satisfies*

$$\left(\frac{\Gamma(2-\delta)\, D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^{\gamma} \in H[1,1] \cap Q$$

and

$$\left(1 - \frac{\sigma\left[\ell(1+(\lambda_2(k-1)))+d\right]}{\ell\lambda_1(1-\delta)}\right)\left(\frac{\Gamma(2-\delta)\, D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^{\gamma}$$
$$+ \frac{\sigma\left[\ell(1+(\lambda_2(k-1)))+d\right]}{\ell\lambda_1(1-\delta)}\left(\frac{\Gamma(2-\delta)\, D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^{\gamma}\left(\frac{D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)$$

be univalent in U. If

$$q_1(z) + \frac{\sigma}{(1-\delta)\gamma} z q_1'(z)$$
$$\prec \left(1 - \frac{\sigma\left[\ell(1+(\lambda_2(k-1)))+d\right]}{\ell\lambda_1(1-\delta)}\right)\left(\frac{\Gamma(2-\delta)\, D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^{\gamma}$$
$$+ \frac{\sigma\left[\ell(1+(\lambda_2(k-1)))+d\right]}{\ell\lambda_1(1-\delta)}\left(\frac{\Gamma(2-\delta)\, D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^{\gamma}\left(\frac{D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)$$
$$\prec q_2(z) + \frac{\sigma}{(1-\delta)\gamma} z q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{\Gamma(2-\delta)\, D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^{\gamma} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Theorem 4.2. *Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$. Suppose q_1 satisfies (3.5) and q_2 satisfies (2.5). Let $f \in A$ satisfies*

$$\left(\frac{D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)^{\gamma} \in H[1,1] \cap Q$$

and $\Psi\left(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ *is   univalent   in   U,   where* $\Psi\left(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ *is given by (2.7). If*

$$\eta_1 + \eta_2 q_1(z) + \eta_3 q_1^2(z) + \eta_4 q_1^3(z) + t\frac{zq_1'(z)}{q_1(z)} \prec \Psi\left(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$$
$$\prec \eta_1 + \eta_2 q_2(z) + \eta_3 q_2^2(z) + \eta_4 q_2^3(z)$$
$$+ t\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)^\gamma \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Theorem 4.3. *Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$. Suppose q_1 satisfies (3.5) and q_2 satisfies (2.5). Let $f \in A$ satisfies*

$$\frac{z^{1-\delta} D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{\Gamma(2-\delta)\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)\right)^2} \in H[1,1] \cap Q$$

and $\Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ *is   univalent   in   U,   where* $\Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ *is given by (2.11). If*

$$\eta_1 + \eta_2 q_1(z) + \eta_3 q_1^2(z) + \eta_4 q_1^3(z) + t\frac{zq_1'(z)}{q_1(z)} \prec \Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$$
$$\prec \eta_1 + \eta_2 q_2(z) + \eta_3 q_2^2(z) + \eta_4 q_2^3(z)$$
$$+ t\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \frac{z^{1-\delta} D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{\Gamma(2-\delta)\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)\right)^2} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Remark 4.1. By specifying the function ϕ and selecting the particular values of $\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, n, m, \lambda_1, \lambda_2, \ell$ and d, we can derive a number of known results. Some of them are given below.

  (1) Taking $\delta = n = \lambda_2 = d = 0$ and $\ell = 1$ in Theorems 2.1, 3.1, 4.1, we get the results obtained by Răducanu and Nechita [10, Theorem 3.1, Theorem 3.6, Theorem 3.9].
  (2) Putting $\delta = n = \lambda_2 = \eta_1 = \eta_3 = \eta_4 = d = 0$, $\eta_2 = \ell = 1$ and $\phi(w) = t$ in Theorems 2.3, 3.3, 4.3, we obtain the results obtained by Nechita [8, Theorem 14, Theorem 19, Corollary 21].

(3) For $\delta = n = \lambda_2 = \eta_1 = \eta_3 = \eta_4 = d = 0$, $\lambda_1 = \eta_2 = \ell = 1$ and $\phi(w) = t$ in Theorems 2.3, 3.3, 4.3, we have the results obtained by Shanmugam et al. [13, Theorem 5.4, Theorem 5.5, Theorem 5.6].

(4) By taking $\delta = n = m = \lambda_2 = \eta_1 = \eta_3 = \eta_4 = d = 0$, $\lambda_1 = \eta_2 = \ell = 1$ and $\phi(w) = t$ in Theorems 2.3, 3.3, 4.3, we get the results obtained by Shanmugam et al. [13, Theorem 3.4, Theorem 3.5, Theorem 3.6].

(5) Putting $\delta = n = \lambda_2 = \eta_1 = \eta_3 = \eta_4 = 0$, $\eta_2 = \ell = 1$ and $\phi(w) = t$ in Theorems 2.3, 3.3, 4.3, we have the results obtained by Shammaky [12, Theorem 3.4, Theorem 3.5, Theorem 3.6].

(6) Taking $\delta = n = m = \lambda_2 = d = 0$ and $\lambda_1 = \ell = 1$ in Theorem 2.1, we obtain the results obtained by Murugusundaramoorthy and Magesh [7, Corollary 3.3].

(7) Putting $\delta = n = m = \lambda_2 = d = 0$ and $\lambda_1 = \ell = 1$ in Theorems 3.1, 4.1, we obtain the results obtained by Răducanu and Nechita [10, Corollary 3.7, Corollary 3.10].

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[1]DEPARTMENT OF MATHEMATICS,
COLLEGE OF SCIENCE,
UNIVERSITY OF AL-QADISIYAH,
DIWANIYA, IRAQ
Email address: abbas.kareem.w@qu.edu.iq

[2]SCHOOL OF MATHEMATICAL SCIENCES,
FACULTY OF SCIENCE AND TECHNOLOGY,
UNIVERSITI KEBANGSAAN MALAYSIA,
BANGI 43600,SELANGOR D. EHSAN, MALAYSIA
Email address: maslina@ukm.edu.my