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LOGARITHMICALLY COMPLETE MONOTONICITY OF RECIPROCAL ARCTAN FUNCTION

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Abstract. We prove the conjecture stated in F. Qi and R. Agarwal, *On complete monotonicity for several classes of functions related to ratios of gamma functions*, J. Inequal. Appl. (2019), that the function $1/\arctan$ is logarithmically completely monotonic on $(0,\infty)$, but not a Stieltjes transform.

1. Introduction

By a *completely monotonic function* (shortly CM) we mean here an infinitely differentiable function $f:(0,\infty)\to\mathbb{R}$, such that

$$(-1)^n f^{(n)} \geq 0, \quad n=0,1,2,\ldots$$

If f' is completely monotonic and $f\geq 0$, then we call f a *Bernstein function*. Here we are mostly interested in *logarithmically completely monotonic functions*, that is, infinitely differentiable functions $f:(0,\infty)\to(0,\infty)$ with the property

$$(-1)^n(\log f)^{(n)} \geq 0, \quad n=1,2,3,\ldots$$

A basic fact concerning CM - functions is the Bernstein theorem: a function f is CM if and only if there exists a non-decreasing function α on $(0,\infty)$ satisfying

$$f(x)=\int_0^\infty e^{-xt}d\alpha(t),$$

for all $x>0$ (see [9, p. 161]). In some occasions it has been proven a stronger property which leads to complete monotonicity of a function f, namely that there exist $a\geq 0$

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and a non-negative Borel measure μ on $[0,\infty)$ for which the equality

$$f(x) = a + \int_0^\infty \frac{d\mu(t)}{x+t}$$

holds for $x > 0$, where the measure μ fulfills the condition

$$\int_0^\infty \frac{d\mu(t)}{1+t} < \infty.$$

Such functions are called *Stieltjes transforms.* We recall that all Stieltjes transforms are logarithmically completely monotonic (see [2] and further generalizations [3]), and the latter are CM (see [5], but also [7] and [8]).

In [6] the authors set the conjecture that the function $f(x) = \frac{1}{\arctan x}$ is logarithmically completely monotonic on $(0,\infty)$, but not a Stieltjes transform. The aim of this paper is to justify these assertions. We will do it in the next section.

2. Formulations and Proofs

Theorem 2.1. *The function $f(x) = \frac{1}{\arctan x}$ is logarithmically completely monotonic on $(0,\infty)$.*

The idea of the proof of Theorem 2.1 is based on the Remark 1 in [1], where the authors suggest employing the residue theorem in an attempt to obtain integral representations of functions under consideration.

Proof. It suffices to prove that

$$g(x) = -(\log f(x))' = \frac{1}{(x^2+1)\arctan x}$$

is CM on $(0,\infty)$. In what follows we always assume that log denotes the principle value of logarithm, i.e., $\log z = \ln|z| + i \arg z$, with $\arg z \in (-\pi,\pi]$.

Let us consider the integral $\int_{\Gamma_{R,r}} G(z)\,dz$, over the "keyhole" contour $\Gamma_{R,r}$ given in Figure 1, where

$$G(z) = \frac{z+1}{z(z-z_0)\log z}$$

and $z_0 = \frac{i-x}{i+x}$ for $x > 0$.

We assume $R > 1$ and $r < 1$. Note that $|z_0| = 1$ and that 1, z_0 are the only singularities of G lying inside $\Gamma_{R,r}$. From the residue theorem, we have

$$\int_{\Gamma_{R,r}} G(z)\,dz = 2\pi i(\operatorname{Res}(G(z);z_0) + \operatorname{Res}(G(z);1)).$$

Since z_0 is a first-order pole, it follows

$$\operatorname{Res}(G(z);z_0) = \frac{1+z_0}{z_0 \log z_0} = \frac{1+\frac{i-x}{i+x}}{\frac{i-x}{i+x}\log\frac{i-x}{i+x}} = \frac{2i}{(i-x)2i\arctan x} = -\frac{(i+x)}{(x^2+1)\arctan x},$$

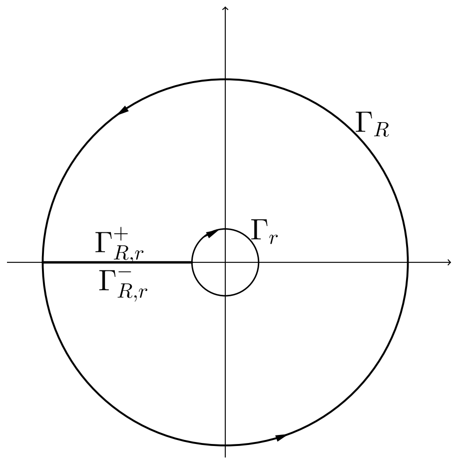

FIGURE 1. Keyhole contour $\Gamma_{R,r}$

where we used the fact that $\arctan x = \frac{1}{2i} \log \frac{1+ix}{1-ix}$, for $x > 0$. Similarly,

$$\operatorname{Res}(G(z); 1) = \lim_{z \to 1} (z-1) \frac{1+z}{z(z-z_0)\log z} = \frac{2}{1-z_0} = \frac{2}{1-\frac{i-x}{i+x}} = \frac{i+x}{x},$$

whence,

$$g(x) = \frac{1}{x} - \frac{1}{2\pi i(x+i)} \int_{\Gamma_{R,r}} G(z)\, dz. \tag{2.1}$$

Now, it remains to calculate the integral $\int_{\Gamma_{R,r}} G(z)\, dz$. In order to accomplish it, we start from the relation

$$\int_{\Gamma_{R,r}} G(z)\, dz = \int_{\Gamma_R} G(z)\, dz + \int_{\Gamma_r} G(z)\, dz + \int_{\Gamma_{R,r}^+} G(z)\, dz + \int_{\Gamma_{R,r}^-} G(z)\, dz. \tag{2.2}$$

The first two integrals vanish as $R \to \infty$ and $r \to 0+$. It follows from the estimates

$$\left| \int_{\Gamma_R} G(z)\, dz \right| \le 2R\pi \max_{|z|=R} \frac{|z+1|}{|z||\log z||z-z_0|} \le 2\pi \frac{R+1}{(\ln R - 2\pi)(R-1)}$$

and

$$\left| \int_{\Gamma_r} G(z)\, dz \right| \le 2r\pi \max_{|z|=r} \frac{|z+1|}{|z||\log z||z-z_0|} \le 2\pi \frac{1+r}{(-\ln r - 2\pi)(1-r)}.$$

We also have for $t < 0$

$$\lim_{\substack{z \to t \\ \Im z > 0}} G(z) = \frac{t+1}{t(\ln(-t) + \pi i)(t - z_0)} = G^+(t)$$

and

$$\lim_{\substack{z\to t\\ \Im z<0}} G(z) = \frac{t+1}{t(\ln(-t)-\pi i)(t-z_0)} = G^-(t).$$

Consequently,

$$\int_{\Gamma^+_{R,r}} G(z)\,dz + \int_{\Gamma^-_{R,r}} G(z)\,dz = \int_{-R}^{-r} [G^+(t)-G^-(t)]\,dt. \tag{2.3}$$

Let us denote $I = \lim\limits_{\substack{R\to\infty\\ r\to 0+}} \int_{\Gamma_{R,r}} G(z)\,dz$. From (2.2) and (2.3) we obtain

$$\begin{aligned} I &= \int_{-\infty}^{0} [G^+(t)-G^-(t)]\,dt \\ &= \int_{-\infty}^{0} \frac{2\pi i(t+1)\,dt}{t(\log^2(-t)+\pi^2)(t-z_0)} \\ &= 2\pi i \int_0^\infty \frac{(1-t)\,dt}{t(\log^2 t+\pi^2)(t+z_0)}. \end{aligned}$$

Using $z_0 = \frac{i-x}{i+x}$, we have

$$\begin{aligned} I &= \int_0^\infty \frac{2\pi i(1-t)\,dt}{t(\log^2 t+\pi^2)(t+\frac{i-x}{i+x})} \\ &= \int_0^\infty \frac{2\pi i(i+x)(1-t)\,dt}{t(\log^2 t+\pi^2)(x(t-1)+i(t+1))} \\ &= -2\pi i(i+x)\int_0^\infty \frac{((1-t)^2 x + i(1-t^2))\,dt}{t(x^2(1-t)^2+(1+t)^2)(\log^2 t+\pi^2)}. \end{aligned}$$

Note that (2.1) implies

$$g(x) = \frac{1}{x} - \frac{1}{2\pi i(x+i)}\, I \tag{2.4}$$

and since $\dfrac{1}{2\pi i(x+i)} I$ is real, we conclude that

$$\int_0^\infty \frac{(1-t^2)\,dt}{t(x^2(1-t)^2+(1+t)^2)(\log^2 t+\pi^2)} = 0.$$

Therefore, from (2.4), it follows

$$g(x) = \frac{1}{x} + \int_0^\infty \frac{(1-t)^2 x\,dt}{t(x^2(1-t)^2+(1+t)^2)(\log^2 t+\pi^2)}. \tag{2.5}$$

Employing

$$\frac{1}{x} = \int_0^\infty \frac{dt}{xt(\log^2 t+\pi^2)},$$

we get

$$g(x) = \int_0^\infty \frac{(2(1-t)^2 x + (1+t)^2)\,dt}{xt(x^2(1-t)^2+(1+t)^2)(\log^2 t+\pi^2)}.$$

The substitution $t \mapsto \frac{1}{t}$ implies

$$\int_0^1 \frac{(2(1-t)^2x+(1+t)^2)\,dt}{xt(x^2(1-t)^2+(1+t)^2)(\log^2 t+\pi^2)} = \int_1^\infty \frac{(2(1-t)^2x+(1+t)^2)\,dt}{xt(x^2(1-t)^2+(1+t)^2)(\log^2 t+\pi^2)}.$$

Hence,

$$g(x) = 2\int_0^1 \frac{(2(1-t)^2x+(1+t)^2)\,dt}{xt(x^2(1-t)^2+(1+t)^2)(\log^2 t+\pi^2)}. \tag{2.6}$$

For $a, b, x > 0$ it is

$$\frac{2a^2x^2+b^2}{x(a^2x^2+b^2)} = \frac{1}{x} + \frac{1}{2}\left(\frac{1}{x+\frac{bi}{a}} + \frac{1}{x-\frac{bi}{a}}\right)$$

and using

$$\frac{1}{x} = \int_0^\infty e^{-xs}\,ds, \quad \frac{1}{x+\frac{bi}{a}} = \int_0^\infty e^{-xs}e^{-\frac{bi}{a}s}\,ds, \quad \frac{1}{x-\frac{bi}{a}} = \int_0^\infty e^{-xs}e^{\frac{bi}{a}s}\,ds,$$

one obtains

$$\frac{2a^2x^2+b^2}{x(a^2x^2+b^2)} = \int_0^\infty e^{-xs}\left(1+\cos\frac{bs}{a}\right)\,ds.$$

Setting $a = 1-t$ and $b = 1+t$ yields

$$\frac{2(1-t)^2x+(1+t)^2}{x(x^2(1-t)^2+(1+t)^2)} = \int_0^\infty e^{-xs}\left(1+\cos\frac{1+t}{1-t}s\right)\,ds.$$

From (2.6), we have

$$g(x) = 2\int_0^1 \left(\int_0^\infty \frac{e^{-xs}(1+\cos\frac{1+t}{1-t}s)\,ds}{t(\ln^2 t+\pi^2)}\right)\,dt,$$

and, finally, after interchanging integration order, we obtain

$$g(x) = \int_0^\infty \left(\int_0^1 \frac{2(1+\cos\frac{1+t}{1-t}s)\,dt}{t(\ln^2 t+\pi^2)}\right) e^{-xs}\,ds. \tag{2.7}$$

Now, it is evident that (2.7) implies complete monotonicity of g. □

Theorem 2.2. *The function $f(x) = \frac{1}{\arctan x}$ is not a Stieltjes transform on $(0,\infty)$.*

For the proof of this theorem, we the use following result on Stieltjes transforms from [4].

Proposition 2.1. *If $f \neq 0$ is a Stieltjes transform, then $\frac{1}{f}$ is a Bernstein function.*

Proof of Theorem 2.2. The function $h(x) = \frac{1}{f(x)} = \arctan x$ is not a Bernstein function, since

$$h^{(3)}(x) = -2\,\frac{3x^2-1}{(1+x^2)^3}$$

changes its sign on $(0,\infty)$. Therefore, according to Proposition 2.1, f is not a Stieltjes transform. □

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