Kragujevac Journal of Mathematics
Volume 43(2) (2019), Pages 293–301.

A NEW PINCHING FOR CLOSED 3-DIMENSIONAL HYPERSURFACES

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Abstract. We will give a new pinching for closed oriented 3-dimensional hypersurfaces immersed in a not necessarily complete space of constant curvature, where we always assume that the hypersurfaces have constant mean curvature H and constant scalar curvature κ. The given assumptions indicate that our result touches the setting of the Chern conjecture for isoparametric hypersurfaces in spheres.

1. Introduction

Let M be a closed oriented 3-dimensional hypersurface with principal curvatures k_1, k_2, k_3 immersed in a space of constant curvature c. In this note we will assume that M has constant mean curvature H and constant scalar curvature κ, given by

$$\begin{cases} 3H = k_1 + k_2 + k_3, \\ 3\kappa = 3c + k_1k_2 + k_1k_3 + k_2k_3. \end{cases} \tag{1.1}$$

The aim of this paper is to establish the following result.

Theorem 1.1. *Let M be a closed oriented 3-dimensional hypersurface with distinct principal curvatures immersed in a not necessarily complete space of constant curvature c. Suppose in addition that M has constant mean curvature H and constant scalar curvature κ. If*

$$6\alpha(H) = -2c - \frac{|H|\left(7|H| + \sqrt{49H^2 + 48c}\right)}{3}$$

then M satisfies one of the following conditions:

a) $\alpha(H) < \kappa < 0$ *and M is not isoparametric;*

Key words and phrases. Minimal surfaces, pinching, constant curvature, Chern conjecture.
2010 *Mathematics Subject Classification.* Primary: 53C42. Secondary: 53C40.
Received: October 31, 2017.
Accepted: Decembar 05, 2017.

b) *$\kappa = 0$ and M is isoparametric.*

For an important space of constant curvature - the sphere - our result is related to Chern conjecture for isoparametric hypersurfaces in spheres which can be stated as follows.

Let M be a closed, minimally immersed hypersurface of the $(n+1)$-dimensional sphere $\mathbb{S}^{n+1}$ with constant scalar curvature. Then M is isoparametric.

It was originally proposed in a less strong version by Chern in [4] and Chern, do Carmo and Kobayashi in [5], in 1968 and 1970 respectively. So far, no proof for the conjecture has been found, although partial results exist in particular for low dimensions and with additional conditions for the curvature functions of M. Its original version relates to the following theorem, first proved by Simons in [14]:

Theorem 1.2. *Let $M \subset \mathbb{S}^{n+1}$ be a closed, minimally immersed hypersurface and S the squared norm of its second fundamental form. Then*

$$\int_M (S-n)S \geq 0.$$

In particular, for $S \leq n$, one has either $S = 0$ or $S = n$ identically on M.

Note that since M is minimally immersed S is constant if and only if the scalar curvature κ is constant. In this case it follows that $S = 0$ or $S \geq n$, which led Chern to propose the following

Conjecture. Consider closed minimal hypersurfaces $M \subset \mathbb{S}^{n+1}$ with constant scalar curvature κ. Then for each n the set of all possible values for κ (or equivalently S) is discrete.

The only known examples for minimal hypersurfaces with constant scalar curvature in $\mathbb{S}^{n+1}$ are isoparametric, i.e. all of their principal curvature functions are constant. From the classification of isoparametric hypersurfaces in spheres, given by Münzner in [10], one obtains that S equals $(g-1)n$, where g is the number of pairwise distinct principal curvatures and can only take the values 1, 2, 3, 4 or 6, which establishes the conjecture in this case. Based on this, Verstraelen, Montiel, Ros and Urbano first formulated the stronger version of the conjecture given initially (see [15]), most of the later results refer to this version.

The first related partial result was achieved by Peng and Terng [11].

Theorem 1.3. *For every $n \geq 3$ there exists a maximal $C(n)$ with the following property: Let $M \subset \mathbb{S}^{n+1}$ be a closed minimal hypersurface with constant $S > n$. Then it follows that $S \geq n + C(n)$ and one has $C(3) = 3$, $C(n) \geq \frac{1}{12n}$.*

The originally shown inequality has since been improved considerably by Yang and Cheng (see [16–18]) to

$$C(n) \geq \frac{26}{61}n - \frac{16}{61} > \frac{1}{3}n$$

and to $C(n) \geq \frac{2n}{3}$ under the additional assumption that the sum of cubes of the principal curvatures f_3 is constant. For further details, we refer to Scherfner, Yau an Weiss [13].

We would like to point out that our result - even if strongly connected to the Chern conjecture - is not restricted to the special case of the sphere as the considered space of constant curvature. The proof of Theorem 1.1 follows.

2. Notation and Preliminaries

Let $x : M \to W$ be an isometric immersion of a closed oriented 3-dimensional hypersurface M in a not necessarily complete space of constant curvature c. Choose a unit normal vector field ν along x, and denote by h the second fundamental form associated to ν. We will always assume that M has principal curvatures k_1, k_2 and k_3 satisfying at every point of M the condition

$$k_1 > k_2 > k_3.$$

At each $p \in M$, they are the eigenvalues of the the linear map $L : T_pM \to T_pM$ associated to the second fundamental form h. From now on we will assume that M has constant mean curvature H and constant scalar curvature κ. They are defined by

$$\begin{cases} 3H = k_1 + k_2 + k_3, \\ 3\kappa = 3c + k_1k_2 + k_1k_3 + k_2k_3. \end{cases}$$

Note that, up to a constant, κ is the trace of the Ricci tensor with respect to the Riemannian metric. In our case,

$$\kappa = c + (9H^2 - S)/6,$$

where $S = k_1^2 + k_2^2 + k_3^2$ is the square of the length of the second fundamental form h. Since H and κ are constant, it follows that S is also a constant function.

Remark. It is known that the 3-dimensional Cartan hypersurface $M \subset \mathbb{S}^4$ has constant principal cuvatures $k_1 = \sqrt{3}$, $k_2 = 0$ and $k_3 = -\sqrt{3}$. Evidently, $\kappa = 0$ and $k_1 > k_2 > k_3$. This is also true for each hypersurface in Cartan's isoparametric family (see [3]). We would like to point out that there are also infinitely many non-isoparametric examples of minimal hypersurfaces in $\mathbb{S}^4$ with three pairwise distinct principal curvatures $k_1 > k_2 = 0 > k_3$ at each point. They can be obtained by taking the boundary of a tube of certain minimal surfaces of $\mathbb{S}^4$ (see [1, 12]). The general case of minimal hypersurfaces with $k_2 = 0$ were also studied by T. Hasanis, A. Savas-Halilaj and T. Vlachos [7–9].

Let φ be the traceless second fundamental form of M, given by

$$\varphi = Hg - h,$$

where g is the Riemannian metric of M. We will denote by Φ the tensor field of type (1,1) corresponding to φ via g. Its eigenvalues are given by $\mu_i = H - k_i$, where

$i = 1, 2, 3$. Note that for some constant $T > 0$

$$|\Phi|^2 = \mathrm{trace}(\Phi^2) = \sum_i \mu_i^2 = S - 3H^2 = 6T^2.$$

We set

$$\begin{cases} a = \varphi/T, \\ A = \Phi/T. \end{cases} \tag{2.1}$$

Note that A is the tensor field of type (1,1) corresponding to a via g. It satisfies the following conditions

$$\begin{cases} \mathrm{trace}\, A = 0, \\ \mathrm{trace}\, A^2 = 6, \\ \mathrm{trace}\, A^3 = f, \end{cases} \tag{2.2}$$

where $f : M \to \mathbb{R}$ is a C^∞ function. We set $I = \{1, 2, 3\}$ and denote by $\lambda_i : M \to \mathbb{R}$ the smooth functions defined by

$$\lambda_i = (H - k_i)/T.$$

They are the eigenvalues of A and satisfies the following condition

$$\lambda_1 < \lambda_2 < \lambda_3.$$

Let $\sigma : I \to I$ be the permutation given by $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$. For each $i \in I$ we define $c_i = \lambda_{i'} - \lambda_{i''}$, where $i' = \sigma(i)$ and $i'' = \sigma(i^{'})$. We have

$$\sum_{i \in I} c_i = 0, \qquad \sum_{i \in I} c_i^2 = 18 \tag{2.3}$$

and

$$\prod_{i \in I} c_i^2 = 3(6^2 - f^2) > 0. \tag{2.4}$$

Now we will take a look at the structure equations of M. From now on, the volume form of M will be denoted by dM.

Definition 2.1. We say that (U, ω) is admissible if

i) U is an open subset of Y;
ii) $\omega = (\omega_1, \omega_2, \omega_3)$ is a smooth orthonormal coframe field on U;
iii) $\omega_1 \wedge \omega_2 \wedge \omega_3 = dM$ on U;
iv) $a = \sum_{i \in I} \lambda_i \omega_i \otimes \omega_i$.

Suppose (U, ω) is admissible. As is well known, there are smooth 1-forms ω_{ij} on U uniquely determined by the equations

$$d\omega_i = -\sum_{j=1}^{3} \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \quad i, j \in I. \tag{2.5}$$

As in [2] there is one and only one well defined 2-form ψ on M such that if (U,ω) is admissible then

$$\psi = \sum_{i<j} \omega_{ij} \wedge *(\omega_i \wedge \omega_j). \tag{2.6}$$

In equation (2.6), we use the Hodge star operator $(*)$ to define the 2-form ψ. The same computation as in [2] shows that

$$d\psi = \left(3\kappa + \frac{|df|^2}{9\gamma^2}\right) dM. \tag{2.7}$$

Here $\gamma = c_1c_2c_3$. The covariant derivative of the tensor field $a = \sum_{i,j\in I} a_{ij}\, \omega_i \otimes \omega_j$ is given by

$$\nabla a = \sum_{i,j,k} a_{ijk}\, \omega_i \otimes \omega_j \otimes \omega_k, \tag{2.8}$$

where

$$\sum_k a_{ijk}\, \omega_k = da_{ij} - \sum_m a_{im}\omega_{mj} - \sum_m a_{mj}\omega_{mi}.$$

The covariant derivatives a_{ijk} are symmetric in any of their indices. Observe that in our case $a_{ij} = \lambda_i \delta_{ij}$. Therefore,

$$d\lambda_i = \sum_{k\in I} a_{iik}\, \omega_k, \quad i \in I. \tag{2.9}$$

On the other hand we know that each pair $(x, \lambda_i(x)) \in M \times \mathbb{R}$ satisfies the polynomial equation $P(x, \lambda) = 0$, where

$$P(x,\lambda) = \prod_{k=1}^{3} (\lambda - \lambda_k) = \lambda^3 - 3\lambda - \frac{f(x)}{3}.$$

Note that for each $i \in I$,

$$P(x,\lambda) = (\lambda - \lambda_i)(\lambda - \lambda_{i'})(\lambda - \lambda_{i''}).$$

Differentiating the equation $P(x, \lambda_i(x)) = 0$, we obtain

$$(\lambda_i - \lambda_{i'})(\lambda_i - \lambda_{i''})d\lambda_i - \frac{1}{3}df = 0.$$

This gives the following identities

$$c_i df = -3\gamma\, d\lambda_i. \tag{2.10}$$

We let $f_i,\ i \in I$ be the differentiable functions given by

$$df = \sum_{i\in I} f_i \omega_i.$$

It follows from equations (2.9) and (2.10) that

$$a_{iik} = -c_i f_k/(3\gamma). \tag{2.11}$$

Notice that

$$|\nabla a|^2 - 6a_{123}^2 = A_1 + A_2 + A_3,$$

where

$$\begin{cases} A_1 = 3a_{331}^2 + 3a_{221}^2 + a_{111}^2, \\ A_2 = 3a_{332}^2 + 3a_{112}^2 + a_{222}^2, \\ A_3 = 3a_{223}^2 + 3a_{113}^2 + a_{333}^2. \end{cases} \tag{2.12}$$

From equation (2.12), it follows that, for $1 \le i \le 3$,

$$A_i = 3(a_{11i}^2 + a_{22i}^2 + a_{33i}^2) - 2a_{iii}^2.$$

Using (2.11) we get

$$9\gamma^2 A_i = 3(c_1^2 + c_2^2 + c_3^2) f_i^2 - 2c_i^2 f_i^2 = (54 - 2c_i^2) f_i^2.$$

It is not difficult to see that $c_i^2 = 12 - 3\lambda_i^2$. Therefore,

$$9\gamma^2 A_i = (30 + 6\lambda_i^2) f_i^2 \ge 30 f_i^2.$$

From this, it follows that

$$\frac{|df|^2}{9\gamma^2} \le \frac{|\nabla a|^2}{30} = \frac{|\nabla \varphi|^2}{30T^2} = \frac{|\nabla \varphi|^2}{5|\Phi|^2}. \tag{2.13}$$

In equation (2.13), we use the fact that $a = \varphi/T$ and $T = |\Phi|/\sqrt{6}$. It is well known that (see, e.g., [6])

$$\frac{1}{2}\Delta|\Phi|^2 = |\nabla\varphi|^2 - |\Phi|^4 + 3\left(c + H^2\right)|\Phi|^2 - 3H \operatorname{trace}\Phi^3.$$

By definition, $A = \Phi/T$, then

$$3 \operatorname{trace}\Phi^3 = 3T^3 f = \sqrt{3/2}\ |\Phi|^3 f/6.$$

Since $|\Phi|^2$ is a constant, it follows that

$$|\nabla\varphi|^2 = |\Phi|^4 - 3\left(c + H^2\right)|\Phi|^2 + \sqrt{3/2}\ |\Phi|^3 H f/6.$$

We know that $|f| < 6$. Therefore,

$$\frac{|\nabla\varphi|^2}{|\Phi|^2} < |\Phi|^2 - 3\left(c + H^2\right) + \sqrt{3/2}\,|H||\Phi|.$$

In our case $\kappa = c + H^2 - |\Phi|^2/6$. Using (2.13) we get

$$3\kappa + \frac{|df|^2}{9\gamma^2} \le 3(c + H^2) - \frac{|\Phi|^2}{2} + \frac{|\nabla\varphi|^2}{5|\Phi|^2}.$$

An easy computation gives

$$3\kappa + \frac{|df|^2}{9\gamma^2} < \frac{-3|\Phi|^2 + \sqrt{6}\,|H||\Phi| + 24\left(c + H^2\right)}{10}. \tag{2.14}$$

3. **Proof of Theorem** 1.1

Proof. Applying Stokes's theorem to equation (2.7) we get

$$\int_M \left(3\kappa + \frac{|df|^2}{9\gamma^2}\right) dM = 0. \tag{3.1}$$

As a consequence of (3.1) we see that if $\kappa \geq 0$, then

$$|df|^2 = \kappa = 0.$$

In this case $\kappa = 0$ and M isoparametric. We will now consider the case $\kappa < 0$. From (2.14) and (3.1) we see that

$$3|\Phi|^2 - \sqrt{6}\,|H|\,|\Phi| - 24(c + H^2) < 0. \tag{3.2}$$

It follows that $|\Phi| < r$ where r is the positive real root of the polynomial

$$Q_H(z) = 3z^2 - \sqrt{6}\,|H|\,z - 24(c + H^2).$$

Note that

$$r = \frac{|H| + \sqrt{49H^2 + 48c}}{\sqrt{6}}.$$

Since $6\kappa = 6(c + H^2) - |\Phi|^2$, then

$$6\kappa > 6(c + H^2) - r^2 = -2c - \frac{|H|}{3}\left(7|H| + \sqrt{49H^2 + 48c}\ \right).$$

From this we see that $\alpha(H) < \kappa < 0$, where

$$\alpha(H) = \frac{1}{6}\left[-2c - \frac{|H|}{3}\left(7|H| + \sqrt{49H^2 + 48c}\ \right)\right].$$

This completes the proof of the theorem. □

Corollary 3.1. *Let M be a closed oriented 3-dimensional hypersurface with distinct principal curvatures minimally immersed in a not necessarily complete space of constant curvature c. Suppose in addition that M has constant scalar curvature κ. Then*

a) $c > 0$;
b) $0 \geq \kappa > -c/3$.

As a consequence, $6c \leq S < 8c$, where S is the square of the length of the second fundamental form of M.

Proof. By assumption the mean curvature H of M is identically zero. Since $H = 0$, it follows from Theorem 1.1, that

$$\kappa = c - \frac{S}{6} \leq 0.$$

If $\kappa = 0$, $c = S/6$. If $\kappa < 0$,

$$-\frac{c}{3} = \alpha(H) < \kappa < 0.$$

In both cases a) and b) obviously holds. As a consequence,

$$-2c < 6c - S \leq 0.$$

It follows that $6c \leq S < 8c$. □

Acknowledgements. This work was partially supported by UFC, CNPq and Universidade Federal do ABC.

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