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A CHARACTERIZATION OF ESSENTIAL PSEUDOSPECTRA INVOLVING POLYNOMIALLY RIESZ OPERATORS

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Abstract. In this article, we investigate the essential pseudospectra associated with a broad class of operators known as polynomially Riesz operators, which extend the classical notion of Riesz operators introduced by Latrach et al. in [19]. We establish several new results concerning the essential pseudospectra of closed linear operators on Banach spaces under perturbations by polynomially Riesz operators. In particular, we examine how these perturbations affect the left (respectively, right) Weyl essential pseudospectra and the left (respectively, right) Fredholm essential pseudospectra. Lastly, we provide a detailed characterization of the essential pseudospectra of the sum of two bounded linear operators, emphasizing the influence of polynomially Riesz perturbations on their spectral behavior.

1. Introduction

Eigenvalue problems hold significant importance across numerous scientific and engineering disciplines. The primary goals when tackling these problems are to extract and localize eigenvalues. However, traditional spectral analysis falls short in achieving both objectives, as it can only identify eigenvalues without localizing them. As a solution, researchers have introduced alternative methods such as the pseudospectrum, first proposed by Varah [25]. The pseudospectrum has found wide-ranging applications in numerous areas of mathematical physics, including engineering (e.g., electrical engineering), aeronautics, ecology, and chemistry. In engineering, for instance, eigenvalues can dictate the precision of a national power grid or an amplifier's

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frequency response. In aeronautical engineering, eigenvalue analysis can help identify whether the airflow across an aircraft wing exhibits laminar or turbulent behavior. In ecological modeling, eigenvalues play a crucial role in assessing the stability of equilibrium states within food web dynamics. In chemistry, they can establish energy states in a stable hydrogen atom. In summary, the pseudospectrum concept has demonstrated its value in addressing eigenvalue problems, allowing researchers to accurately extract and localize eigenvalues, thus contributing to significant progress in diverse areas of science and engineering. Motivated by the concept of pseudospectra, F. Abdmouleh et al. [1] developed the notion of the pseudo Browder essential spectrum for densely defined closed linear operators on Banach spaces. Later, in [2, 3], F. Abdmouleh and B. Elgabeur introduced the concepts of pseudo left and right Fredholm and Browder operators, and examined their associated spectra in the context of bounded linear operators. A key part of their investigation focused on the stability of these pseudo-essential spectra under perturbations by Riesz operators in Banach spaces. They also analyzed the behavior of the pseudo left and right Fredholm and Browder essential spectra for the sum of two bounded linear operators. In related contributions, A. Ammar, A. Jeribi and K. Mahfoudhi, [4, 5] extended this line of research by examining the essential pseudospectra of bounded operators and providing a formulation of the pseudo-Fredholm operator concept along with its corresponding essential pseudospectrum.

In this work, we extend the analysis of essential pseudospectra in Banach spaces to a broader class of operators known as polynomially Riesz operators, which are considered to be generalizations of some well-known classes Fredholm perturbations, polynomially Fredholm perturbations, polynomially strictly singular operators and polynomially compact operators. This class of operators has drawn significant attention from various researchers due to its relevance in deriving meaningful results within spectral theory. The reader may find the following references useful: Dehici et al. in [10], K. Latrach et al. in [18], S. C. Živković Zlatanović et al. in [27]. The primary objective of this paper is to generalize the stability results of essential pseudospectra under Riesz operator perturbations, as established in [1–7], by considering perturbations through polynomially Riesz operators acting on closed, densely defined linear operators. The second aim of this work, is to describe the essential pseudospectrum of the sum of two bounded linear operators with the new concept of polynomially Riesz operator.

Let us now outline the structure of this paper. In Section 2, we begin by recalling some essential notations and definitions related to Fredholm operators and their corresponding essential spectra. We also introduce the notion of polynomially Riesz operators and present several preliminary results relevant to our study. Section 3 is devoted to establishing new stability results and to deriving alternative characterizations of the left (respectively right) Weyl and Fredholm essential pseudospectra within the framework of bounded linear operators on Banach spaces. Finally, in Section 4, we present a main result concerning the essential pseudospectra of the sum of two bounded linear operators, inspired by the theory of polynomially Riesz perturbations.

2. Notations and Definitions

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two Banach spaces. By an operator $\mathcal{A}_1$ from $\mathcal{X}_1$ into $\mathcal{X}_2$ we mean a linear operator with domain $\mathcal{D}(A) \subseteq \mathcal{X}_1$ and range contained in $\mathcal{X}_2$. We denote by $\mathcal{C}(\mathcal{X}_1, \mathcal{X}_2)$ (resp., $\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$) the set of all closed, densely defined (resp., bounded) linear operators from $\mathcal{X}_1$ to $\mathcal{X}_2$. The subset of all compact operators of $\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$ is designated by $\mathcal{K}(\mathcal{X}_1, \mathcal{X}_2)$. If $\mathcal{A}_1 \in \mathcal{C}(\mathcal{X}_1, \mathcal{X}_2)$, we write $N(\mathcal{A}_1) \subset \mathcal{X}_1$ and $R(\mathcal{A}_1) \subset \mathcal{X}_2$ for the null space and the range of $\mathcal{A}_1$. We set $\alpha(\mathcal{A}_1) := \dim N(\mathcal{A}_1)$ and $\beta(A) := \operatorname{codim} R(\mathcal{A}_1)$. Let $\mathcal{A}_1 \in \mathcal{C}(\mathcal{X}_1, \mathcal{X}_2)$ with closed range. Then, $\mathcal{A}_1$ is a Φ_+-operator ($\mathcal{A}_1 \in \Phi_+(\mathcal{X}_1, \mathcal{X}_2)$) if $\alpha(\mathcal{A}_1) < +\infty$, and then $\mathcal{A}_1$ is a Φ_--operator ($\mathcal{A}_1 \in \Phi_-(\mathcal{X}_1, \mathcal{X}_2)$) if $\beta(A) < +\infty$. $\Phi(\mathcal{X}_1, \mathcal{X}_2) = \Phi_+(\mathcal{X}_1, \mathcal{X}_2) \cap \Phi_-(\mathcal{X}_1, \mathcal{X}_2)$ is the class of Fredholm operators while $\Phi_\pm(\mathcal{X}_1, \mathcal{X}_2)$ denotes the set $\Phi_\pm(\mathcal{X}_1, \mathcal{X}_2) = \Phi_+(\mathcal{X}_1, \mathcal{X}_2) \cup \Phi_-(\mathcal{X}_1, \mathcal{X}_2)$. For $A \in \Phi(\mathcal{X}_1, \mathcal{X}_2)$, the index of $\mathcal{A}_1$ is defined by $ind(\mathcal{A}_1) = \alpha(\mathcal{A}_1) - \beta(\mathcal{A}_1)$. If $\mathcal{X}_1 = \mathcal{X}_2$, then $\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2), \mathcal{K}(\mathcal{X}_1, \mathcal{X}_2), \mathcal{C}(\mathcal{X}_1, \mathcal{X}_2), \Phi_+(\mathcal{X}_1, \mathcal{X}_2), \Phi_\pm(\mathcal{X}_1, \mathcal{X}_2)$ and $\Phi(\mathcal{X}_1, \mathcal{X}_2)$ are replaced, respectively, by $\mathcal{L}(\mathcal{X}_1)$, $\mathcal{K}(\mathcal{X}_1)$, $\mathcal{C}(\mathcal{X}_1)$, $\Phi_+(\mathcal{X}_1)$, $\Phi_\pm(\mathcal{X}_1)$ and $\Phi(\mathcal{X}_1)$. Let $\mathcal{A}_1 \in \mathcal{C}(\mathcal{X}_1)$, the spectrum of $\mathcal{A}_1$ will be denoted by $\sigma(\mathcal{A}_1)$. The resolvent set of $\mathcal{A}_1, \rho(\mathcal{A}_1)$, is the complement of $\sigma(\mathcal{A}_1)$ in the complex plane. A complex number λ is in $\Phi_{+\mathcal{A}_1}, \Phi_{-\mathcal{A}_1}, \Phi_{\pm A}$ or $\Phi_{\mathcal{A}_1}$ if $\lambda - \mathcal{A}_1$ is in $\Phi_+(\mathcal{X}_1), \Phi_-(\mathcal{X}_1), \Phi_\pm(\mathcal{X}_1)$ or $\Phi(\mathcal{X}_1)$, respectively. Let $F \in \mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$. F is called a Fredholm perturbation if $U + F \in \Phi(\mathcal{X}_1, \mathcal{X}_2)$ whenever $U \in \Phi(\mathcal{X}_1, \mathcal{X}_2)$. F is called an upper (resp., lower) Fredholm perturbation if $U + F \in \Phi_+(\mathcal{X}_1, \mathcal{X}_2)$ (resp., $U + F \in \Phi_-(\mathcal{X}_1, \mathcal{X}_2)$) whenever $U \in \Phi_+(\mathcal{X}_1, \mathcal{X}_2)$ (resp., $U \in \Phi_-(\mathcal{X}_1, \mathcal{X}_2)$). The set of Weyl operators is defined as $\mathcal{W}(\mathcal{X}_1, \mathcal{X}_2) = \{\mathcal{A}_1 \in \Phi(\mathcal{X}_1, \mathcal{X}_2) : ind(\mathcal{A}_1) = 0\}$. Sets of left and right Fredholm operators, respectively, are defined as:

$$\begin{aligned}\Phi_l(\mathcal{X}_1) :=& \{\mathcal{A}_1 \in \mathcal{L}(\mathcal{X}_1) : R(\mathcal{A}_1) \text{ is a closed and complemented subspace}\\ &\text{of } \mathcal{X}_1 \text{ and } \alpha(\mathcal{A}_1) < +\infty\},\\ \Phi_r(\mathcal{X}_1) :=& \{\mathcal{A}_1 \in \mathcal{L}(\mathcal{X}_1) : N(\mathcal{A}_1) \text{ is a closed and complemented subspace}\\ &\text{of } \mathcal{X}_1 \text{ and } \beta(\mathcal{A}_1) < +\infty\}.\end{aligned}$$

An operator $\mathcal{A}_1 \in \mathcal{L}(\mathcal{X}_1)$ is left (right) Weyl if $\mathcal{A}_1$ is left (right) Fredholm operator and $ind(\mathcal{A}_1) \leq 0$ ($ind(\mathcal{A}_1) \geq 0$). We use $\mathcal{W}_l(\mathcal{X}_1)$ ($\mathcal{W}_r(\mathcal{X}_1)$) to denote the set of all left (right) Weyl operators. It is known that the sets $\Phi_l(\mathcal{X}_1)$ and $\Phi_r(\mathcal{X}_1)$ are open satisfying the following inclusions:

$$\Phi(X) \subset \mathcal{W}_l(\mathcal{X}_1) \subset \Phi_l(\mathcal{X}_1) \quad \text{and} \quad \Phi(\mathcal{X}_1) \subset \mathcal{W}_r(\mathcal{X}_1) \subset \Phi_r(\mathcal{X}_1).$$

The sets of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(\mathcal{X}_1, \mathcal{X}_2)$, $\mathcal{F}_+(\mathcal{X}_1, \mathcal{X}_2)$ and $\mathcal{F}_-(\mathcal{X}_1, \mathcal{X}_2)$, respectively. In general, we have

$$\begin{aligned}\mathcal{K}(\mathcal{X}_1, \mathcal{X}_2) \subseteq \mathcal{F}_+(\mathcal{X}_1, \mathcal{X}_2) \subseteq \mathcal{F}(\mathcal{X}_1, \mathcal{X}_2),\\ \mathcal{K}(\mathcal{X}_1, \mathcal{X}_2) \subseteq \mathcal{F}_-(\mathcal{X}_1, \mathcal{X}_2) \subseteq \mathcal{F}(\mathcal{X}_1, \mathcal{X}_2).\end{aligned}$$

If $X=Y$, we write $\mathcal{F}(\mathcal{X}_1)$, $\mathcal{F}_+(\mathcal{X}_1)$ and $\mathcal{F}_-(\mathcal{X}_1)$ for $\mathcal{F}(\mathcal{X}_1)$, $\mathcal{F}_+(\mathcal{X}_1)$ and $\mathcal{F}_-(\mathcal{X}_1)$, respectively. Let $\Phi^b(\mathcal{X}_1,\mathcal{X}_2)$, $\Phi^b_+(\mathcal{X}_1,\mathcal{X}_2)$ and $\Phi^b_-(\mathcal{X}_1,\mathcal{X}_2)$ denote the sets $\Phi(\mathcal{X}_1,\mathcal{X}_2)\cap\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)$, $\Phi_+(\mathcal{X}_1,\mathcal{X}_2)\cap\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)$ and $\Phi_-(\mathcal{X}_1,\mathcal{X}_2)\cap\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)$, respectively. If in Definition 1.1 we replace $\Phi(\mathcal{X}_1,\mathcal{X}_2)$, $\Phi_+(\mathcal{X}_1,\mathcal{X}_2)$ and $\Phi_-(\mathcal{X}_1,\mathcal{X}_2)$ by $\Phi^b(\mathcal{X}_1,\mathcal{X}_2)$, $\Phi^b_+(\mathcal{X}_1,\mathcal{X}_2)$ and $\Phi^b_-(\mathcal{X}_1,\mathcal{X}_2)$, we obtain the sets $\mathcal{F}^b(\mathcal{X}_1,\mathcal{X}_2)$, $\mathcal{F}^b_+(\mathcal{X}_1,\mathcal{X}_2)$ and $\mathcal{F}^b_-(\mathcal{X}_1,\mathcal{X}_2)$. These classes of operators were introduced and investigated in [6]. In particular, it is shown that $\mathcal{F}^b(\mathcal{X}_1,\mathcal{X}_2)$ is a closed subset of $\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)$ and $\mathcal{F}^b(\mathcal{X}_1)$ is a closed two-sided ideal of $\mathcal{L}(\mathcal{X}_1)$. In general we have

$$\begin{aligned}\mathcal{K}(\mathcal{X}_1,\mathcal{X}_2)\subseteq\mathcal{F}^b_+(\mathcal{X}_1,\mathcal{X}_2)\subseteq\mathcal{F}^b(\mathcal{X}_1,\mathcal{X}_2),\\ \mathcal{K}(\mathcal{X}_1,\mathcal{X}_2)\subseteq\mathcal{F}^b_-(\mathcal{X}_1,\mathcal{X}_2)\subseteq\mathcal{F}^b(\mathcal{X}_1,\mathcal{X}_2).\end{aligned}$$

Let $A\in\mathcal{C}(\mathcal{X}_1)$. It follows from the closeness of $\mathcal{A}_1$ that $\mathcal{D}(\mathcal{A}_1)$ endowed with the graph norm $\|\cdot\|_{\mathcal{A}_1}$ ($\|x\|_{\mathcal{A}_1}=\|x\|+\|\mathcal{A}_1x\|$) is a Banach space denoted by $X_{\mathcal{A}_1}$. Clearly, for $x\in\mathcal{D}(\mathcal{A}_1)$ we have $\|Ax\|\leqslant\|x\|_{\mathcal{A}_1}$, so $\mathcal{A}_1\in\mathcal{L}\left(\mathcal{X}_{\mathcal{A}_1},\mathcal{X}_1\right)$. Furthermore, we have the obvious relations

$$\begin{aligned}&\alpha(\hat{\mathcal{A}}_1)=\alpha(\mathcal{A}_1),\quad\beta(\hat{\mathcal{A}}_1)=\beta(\mathcal{A}_1),\quad R(\hat{\mathcal{A}}_1)=R(\mathcal{A}_1),\\ &\alpha(\hat{\mathcal{A}}_1+\hat{\mathcal{A}}_2)=\alpha(\mathcal{A}_1+\mathcal{A}_2),\\ &\beta(\hat{\mathcal{A}}_1+\hat{\mathcal{A}}_2)=\beta(\mathcal{A}_1+\mathcal{A}_2)\quad\text{and}\quad R(\hat{\mathcal{A}}_1+\hat{\mathcal{A}}_2)=R(\mathcal{A}_1+\mathcal{A}_2).\end{aligned}\tag{2.1}$$

In this paper we are concerned with the following essential spectra of $\mathcal{A}_1\in\mathcal{C}(\mathcal{X}_1)$:

$$\begin{aligned}\sigma_e(\mathcal{A}_1)&:=\{\mu\in\mathbf{C}:\mathcal{A}_1-\mu\notin\Phi(\mathcal{X}_1)\}:\text{the Fredholm spectrum of }\mathcal{A}_1,\\ \sigma^l_e(\mathcal{A}_1)&:=\{\lambda\in\mathbf{C}:\mathcal{A}_1-\mu\notin\Phi_l(\mathcal{X}_1)\}:\text{the left Fredholm spectrum of }\mathcal{A}_1,\\ \sigma^r_e(\mathcal{A}_1)&:=\{\lambda\in\mathbf{C}:\mathcal{A}_1-\mu\notin\Phi_r(\mathcal{X}_1)\}:\text{the right Fredholm spectrum of }\mathcal{A}_1,\\ \sigma_w(\mathcal{A}_1)&:=\{\mu\in\mathbb{C}:\mathcal{A}_1-\mu\notin\mathcal{W}(\mathcal{X}_1)\}:\text{the Weyl spectrum of }\mathcal{A}_1,\\ \sigma^l_w(\mathcal{A}_1)&:=\{\mu\in\mathbb{C}:\mathcal{A}_1-\mu\notin\mathcal{W}_l(\mathcal{X}_1)\}:\text{the left Weyl spectrum of }\mathcal{A}_1,\\ \sigma^r_w(\mathcal{A}_1)&:=\{\mu\in\mathbb{C}:\mathcal{A}_1-\lambda\notin\mathcal{W}_r(\mathcal{X}_1)\}:\text{the right Weyl spectrum of }\mathcal{A}_1,\\ \sigma_{\text{eap}}(\mathcal{A}_1)&:=\mathrm{C}\backslash\rho_{\text{eap}}(\mathcal{A}_1):\text{the essential approximate point spectrum of }\mathcal{A}_1,\\ \sigma_{e\delta}(\mathcal{A}_1)&:=\mathrm{C}\backslash\rho_{e\delta}(\mathcal{A}_1):\text{the essential defect spectrum of }\mathcal{A}_1,\end{aligned}$$

where

$$\rho_{\text{eap}}(\mathcal{A}_1):=\{\mu\in\mathrm{C}\text{ such that }\mu-\mathcal{A}_1\in\Phi_+(\mathcal{X}_1)\text{ and }i(\mu-\mathcal{A}_1)\leq 0\}$$

and

$$\rho_{e\delta}(\mathcal{A}_1):=\{\mu\in\mathrm{C}\text{ such that }\mu-\mathcal{A}_1\in\Phi_-(\mathcal{X}_1)\text{ and }i(\mu-\mathcal{A}_1)\geq 0\}.$$

The definition of pseudospectrum of a closed densely linear operator $\mathcal{A}_1$ for every $\varepsilon>0$ is given by:

$$\sigma_\varepsilon(\mathcal{A}_1):=\sigma(\mathcal{A}_1)\cup\left\{\mu\in\mathbb{C}:\left\|(\mu-\mathcal{A}_1)^{-1}\right\|>\frac{1}{\varepsilon}\right\}.$$

By convention, we write $\|(\mu - \mathcal{A}_1)^{-1}\| = +\infty$ if $(\mu - \mathcal{A}_1)^{-1}$ is unbounded or nonexistent, i.e., if μ is in the spectrum $\sigma(\mathcal{A}_1)$. In [9], Davies defined another equivalent of the pseudospectrum, one that is in terms of perturbations of the spectrum. In fact for $\mathcal{A}_1 \in \mathbb{C}(\mathfrak{X}_1)$, we have

$$\sigma_\varepsilon(\mathcal{A}_1) := \bigcup_{\|D\|<\varepsilon} \sigma(\mathcal{A}_1 + D).$$

Inspired by the notion of pseudospectra, Ammar and Jeribi in their works [4,5], aimed to extend these results for the essential pseudo-spectra of bounded linear operators on a Banach space and give the definitions of pseudo-Fredholm operator as follows: for $\mathcal{A}_1 \in \mathcal{L}(\mathfrak{X}_1)$ and for all $D \in \mathcal{L}(\mathfrak{X}_1)$ such that $\|D\| < \varepsilon$ we have $\mathcal{A}_1$ is called a pseudo-upper (resp. lower) semi-Fredholm operator if $\mathcal{A}_1 + D$ is an upper (resp. lower) semi-Fredholm operator and it is called a pseudo semi-Fredholm operator if $\mathcal{A}_1 + D$ is a semi-Fredholm operator. $\mathcal{A}_1$ is called a pseudo-Fredholm operator if $\mathcal{A}_1 + D$ is a Fredholm operator. They are noted by $\Phi^\varepsilon(\mathfrak{X}_1)$ the set of pseudo-Fredholm operators, by $\Phi^\varepsilon_\pm(\mathfrak{X}_1)$ the set of pseudo-semi-Fredholm operator and by $\Phi^\varepsilon_+(\mathfrak{X}_1)$ (resp. $\Phi^\varepsilon_-(\mathfrak{X}_1)$) the set of pseudo-upper semi-Fredholm (resp. lower semi-Fredholm) operator. A complex number μ is in $\Phi^\varepsilon_{\pm A}$, $\Phi^\varepsilon_{+\mathcal{A}_1}$, $\Phi^\varepsilon_{-\mathcal{A}_1}$ or $\Phi^\varepsilon_{\mathcal{A}_1}$ if $\mu - \mathcal{A}_1$ is in $\Phi^\varepsilon_\pm(\mathfrak{X}_1)$, $\Phi^\varepsilon_+(\mathfrak{X}_1)$, $\Phi^\varepsilon_-(\mathfrak{X}_1)$ or $\Phi^\varepsilon(\mathfrak{X}_1)$.

F. Abdmouleh and B. Elgabeur in [3] defining the concept of pseudo left (resp. right)-Fredholm, for $\mathcal{A}_1 \in \mathcal{L}(\mathfrak{X}_1)$ and for all $D \in \mathcal{L}(\mathfrak{X}_1)$ such that $\|D\| < \varepsilon$ we have $\mathcal{A}_1$ is called a pseudo left (resp. right) Fredholm operator if $\mathcal{A}_1 + D$ is an left (resp. right) Fredholm operator they are denoted by $\Phi^\varepsilon_l(\mathfrak{X}_1)$ (resp. $\Phi^\varepsilon_r(\mathfrak{X}_1)$).

In this paper we are concerned with the following essential pseudospectra of $\mathcal{A}_1 \in \mathbb{C}(\mathfrak{X}_1)$:

$$\begin{aligned}
\sigma_{e1,\varepsilon}(\mathcal{A}_1) &:= \left\{\mu \in \mathbb{C} : \mu - \mathcal{A}_1 \notin \Phi^\varepsilon_+(\mathfrak{X}_1)\right\} = \mathbb{C}\backslash\Phi^\varepsilon_{+\mathcal{A}_1},\\
\sigma_{e2,\varepsilon}(\mathcal{A}_1) &:= \left\{\mu \in \mathbb{C} : \mu - \mathcal{A}_1 \notin \Phi^\varepsilon_-(\mathfrak{X}_1)\right\} = \mathbb{C}\backslash\Phi^\varepsilon_{-A},\\
\sigma_{e3,\varepsilon}(\mathcal{A}_1) &:= \left\{\mu \in \mathbb{C} : \mu - \mathcal{A}_1 \notin \Phi^\varepsilon_\pm(\mathfrak{X}_1)\right\} = \mathbb{C}\backslash\Phi^\varepsilon_{\pm\mathcal{A}_1},\\
\sigma_{e,\varepsilon}(\mathcal{A}_1) &:= \{\mu \in \mathbb{C} : \mu - \mathcal{A}_1 \notin \Phi^\varepsilon(\mathfrak{X}_1)\} = \mathbb{C}\backslash\Phi^\varepsilon_{\mathcal{A}_1},\\
\sigma_{eap,\varepsilon}(\mathcal{A}_1) &:= \sigma_{e2,\varepsilon}(\mathcal{A}_1) \bigcup \{\mu \in \mathbb{C} : ind(\mu - \mathcal{A}_1 - D) > 0,\ \text{for all } \|D\| < \varepsilon\},\\
\sigma_{\delta,\varepsilon}(\mathcal{A}_1) &:= \sigma_{e2,\varepsilon}(\mathcal{A}_1) \bigcup \{\mu \in \mathbb{C} : ind(\mu - \mathcal{A}_1 - D) < 0,\ \text{for all } \|D\| < \varepsilon\},\\
\sigma^l_{e,\varepsilon}(\mathcal{A}_1) &:= \{\mu \in \mathbb{C} : \mathcal{A}_1 - \mu \notin \Phi^\varepsilon_l(\mathfrak{X}_1)\},\\
\sigma^r_{e,\varepsilon}(\mathcal{A}_1) &:= \{\mu \in \mathbb{C} : \mathcal{A}_1 - \mu \notin \Phi^\varepsilon_r(\mathfrak{X}_1)\},\\
\sigma^l_{\mathcal{W},\varepsilon}(\mathcal{A}_1) &:= \sigma^l_{e,\varepsilon}(\mathcal{A}_1) \bigcup \{\mu \in \mathbb{C} : ind(\mu - \mathcal{A}_1 - D) > 0,\ \text{for all } \|D\| < \varepsilon\},\\
\sigma^r_{\mathcal{W},\varepsilon}(\mathcal{A}_1) &:= \sigma^l_{e,\varepsilon}(\mathcal{A}_1) \bigcup \{\mu \in \mathbb{C} : ind(\mu - \mathcal{A}_1 - D) < 0,\ \text{for all } \|D\| < \varepsilon\}.
\end{aligned}$$

Note that if ε tends to 0, we recover the usual definition of the essential spectra of a closed operator $\mathcal{A}_1$. The subsets σ_{e1} and σ_{e2} are the Gustafson and Weidmann essential spectra [12], σ_{e3} is the Kato essential spectrum, [15] σ_e is the Wolf essential spectrum

[12], σ_{e5} is the Schechter essential spectrum [23], σ_{eap} is the essential approximate point spectrum [21], $\sigma_{e\delta}$ is the essential defect spectrum [22], $\sigma_e^l(A)$ (resp. $\sigma_e^r(A)$) is the left (resp. right) Fredholm essential spectra and $\sigma_w^l(\mathcal{A}_1)$ (resp. $\sigma_w^r(\mathcal{A}_1)$) is the left (resp. right) Weyl essential spectra [11].

As a concept, pseudospectra and essential pseudospectra are interesting because they offer more information than spectra, especially about transients rather than just asymptotic behavior. Moreover, they perform more efficiently than spectra in terms of convergence and approximation. These include the existence of approximate eigenvalues far from the spectrum and the instability of the spectrum even under small perturbations. Various applications of pseudospectra and essential pseudospectra have been developed as a result of the analysis of pseudospectra and essential pseudospectra.

We now list some of the known facts about left and right Fredholm operators in Banach space which will be used in the sequel.

Proposition 2.1. ([14, Propositon 2.3]). *Let $\mathcal{X}_1, \mathcal{Y}_1$ and $\mathcal{Z}_1$ be three Banach spaces.*

(i) If $\mathcal{A}_1 \in \Phi^b(\mathcal{Y}_1, \mathcal{Z}_1)$ and $\mathcal{T}_1 \in \Phi_l^b(\mathcal{X}_1, \mathcal{Y}_1)$ (resp. $\mathcal{T}_1 \in \Phi_r^b(\mathcal{X}_1, \mathcal{Y}_1)$), then $\mathcal{A}_1\mathcal{T}_1 \in \Phi_l^b(\mathcal{X}_1, \mathcal{Z}_1)$ (resp. $\mathcal{A}_1\mathcal{T}_1 \in \Phi_r^b(\mathcal{X}_1, \mathcal{Z}_1)$).

(ii) If $\mathcal{A}_1 \in \Phi^b(\mathcal{Y}_1, \mathcal{Z}_1)$ and $T \in \Phi_l^b(\mathcal{X}_1, \mathcal{Y}_1)$ (resp. $\mathcal{T}_1 \in \Phi_r^b(\mathcal{X}_1, \mathcal{Y}_1)$), then $\mathcal{T}_1\mathcal{A}_1 \in \Phi_l^b(\mathcal{X}_1, \mathcal{Z}_1)$ (resp. $\mathcal{T}_1\mathcal{A}_1 \in \Phi_r^b(\mathcal{X}_1, \mathcal{Z}_1)$).

Theorem 2.1 ([20, 23])**.** *Let $\mathcal{X}_1, \mathcal{Y}_1$ and $\mathcal{Z}_1$ be three Banach spaces, $\mathcal{A}_1 \in \mathcal{L}(\mathcal{Y}_1, \mathcal{Z}_1)$ and $\mathcal{T}_1 \in \mathcal{L}(\mathcal{X}_1, \mathcal{Y}_1)$.*

(i) If $\mathcal{A}_1 \in \Phi^b(\mathcal{Y}_1, \mathcal{Z}_1)$ and $\mathcal{T}_1 \in \Phi^b(\mathcal{X}_1, \mathcal{Y}_1)$, then $AT \in \Phi^b(\mathcal{X}_1, \mathcal{Z}_1)$ and $ind(\mathcal{A}_1\mathcal{T}_1) = ind(\mathcal{A}_1) + ind(\mathcal{T}_1)$.

(ii) If $\mathcal{X}_1 = \mathcal{Y}_1 = \mathcal{Z}_1$, $\mathcal{A}_1\mathcal{T}_1 \in \Phi^b(\mathcal{X}_1)$ and $\mathcal{T}_1\mathcal{A}_1 \in \Phi^b(\mathcal{X}_1)$, then $\mathcal{A}_1 \in \Phi^b(\mathcal{X}_1)$ and $\mathcal{T}_1 \in \Phi^b(\mathcal{X}_1)$.

Lemma 2.1. ([11, Theorem 2.3]). *Let $\mathcal{A}_1 \in \mathcal{L}(\mathcal{X}_1)$. Then,*

(i) $\mathcal{A}_1 \in \Phi_l^b(\mathcal{X}_1)$ if and only if there exist $\mathcal{A}_l \in \mathcal{L}(\mathcal{X}_1)$ and $\mathcal{K}_1 \in \mathcal{K}(\mathcal{X}_1)$ such that $\mathcal{A}_l\mathcal{A}_1 = I - \mathcal{A}_1$;

(ii) $\mathcal{A}_1 \in \Phi_r^b(\mathcal{X}_1)$ if and only if there exist $\mathcal{A}_r \in \mathcal{L}(\mathcal{X}_1)$ and $\mathcal{K}_1 \in \mathcal{K}(\mathcal{X}_1)$ such that $\mathcal{A}_1\mathcal{A}_r = I - \mathcal{K}_1$.

Lemma 2.2. ([11, Theorem 2.7]). *Let $\mathcal{A}_1 \in \mathcal{L}(\mathcal{X}_1)$. If $\mathcal{A}_1 \in \Phi_l^b(\mathcal{X}_1)$ (resp. $\Phi_r^b(\mathcal{X}_1)$) and $\mathcal{K}_1 \in \mathcal{K}(\mathcal{X}_1)$, then $\mathcal{A}_1 + \mathcal{K}_1 \in \Phi_l^b(\mathcal{X}_1)$ (resp. $\Phi_r^b(\mathcal{X}_1)$) and $ind(\mathcal{A}_1 + \mathcal{K}_1) = ind(\mathcal{A}_1)$.*

Lemma 2.3. ([11, Theorem 2.5]). *Let $\mathcal{A}_1, \mathcal{B}_1 \in \mathcal{L}(\mathcal{X}_1)$. If $\mathcal{A}_1 \in \Phi_l^b(\mathcal{X}_1)$ (resp. $\Phi_r^b(\mathcal{X}_1)$) and $\mathcal{B}_1 \in \Phi_l^b(\mathcal{X}_1)$ (resp. $\Phi_r^b(\mathcal{X}_1)$), then $\mathcal{A}_1\mathcal{B}_1 \in \Phi_l^b(\mathcal{X}_1)$ (resp. $\Phi_r^b(\mathcal{X}_1)$) and $ind(\mathcal{A}_1 + \mathcal{B}_1) = ind(\mathcal{A}_1) + ind(\mathcal{B}_1)$.*

We close with the following lemma.

Lemma 2.4. ([8, Lemma 3.4]). *Let $\mathcal{A}_1 \in \mathcal{L}(\mathcal{X}_1)$.*

(i) If $\mathcal{A}_1\mathcal{B}_1 \in \Phi_l^b(\mathcal{X}_1)$, then $\mathcal{B}_1 \in \Phi_l^b(\mathcal{X}_1)$.

(ii) If $\mathcal{A}_1\mathcal{B}_1 \in \Phi_r^b(\mathcal{X}_1)$, then $\mathcal{A}_1 \in \Phi_r^b(\mathcal{X}_1)$.

Definition 2.1. Let $\mathcal{X}_1$ be a Banach space.

(i) An operator $\mathcal{A}_1 \in \mathcal{L}(\mathcal{X}_1)$ is said to have a left Fredholm inverse if there exists $\mathcal{A}_l \in \mathcal{L}(\mathcal{X}_1)$ such that $I - \mathcal{A}_l\mathcal{A}_1 \in \mathcal{K}(\mathcal{X}_1)$.

(ii) An operator $\mathcal{A}_1 \in \mathcal{L}(\mathcal{X}_1)$ is said to have a right Fredholm inverse if there exists $\mathcal{A}_r \in \mathcal{L}(\mathcal{X}_1)$ such that $I - \mathcal{A}_1\mathcal{A}_r \in \mathcal{K}(\mathcal{X}_1)$.

We know by the classical theory of Fredholm operators, see for example [15], that $\mathcal{A}_1$ belong to $\Phi(\mathcal{X}_1)$ if it possesses a left, right or two-sided Fredholm inverse, respectively.

We define these sets $\mathcal{I}nvF^l_{\mathcal{A}}(\mathcal{X}_1)$ and $\mathcal{I}nvF^r_{\mathcal{A}}(\mathcal{X}_1)$ by:

$$\mathcal{I}nv^F_{\mathcal{A}_1,l}(\mathcal{X}_1) :=\{\mathcal{A}_l \in \mathcal{L}(\mathcal{X}_1) : \mathcal{A}_l \text{ is a left Fredholm inverse of } \mathcal{A}_1\},$$
$$\mathcal{I}nv^F_{\mathcal{A}_1,r}(\mathcal{X}_1) :=\{\mathcal{A}_r \in \mathcal{L}(\mathcal{X}_1) : \mathcal{A}_r \text{ is a right Fredholm inverse of } \mathcal{A}_1\}.$$

Definition 2.2 ([23])**.** Let $\mathcal{X}_1$ be a Banach space and $\mathcal{F}_1 \in \mathcal{L}(\mathcal{X}_1)$. F is called a Riesz operator if $\mu - \mathcal{F}_1 \in \Phi(\mathcal{X}_1)$ for all scalars $\mu \neq 0$, and denote by $\mathcal{R}(\mathcal{X}_1)$ the class of all Riesz operators.

Definition 2.3. An minimal polynomial P is the unitary polynomial of smaller degree which cancels an endomorphism, that is to say a linear application of a vector space in itself.

We say that $\mathcal{A}_1 \in \mathcal{L}(\mathcal{X}_1)$ is polynomially Riesz if there exists a nonzero complex polynomial $p(\cdot)$ such that the operator $p(\mathcal{A}_1) \in \mathcal{R}(\mathcal{X}_1)$. The set of polynomially Riesz operators will be denoted by $\mathcal{P}_{\mathcal{R}}(\mathcal{X}_1)$.

If $\mathcal{A}_1$ belongs $\mathcal{P}_{\mathcal{R}}(\mathcal{X}_1)$, then there exists a nonzero polynomial $p(\cdot)$ such that $p(\mathcal{A}_1) \in \mathcal{R}(\mathcal{X}_1)$.

In the following, $\mathcal{E}_{\mathcal{PR}}(\mathcal{X}_1)$ will denote the subset of $\mathcal{P}_{\mathcal{R}}(\mathcal{X}_1)$ defined by:

$$\mathcal{E}_{\mathcal{PR}}(\mathcal{X}_1) :=\Big\{\mathcal{A}_1 \in \mathcal{P}_{\mathcal{R}}(\mathcal{X}_1) \text{ such that the minimal polynomial } p(\cdot) \text{ of } \mathcal{A}_1 \text{ satisfies } p(-1) \neq 0\Big\}.$$

Let us recall the following results which are fundamental for the proofs of the main results.

Proposition 2.2. ([19, Lemma 2.3]). *If* $\mathcal{F}_1 \in \mathcal{E}_{\mathcal{PR}}(\mathcal{X}_1)$, *then* $I + \mathcal{F}_1 \in \Phi(\mathcal{X}_1)$ *and* $ind(I + \mathcal{F}_1) = 0$.

3. Stability of Essential Pseudospectra by Means of Polynomially Riesz Perturbations Operators

The following theorem provides a practical criterion for the stability of some essential pseudospectra for perturbed linear operators.

Theorem 3.1. *Let* $\varepsilon > 0$ *and consider* $\mathcal{A}_1, \mathcal{B}_1 \in \mathcal{C}(\mathcal{X}_1)$. *Assume that there are* $\mathcal{A}_0, \mathcal{B}_0 \in \mathcal{L}(\mathcal{X}_1)$ *and* $R_1, R_2 \in \mathcal{E}_{\mathcal{PR}}(\mathcal{X}_1)$ *such that*

$$\mathcal{A}_1\mathcal{A}_0 = I - R_1, \tag{3.1}$$

$$\mathcal{B}_1\mathcal{B}_0 = I - R_2. \tag{3.2}$$

(i) If $0 \in \Phi_{\mathcal{A}_1} \cap \Phi_{\mathcal{B}_1}$, $\mathcal{A}_0 - \mathcal{B}_0 \in \mathcal{F}_+(\mathcal{X}_1)$ *and* $ind(\mathcal{A}_1) = ind(\mathcal{B}_1)$, *then*

$$\sigma_{\mathrm{eap},\varepsilon}(\mathcal{A}_1) = \sigma_{\mathrm{eap},\varepsilon}(\mathcal{B}_1). \tag{3.3}$$

(ii) If $0 \in \Phi_{\mathcal{A}_1} \cap \Phi_{\mathcal{B}_1}$, $\mathcal{A}_0 - \mathcal{B}_0 \in \mathcal{F}_-(\mathcal{X}_1)$ *and* $ind(\mathcal{A}_1) = ind(\mathcal{B}_1)$, *then*

$$\sigma_{e\delta,\varepsilon}(\mathcal{A}_1) = \sigma_{e\delta,\varepsilon}(\mathcal{B}_1). \tag{3.4}$$

(iii) If $\mathcal{A}_0 - \mathcal{B}_0 \in \mathcal{F}(\mathcal{X}_1)$, *then*

$$\sigma_{e,\varepsilon}(\mathcal{A}_1) = \sigma_{e,\varepsilon}(\mathcal{B}_1).$$

If, further, $0 \in \Phi_{\mathcal{A}_1} \cap \Phi_{\mathcal{B}_1}$ *such that* $ind(\mathcal{A}_1) = ind(\mathcal{B}_1)$, *then*

$$\sigma_{w,\varepsilon}(\mathcal{A}_1) = \sigma_{w,\varepsilon}(\mathcal{B}_1). \tag{3.5}$$

Proof. Let μ be a complex number. Equations (3.1) and (3.2) imply

$$(\mu - \mathcal{A}_1 - D)\mathcal{A}_0 - (\mu - \mathcal{B}_1 - D)\mathcal{B}_0 = R_1 - R_2 + (\mu - D)(\mathcal{A}_0 - \mathcal{B}_0). \tag{3.6}$$

(i) Let $\mu \notin \sigma_{\mathrm{eap},\varepsilon}(\mathcal{B}_1)$. Then, $\mu \in \Phi^{\varepsilon}_{+\mathcal{B}_1}$ such that $i(\mu - \mathcal{B}_1 - D) \leq 0$, for all $D \in \mathcal{L}(\mathcal{X}_1)$ such that $\|D\| < \varepsilon$. Since $\mathcal{B}_1 + D$ is closed and $\mathcal{D}(\mathcal{B}_1 + D) = \mathcal{D}(\mathcal{B}_1)$ endowed with the graph norm is a Banach space denoted by $X_{\mathcal{B}_1+D}$. We can regard $\mathcal{B}_1 + D$ an operator from $X_{\mathcal{B}_1+D}$ into $\mathcal{X}_1$. This will be denoted by $\widehat{\mathcal{B}_1 + D}$. Using (2.1) we can show that

$$\mu - \widehat{\mathcal{B}_1 + D} \in \Phi^b_+(X_{\mathcal{B}_1}, \mathcal{X}_1) \quad \text{and} \quad ind(\mu - \widehat{\mathcal{B}_1 + D}) \leq 0.$$

Moreover, since $R_2 \in \mathcal{E}_{\mathcal{PR}}(\mathcal{X}_1)$, applying Proposition 2.2, we obtain $I - R_2 \in \Phi(\mathcal{X}_1)$.

Applying [23, Theorem 2.7, p. 171] and (3.2), we get $\mathcal{B}_0 \in \Phi^b(\mathcal{X}_1, X_{\mathcal{B}_1})$.That is $(\mu - \widehat{\mathcal{B}_1 + D})\mathcal{B}_0 \in \Phi^b_+(\mathcal{X}_1)$. Remembering that $\mathcal{A}_0 - \mathcal{B}_0 \in \mathcal{F}_+(\mathcal{X}_1)$ and taking into account (3.6), asserts that $(\mu - \widehat{\mathcal{A}_1 + D})\mathcal{A}_0 \in \Phi^b_+(\mathcal{X}_1)$ and

$$i((\mu - \widehat{\mathcal{A}_1 + D})\mathcal{A}_0) = i((\mu - \widehat{\mathcal{B}_1 + D})\mathcal{B}_0). \tag{3.7}$$

A similar reasoning as before combining (2.1) and (3.1), Proposition 2.2 and [23, Corollary 1.6, p. 166], [23, Theorem 2.6, p. 170] show that $\mathcal{A}_0 \in \Phi^b(\mathcal{X}_1, \mathcal{X}_{\mathcal{A}_1})$, where $\mathcal{X}_{\mathcal{A}_1} := (\mathcal{D}(\mathcal{A}_1), \|\cdot\|_{\mathcal{A}_1})$. By [23, Theorem 1.4, p. 108] one sees that

$$\mathcal{A}_0 S = I - F \quad \text{on } \mathcal{X}_{\mathcal{A}_1}, \tag{3.8}$$

where $S \in \mathcal{L}(\mathcal{X}_{\mathcal{A}_1}, \mathcal{X}_1)$ and $F \in \mathcal{K}(\mathcal{X}_{\mathcal{A}_1})$, by (3.2) we have

$$(\mu - \widehat{\mathcal{B}_1 + D})\mathcal{A}_0 S = (\mu - \widehat{\mathcal{A}_1 + D}) - (\mu - \widehat{\mathcal{A}_1 + D})F. \tag{3.9}$$

Combining the fact that $S \in \Phi^b(\mathcal{X}_{\mathcal{A}_1}, \mathcal{X}_1)$, with [23, Theorem 6.6, p. 129], we show that $(\mu - \widehat{\mathcal{A}_1 + D})\mathcal{A}_0 S \in \Phi^b_+(\mathcal{X}_{\mathcal{A}_1}, \mathcal{X}_1)$. Following [23, Theorem 6.3, p. 128], we derive $(\mu - \widehat{\mathcal{A}_1 + D}) \in \Phi^b_+(\mathcal{X}_{1\mathcal{A}_1}, \mathcal{X}_1)$. Thus, (2.1) asserts that

$$(\mu - \mathcal{A}_1 - D) \in \Phi_+(\mathcal{X}_1). \tag{3.10}$$

On the other hand, the assumptions $R_1, R_2 \in \mathcal{E}_{\mathcal{PR}}(\mathcal{X}_1)$, (3.1), (3.2) and Proposition 2.1, [23, Theorem 2.3, p. 111] reveals that

$$ind(\mathcal{A}_1) + ind(\mathcal{A}_0) = ind(I - R_1) = 0 \quad \text{and} \quad ind(\mathcal{B}_1) + ind(\mathcal{B}_0) = ind(I - R_2) = 0,$$

since $ind(\mathcal{A}_1) = ind(\mathcal{B}_1)$. That is $ind(\mathcal{A}_0) = ind(\mathcal{B}_0)$. Using (3.7) and [20, Theorem 2.3, p. 111], we can write

$$ind(\mu - \mathcal{A}_1 - D) + ind(\mathcal{A}_0) = ind(\mu - \mathcal{B}_1 - D) + ind(\mathcal{B}_0).$$

Therefore,

$$ind(\mu - \mathcal{A}_1 - D) \leq 0, \quad \text{for all } D \in \mathcal{L}(\mathcal{X}_1), \|D\| < \varepsilon. \tag{3.11}$$

Using (3.10) and (3.11), we conclude that

$$\mu \notin \sigma_{eap,\varepsilon}(\mathcal{A}_1).$$

Therefore, we prove the inclusion $\sigma_{eap,\varepsilon}(\mathcal{A}_1) \subset \sigma_{eap,\varepsilon}(\mathcal{B}_1)$. The opposite inclusion follows from symmetry and we obtain (3.3).

(ii) The proof of (3.4) may be checked in a similar way to that in (i). It suffices to replace $\sigma_{eap,\varepsilon}(\cdot), \Phi_+(\cdot), i(\cdot) \leq 0$ [23, Theorem 6.6, p. 129], [23, Theorem 6.3, p. 128], by $\sigma_{e\delta,\varepsilon}(\cdot)$, $\Phi_-(\cdot)$, $i(\cdot) \geq 0$ [20, Theorem 5 (i), p. 150], [23, Theorem 6.7, p. 129], respectively. The details are therefore omitted.

(iii) If $\mu \notin \sigma_{e,\varepsilon}(\mathcal{B}_1)$, then $\mu - \mathcal{B}_1 - D \in \Phi(X)$. Since $\mathcal{B}_1$ is closed, its domain $\mathcal{D}(\mathcal{B}_1)$ becomes a Banach space $\mathcal{X}_{\mathcal{B}_1}$ for the graph norm $\|\cdot\|_{\mathcal{B}_1}$. The use of (2.1) leads to $\mu - \widehat{\mathcal{B}_1 + D} \in \Phi^b(X_{\mathcal{B}_1}, \mathcal{X}_1)$. Moreover, (3.2), Proposition 2.1 and [23, Theorem 5.13], reveal that $B_0 \in \Phi^b(\mathcal{X}_1, \mathcal{X}_{\mathcal{B}_1})$ and consequently $(\mu - \widehat{\mathcal{B}_1 + D})\mathcal{B}_0 \in \Phi^b(\mathcal{X}_1)$. Following with the assumption, (3.6) and [23, Theorem 5.13], lead to estimate $(\mu - \widehat{\mathcal{A}_1 + D})\mathcal{A}_0 \in \Phi^b(\mathcal{X}_1)$ with

$$ind\left((\mu - \widehat{\mathcal{A}_1 + D})\mathcal{A}_0\right) = ind\left((\mu - \widehat{\mathcal{B}_1 + D})\mathcal{B}_0\right). \tag{3.12}$$

Since $\mathcal{A}_1 \in \mathcal{C}(\mathcal{X}_1)$, proceeding as above, (3.1) implies that $\mathcal{A}_0 \in \Phi^b(\mathcal{X}_1, \mathcal{X}_{\mathcal{A}_1})$. By [23, Theorem 5.4], we can write

$$\mathcal{A}_{10}S = I - F, \quad \text{on } \mathcal{X}_{\mathcal{A}_1}, \tag{3.13}$$

where $S \in \mathcal{L}(\mathcal{X}_{\mathcal{A}_1}, \mathcal{X}_1)$ and $F \in \mathcal{F}(\mathcal{X}_{\mathcal{A}_1})$. Taking into account (3.13) we infer that

$$(\mu - \widehat{\mathcal{A}_1 + D})\mathcal{A}_0 S = (\mu - \widehat{\mathcal{A} + D}) - (\mu - \widehat{\mathcal{A}_1 + D})F.$$

Therefore, since $S \in \Phi^b(\mathcal{X}_{\mathcal{A}_1}, \mathcal{X}_1)$, the use of [23, Theorem 6.6], amounts to

$$(\mu - \widehat{\mathcal{A}_1 + D})\mathcal{A}_0 S \in \Phi^b(X_A, \mathcal{X}_1).$$

Applying [23, Theorem 6.3], we prove that $(\mu - \widehat{\mathcal{A}_1 + D}) \in \Phi^b(\mathcal{X}_{\mathcal{A}_1}, \mathcal{X}_1)$ and consequently,

$$(\mu - \mathcal{A}_1 - D) \in \Phi(\mathcal{X}_1).$$

Thus, $\mu \notin \sigma_{e,\varepsilon}(\mathcal{A}_1)$. This implies that $\sigma_{e,\varepsilon}(\mathcal{A}_1) \subset \sigma_{e,\varepsilon}(\mathcal{B}_1)$. Conversely, if $\mu \notin \sigma_{e,\varepsilon}(\mathcal{A}_1)$, we can easily derive the opposite inclusion.

Now, we prove (3.5). If $\mu \notin \sigma_{w,\varepsilon}(\mathcal{B}_1)$, then $\mu \in \Phi_B^\varepsilon$ and $i(\mu - \mathcal{B}_1 - D) = 0$, for all $D \in \mathcal{L}(\mathcal{X}_1)$ with $\|D\| < \varepsilon$. On the other hand, since $R_1, R_2 \in \mathcal{E}_{\mathcal{PR}}(\mathcal{X}_1)$ and $ind(\mathcal{A}_1) = ind(\mathcal{B}_1) = 0$, using the Atkinson theorem, we obtain $ind(\mathcal{A}_0) = ind(\mathcal{B}_1) = 0$. This together with (3.12) gives $ind(\mu - \widehat{\mathcal{A}_1 + D}) = ind(\mu - \widehat{\mathcal{B}_1 + D})$. Consequently, $ind(\mu - \mathcal{A}_1 - D) = 0$, for all $D \in \mathcal{L}(\mathcal{X}_1)$ with $\|D\| < \varepsilon$. Hence, $\mu \notin \sigma_{w,\varepsilon}(\mathcal{A}_1)$, which proves the inclusion $\sigma_{w,\varepsilon}(\mathcal{A}_1) \subset \sigma_{w,\varepsilon}(\mathcal{B}_1)$. The opposite inclusion follows by symmetry. □

In the following theorems we give some perturbation results of the pseudo left, pseudo right Fredholm and pseudo-left, pseudo-right Weyl spectra for bounded linear operators in Banach space.

Theorem 3.2. *Let $\mathcal{A}_1$ and $\mathcal{B}_1$ be two operators in $\mathcal{L}(\mathcal{X}_1)$ and $\mu \in \mathbb{C}$. For all $D \in \mathcal{L}(\mathcal{X}_1)$ such that $\|D\| < \varepsilon$, the following statements hold.*

(*i*) *Assume that for every μ satisfying*

$$\mu - \mathcal{A}_1 \in \Phi_l(\mathcal{X}_1),$$

there exists $\mathcal{A}_l \in \mathcal{I}nv^F_{\mu-\mathcal{A}_1-D,l}(\mathcal{X}_1)$ such that $\mathcal{B}_1\mathcal{A}_l \in \mathcal{E}_{\mathcal{PR}}(\mathcal{X}_1)$. Then,

$$\sigma^l_{e,\varepsilon}(\mathcal{A}_1 + \mathcal{B}_1) \subseteq \sigma^l_{e,\varepsilon}(\mathcal{A}_1).$$

(*ii*) *Assume that for every μ satisfying $\mu - \mathcal{A}_1 \in \Phi_r(\mathcal{X}_1)$, there exists $A_r \in \mathcal{I}nv^F_{\mu-\mathcal{A}_1-D,r}(\mathcal{X}_1)$ such that $A_r\mathcal{B}_1 \in \mathcal{E}_{\mathcal{PR}}(\mathcal{X}_1)$. Then,*

$$\sigma^r_{e,\varepsilon}(\mathcal{A}_1 + \mathcal{B}_1) \subseteq \sigma^r_{e,\varepsilon}(\mathcal{A}_1).$$

Proof. (*i*) Let $\mu \notin \sigma^{\text{left}}_{\text{e},\varepsilon}(\mathcal{A}_1)$, $\mu - \mathcal{A}_1 - D \in \Phi^\varepsilon_l(\mathcal{X}_1)$. As $\mathcal{A}_l$ is a left Fredholm inverse of $\mu - \mathcal{A}_1 - D$, for all $D \in \mathcal{L}(\mathcal{X}_1)$ such that $\|D\| < \varepsilon$, then, by Lemma 2.1, there exists a compact operator $\mathcal{K}_1 \in \mathcal{K}(\mathcal{X}_1)$ such that

$$\mathcal{A}_l(\mu - \mathcal{A}_1 - D) + \mathcal{K}_1 = I.$$

Then, we can write

$$\mu - \mathcal{A}_1 - \mathcal{B}_1 - D = (I - \mathcal{B}_1\mathcal{A}_l)(\mu - \mathcal{A}_1 - D) - \mathcal{B}_1\mathcal{K}_1. \tag{3.14}$$

Using the fact that $\mathcal{B}_1\mathcal{A}_l \in \mathcal{E}_{\mathcal{PR}}(\mathcal{X}_1)$ and according to Proposition 2.2, we have $I - \mathcal{B}_1\mathcal{A}_l \in \Phi(\mathcal{X}_1)$. Consequently, by Lemma 2.3, we get

$$(I - \mathcal{B}_1\mathcal{A}_l)(\mu - \mathcal{A}_1 - D) \in \Phi_l(\mathcal{X}_1), \quad \text{for all } D \in \mathcal{L}(\mathcal{X}_1), \|D\| < \varepsilon.$$

Thus, combining the fact that $\mathcal{B}_1\mathcal{K}_1 \in \mathcal{K}(\mathcal{X}_1)$ with the use of (3.14) and Lemma 2.2, we have $\mu - \mathcal{A}_1 - \mathcal{B}_1 - D \in \Phi_l(\mathcal{X}_1)$, for all $D \in \mathcal{L}(\mathcal{X}_1)$ such that $\|D\| < \varepsilon$. Therefore, $\mu \notin \sigma^{\text{left}}_{\text{e},\varepsilon}(\mathcal{A}_1 + \mathcal{B}_1)$ as required.

(*ii*) Let $\mu \notin \sigma^{\text{right}}_{\text{e},\varepsilon}(\mathcal{A}_1)$. Then, $\mu - \mathcal{A}_1 - D \in \Phi_{\text{r}}(\mathcal{X}_1)$, for all $D \in \mathcal{L}(\mathcal{X}_1)$ such that $\|D\| < \varepsilon$. Since A_{r} is a right Fredholm inverse of $\mu - \mathcal{A}_1 - D$. From Lemma 2.1 we infer there exists a compact operator $\mathcal{K}_1 \in \mathcal{K}(\mathcal{X}_1)$ such that

$$(\mu - \mathcal{A}_1 - D)\mathcal{A}_{\text{r}} = I - \mathcal{K}_1, \quad \text{for all } D \in \mathcal{L}(\mathcal{X}_1), \|D\| < \varepsilon.$$

Then, we can write $\mu - \mathcal{A}_1 - \mathcal{B}_1 - D$ in the following form
(3.15)
$$\mu - \mathcal{A}_1 - \mathcal{B}_1 - D = (\mu - \mathcal{A}_1 - D)\,(I - \mathcal{A}_{\mathrm{r}}\mathcal{B}_1) - \mathcal{K}_1\mathcal{B}_1, \quad \text{for all } D \in \mathcal{L}(\mathcal{X}_1), \|D\| < \varepsilon.$$

Since $\mathcal{A}_{\mathrm{r}}\mathcal{B}_1 \in \mathcal{E}_{\mathcal{PR}}(\mathcal{X}_1)$ then, according to Proposition 2.2, we have $I - \mathcal{A}_{\mathrm{r}}\mathcal{B}_1 \in \Phi(\mathcal{X}_1)$. Consequently, by Lemma 2.3, we get

$$(\mu - \mathcal{A}_1 - D)\,(I - \mathcal{A}_{\mathrm{r}}\mathcal{B}_1) \in \Phi_{\mathrm{r}}(\mathcal{X}_1), \quad \text{for all } D \in \mathcal{L}(\mathcal{X}_1), \|D\| < \varepsilon.$$

On the other hand, from (3.15) and Lemma 2.2 and the fact $\mathcal{B}_1\mathcal{K}_1 \in \mathcal{K}(\mathcal{X}_1)$ we show that $\mu - \mathcal{A}_1 - \mathcal{B}_1 - D \in \Phi_{\mathrm{r}}(\mathcal{X}_1)$, for all $D \in \mathcal{L}(\mathcal{X}_1)$ and $\|D\| < \varepsilon$. We deduce that $\mu \notin \sigma^{\mathrm{right}}_{\mathrm{e},\varepsilon}(\mathcal{A}_1 + \mathcal{B}_1)$. ☐

Theorem 3.3. *Let $\mathcal{A}_1$ and $\mathcal{B}_1$ be two operators in $\mathcal{L}(\mathcal{X}_1)$ and $\mu \in \mathbb{C}$. For all $D \in \mathcal{L}(\mathcal{X}_1)$ with $\|D\| < \varepsilon$ the following statements hold.*

(i) Assume that for every μ satisfying $\mu - \mathcal{A}_1 \in \Phi_l(\mathcal{X}_1)$, there exists

$$\mathcal{A}_l \in \mathcal{I}nv^F_{\mu-\mathcal{A}_1-D,l}(\mathcal{X}_1)$$

such that $\mathcal{B}_1\mathcal{A}_l \in \mathcal{E}_{\mathcal{PR}}(\mathcal{X}_1)$. Then,

$$\sigma^l_{w,\varepsilon}(\mathcal{A}_1 + \mathcal{B}_1) \subseteq \sigma^l_{w,\varepsilon}(\mathcal{A}_1).$$

(ii) Assume that for every μ satisfying $\mu - \mathcal{A}_1 \in \Phi_r(\mathcal{X}_1)$, there exists $\mathcal{A}_r \in \mathcal{I}nv^F_{\mu-\mathcal{A}_1-D,r}(\mathcal{X}_1)$ such that $\mathcal{A}_r\mathcal{B}_1 \in \mathcal{E}_{\mathcal{PR}}(\mathcal{X}_1)$. Then,

$$\sigma^r_{w,\varepsilon}(\mathcal{A}_1 + \mathcal{B}_1) \subseteq \sigma^r_{w,\varepsilon}(\mathcal{A}_1).$$

Proof. (i) Assume that $\mu \notin \sigma^{\mathrm{l}}_{\mathrm{w},\varepsilon}(\mathcal{A}_1)$. Then, we have $\mu - \mathcal{A}_1 - D \in \Phi_l(\mathcal{X}_1)$ and $ind(\mu - \mathcal{A}_1 - D) \leq 0$. A similar reasoning as above gives $\mu - \mathcal{A}_1 - \mathcal{B}_1 - D \in \Phi_l(\mathcal{X}_1)$ and it suffices to prove that $ind(\mu - \mathcal{A}_1 - \mathcal{B}_1 - D) \leq 0$. Since $\mathcal{B}_1\mathcal{K}_1 \in \mathcal{K}(\mathcal{X}_1)$, then using (3.14) together with Lemmas 2.2 and 2.3, we obtain that

$$ind(\mu - \mathcal{A}_1 - \mathcal{B}_1 - D) = ind\,(I - \mathcal{B}_1\mathcal{A}_l) + ind(\mu - \mathcal{A}_1 - D).$$

Now, since $\mathcal{B}_1\mathcal{A}_l \in \mathcal{E}_{\mathcal{PR}}(\mathcal{X}_1)$, we get by Proposition 2.2, that $i\,(I - \mathcal{B}_1\mathcal{A}_l) = 0$. We deduce that

$$ind(\mu - \mathcal{A}_1 - \mathcal{B}_1 - D) = ind\,(\mu - \mathcal{A}_1 - D) \leq 0.$$

Finally, we conclude that $\mu - \mathcal{A}_1 - \mathcal{B}_1 - D \in \mathcal{W}_l(\mathcal{X}_1)$, which entails that $\mu \notin \sigma^{\mathrm{l}}_{\mathrm{w},\varepsilon}(\mathcal{A}_1 + \mathcal{B}_1)$.

(ii) With the same reasoning of (i). Let $\mu \notin \sigma^{\mathrm{r}}_{\mathrm{w},\varepsilon}(\mathcal{A}_1)$. Then, we have $\mu - \mathcal{A}_1 - D \in \Phi_r(\mathcal{X}_1)$ and $i(\mu - \mathcal{A}_1 - D) \geq 0$. Proceeding as the proof above, we establish that $\mu - \mathcal{A}_1 - \mathcal{B}_1 - D \in \Phi_r(\mathcal{X}_1)$ and $ind(\mu - \mathcal{A}_1 - \mathcal{B}_1 - D) \geq 0$. Therefore, $\mu - \mathcal{A}_1 - \mathcal{B}_1 - D \in \mathcal{W}_r(\mathcal{X}_1)$ and we deduce that $\mu \notin \sigma^{\mathrm{r}}_{\mathrm{w},\varepsilon}(\mathcal{A}_1 + \mathcal{B}_1)$. ☐

Remark 3.1. The results of Theorems 3.1, 3.2 and 3.3 are an extension and an improvement of the results in [1–5] to a large class of polynomially Riesz perturbation operators.

4. Characterization Essential Spectrum of Two Linears Bounded Operators

The aim of this section is to establish new criteria for investigating the spectral properties of the sum of two bounded linear operators. We begin by stating the following lemma, which will be used in the sequel.

Lemma 4.1. ([8, Lemma 4.1]) *Let X be a Banch space and $A \in \mathcal{L}(X)$.*
(i) If $C\sigma_e^l(A)$ is connected, then $\sigma_e^l(A) = \sigma_w^l(A)$.
(ii) If $C\sigma_e^r(A)$ is connected, then $\sigma_e^r(A) = \sigma_w^r(A)$.

Theorem 4.1. *Let $\mathcal{A}$, $B \in \mathcal{L}(X)$ and $\mu \in \mathbb{C}^*$. For all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, the following statements hold.*

(i) Assume that the subsets $C\sigma_e^l(A)$ and $C\sigma_e^l(B)$ are connected, $-\mu^{-1}ABQ_l \in \mathcal{E}_{\mathcal{PR}}(X)$ and $-\mu^{-1}BAQ_l \in \mathcal{E}_{\mathcal{PR}}(X)$, for every $Q_l \in \mathcal{I}nv^F_{\mu-A-B-D,l}(X)$. Then, we have

$$\left(\sigma_w^l(A) \cup \sigma_{w,\varepsilon}^l(B)\right) \setminus \{0\} \subseteq \sigma_{w,\varepsilon}^l(A+B) \setminus \{0\}.$$

(ii) Assume that the subsets $C\sigma_e^r(A)$ and $C\sigma_e^r(B)$ are connected, $-\mu^{-1}Q_rAB \in \mathcal{E}_{\mathcal{PR}}(X)$ and $-\mu^{-1}Q_rBA \in \mathcal{E}_{\mathcal{PR}}(X)$, for every $Q_r \in \mathcal{I}nv^F_{\mu-A-B-D,r}(X)$. Then, we have

$$\left(\sigma_w^r(A) \cup \sigma_{w,\varepsilon}^r(B)\right) \setminus \{0\} \subseteq \sigma_{w,\varepsilon}^r(A+B) \setminus \{0\}.$$

(iii) Assume that the subsets $C\sigma_e^l(A)$, $C\sigma_e^l(B)$, $C\sigma_e^r(A)$ and $C\sigma_e^r(B)$ are connected, $-\mu^{-1}ABQ_l \in \mathcal{E}_{\mathcal{PR}}(X)$, $-\mu^{-1}BAQ_l \in \mathcal{E}_{\mathcal{PR}}(X)$, $-\mu^{-1}Q_rAB \in \mathcal{E}_{PR}(X)$ and

$$-\mu^{-1}Q_rBA \in \mathcal{E}_{PR}(X),$$

for $Q_l \in \mathcal{I}nv^F_{\mu-A-B-D,l}(X)$ and $Q_r \in \mathcal{I}nv^F_{\mu-A-B-D,r}(X)$. Then, we have

$$(\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)) \setminus \{0\} \subseteq \sigma_{w,\varepsilon}(A+B) \setminus \{0\}.$$

Proof. First, we note two equalities that will be used repeatedly

$$(\mu - A)(\mu - B - D) = A(B + D) + \mu(\mu - A - B - D), \tag{4.1}$$

$$(\mu - B - D)(\mu - A) = (B + D)A + \mu(\mu - A - B - D). \tag{4.2}$$

(i) Let $\mu \notin \sigma_{w,\varepsilon}^l(A+B) \cup \{0\}$ so we have $\mu - A - B - D \in \Phi_l(X)$ and $i(\mu - A - B - D) \leq 0$. Then, following Lemma 2.1, there exist $Q_l \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $Q_l(\mu - A - B - D) = I - K$. So, when we use (4.1) we obtain

$$\begin{aligned}(\mu - A)(\mu - B - D) &= A(B + D) + \mu(\mu - A - B - D)\\ &= AB\left(Q_l(\mu - A - B - D) + K\right) + \mu(\mu - A - B - D)\\ &= \left(ABQ_l + \mu I\right)(\mu - A - B - D) + ABK\\ &= \mu\left(\mu^{-1}ABQ_l + I\right)(\mu - A - B - D) + ABK.\end{aligned}$$

Since $\mu\left(\mu^{-1}ABQ_l + I\right) \in \Phi(X)$ and $(\mu - A - B - D) \in \Phi_l(X)$ it follows from Proposition 2.1 that $\mu\left(\mu^{-1}ABQ_l + I\right)(\mu - A - B - D) \in \Phi_l(X)$. Since $ABK \in \mathcal{K}(X)$, this implies by the use of Lemma 2.2 that

$$\mu\left(\mu^{-1}ABQ_l + I\right)(\mu - A - B - D) + ABQ_lK \in \Phi_l(X).$$

So, $(\mu - A)(\mu - B - D) \in \Phi_l(X)$ and as a direct consequence of Lemma 2.4 we obtain

$$\mu - B - D \in \Phi_l(X), \quad \text{for all } D \in \mathcal{L}(X), \|D\| < \varepsilon. \tag{4.3}$$

In the other hand, when we use the (4.2) we have

$$\begin{aligned}(\mu - B - D)(\mu - A) =& BA + \mu(\mu - A - B - D),\\ =& BA\left(Q_l(\mu - A - B - D) + K\right) + \mu(\mu - A - B - D)\\ =& \left(BAQ_l + \mu I\right)(\mu - A - B - D) + BAK\\ =& \mu\left(\mu^{-1}BAQ_l + I\right)(\mu - A - B - D) + BAK.\end{aligned}$$

Since $\mu\left(\mu^{-1}BAQ_l + I\right) \in \Phi(X)$ and $(\mu - A - B - D) \in \Phi_l(X)$ it follows, from Proposition 2.1, that

$$\mu\left(\mu^{-1}BAQ_l + I\right)(\mu - A - B - D) \in \Phi_l(X).$$

Obviously, since $BAK \in \mathcal{K}(X)$ and applying Lemma 2.2, we find that

$$\mu\left(\mu^{-1}BAQ_l + I\right)(\mu - A - B - D) + BAK \in \Phi_l(X).$$

So, $(\mu - B - D)(\mu - A) \in \Phi_l(X)$. Therefore, using Lemma 2.4, we obtain

$$\mu - A \in \Phi_l(X). \tag{4.4}$$

Now, to check the index we must have a discussion according to the sign, thus using the above we have

$$ind(\mu - A) + ind(\mu - B - D) = ind(\mu - A - B - D) \le 0.$$

Case 1: $ind(\mu - A) \le 0$.

Using Lemma 4.1, the index $ind(\mu - B - D)$ must be negative. Therefore, adding this condition to (4.3) and (4.4) we obtain

$$\mu \notin \left(\sigma^l_w(A) \cup \sigma^l_{w,\varepsilon}(B)\right) \cup \{0\}.$$

Case 2: $ind(\mu - B - D) \le 0$.

Following Lemma 4.1, the index $ind(\mu - A)$ must be negative. Then, adding this condition to (4.3) and (4.4) we assert

$$\mu \notin \left(\sigma^l_w(A) \cup \sigma^l_{w,\varepsilon}(B)\right) \cup \{0\}.$$

Case 3: $ind(\mu - A) > 0$.

Following Lemma 4.1, the index $ind(\mu - B - D)$ should be positive which contradicts the fact that $ind(\mu - A - B - D) \le 0$.

Case 4: $ind(\mu - B - D) > 0$.

Following Lemma 4.1, the index $i(\mu - A)$ must be positive which contradicts the fact that $ind(\mu - A - B - D) \le 0$.

(ii) Let $\mu \notin \sigma^r_{w,\varepsilon}(A+B) \cup \{0\}$. Then, $\mu - A - B - D \in \Phi_r(X)$ and $i(\mu - A - B - D) \leq 0$. So, by Lemma 2.1, there exist $Q_r \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $(\mu - A - B - D)Q_r = I - K$. So, following (4.1) we have

$$\begin{aligned}(\mu - A)(\mu - B - D) =& AB + \mu(\mu - A - B - D)\\ =& \left((\mu - A - B - D)Q_r + K\right) AB + \mu(\mu - A - B - D)\\ =& (\mu - A - B - D)\left(Q_r AB + \mu I\right) + ABK\\ =& \mu(\mu - A - B - D)\left(\mu^{-1}Q_r AB + I\right) + ABK.\end{aligned}$$

Since $\mu\left(\mu^{-1}Q_r AB + I\right) \in \Phi(X)$ and $(\mu - A - B - D) \in \Phi_r(X)$, it follows by Proposition 2.1 that

$$\mu\left(\mu^{-1}Q_r AB + I\right)(\mu - A - B - D) \in \Phi_r(X).$$

Since $ABK \in \mathcal{K}(X)$, then

$$\mu\left(\mu^{-1}Q_r AB + I\right)(\mu - A - B - D) + ABK \in \Phi_r(X).$$

So, $(\mu - A)(\mu - B - D) \in \Phi_r(X)$, following Lemma 2.4 we infer that

$$\mu - A \in \Phi_r(X).$$

In the other hand, the use of (4.2) assert

$$\begin{aligned}(\mu - B - D)(\mu - A) =& BA + \mu(\mu - A - B - D)\\ =& BA\left((\mu - A - B - D)Q_r + K\right) BA + \mu(\mu - A - B - D)\\ =& (\mu - A - B - D)\left(Q_r BA + \mu I\right) + KBA\\ =& \mu(\mu - A - B - D)\left(\mu^{-1}Q_r BA + I\right) + KBA.\end{aligned}$$

By hypothesis $(\mu^{-1}Q_r BA + I) \in \Phi(X)$ and $(\mu - A - B - D) \in \Phi_r(X)$ we have by Proposition 2.1

$$\mu(\mu - A - B - D)\left(\mu^{-1}Q_r BA + I\right) \in \Phi_r(X).$$

Since $KBA \in \mathcal{K}(X)$, we obtain

$$\mu(\mu - A - B - D)\left(\mu^{-1}Q_r BA + I\right) + KBA \in \Phi_r(X).$$

So, $(\mu - B - D)(\mu - A) \in \Phi_r(X)$. Then, the use of Lemma 2.4 infer that

$$\mu - B - D \in \Phi_r(X), \quad \text{for all } D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{4.5}$$

Now, to check the index we must have a discussion according to the sign. Thus, using the above we have

$$ind(\mu - A) + ind(\mu - B - D) = ind(\mu - A - B - D) \geq 0.$$

Case 1: $ind(\mu - A) \geq 0$.

Using Lemma 4.1, the index $ind(\mu - B - D)$ must be positive. Therefore, adding this condition to (3.8) and (4.5) we get

$$\mu \notin \left(\sigma^r_w(A) \cup \sigma^r_{w,\varepsilon}(B)\right) \cup \{0\}.$$

Case 2: $ind(\mu - B - D) \geq 0$.

Following Lemma 4.1, the index $ind(\mu - A)$ must be positive. Then adding this condition to (4.3) and (4.4) we obtain

$$\mu \notin \left(\sigma^r_w(A) \cup \sigma^r_{w,\varepsilon}(B)\right) \cup \{0\}.$$

Case 3: $ind(\mu - A) < 0$.

Following Lemma 4.1, the index $i(\mu - B - D)$ should be negative which contradicts the fact that $ind(\mu - A - B - D) \geq 0$.

Case 4: $ind(\mu - B - D) < 0$.

Following Lemma 4.1, the index $ind(\mu - A)$ should be negative which contradicts the fact that $ind(\mu - A - B - D) \geq 0$.

(iii) Let $\mu \notin \sigma_{w,\varepsilon}(A+B) \cup \{0\}$. Therefore, $\mu - A - B - D \in \Phi(X)$ and $ind(\mu - A - B - D) = 0$ then there exist $Q_l, Q_r \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $Q_l(\mu - A - B - D) = I - K$ and $(\mu - A - B - D)Q_r = I - K$. Now, according to items (i) and (ii) we get

$$(\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)) \setminus \{0\} \subseteq \sigma_{w,\varepsilon}(A + B) \setminus \{0\}.$$

□

Theorem 4.2. *Let A, $B \in \mathcal{L}(X)$ such that $AB = BA$ and $\mu \in \mathbb{C}^*$. For all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, the following statements hold.*

(i) If there exists $Q_l \in \mathcal{I}nv^F_{\mu-A-B-D,l}(X)$ such that $-\mu^{-1}ABQ_l \in \mathcal{E}_{\mathcal{PR}}(X)$, then

$$\sigma^l_{e,\varepsilon}(A + B) \setminus \{0\} = \left(\sigma^l_e(A) \cup \sigma^l_{e,\varepsilon}(B)\right) \setminus \{0\}.$$

(ii) If there exists $Q_r \in \mathcal{I}nv^F_{\mu-A-B-D,r}(X)$ such that $-\mu^{-1}Q_rAB \in \mathcal{E}_{\mathcal{PR}}(X)$, then

$$\sigma^r_{e,\varepsilon}(A + B) \setminus \{0\} = \left(\sigma^r_e(A) \cup \sigma^r_{e,\varepsilon}(B)\right) \setminus \{0\}.$$

(iii) If there exists $Q \in \mathcal{I}nv^F_{\mu-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\mu-A-B-D,r}(X)$ such that $-\mu^{-1}QAB \in \mathcal{E}_{\mathcal{PR}}(X)$ and $-\mu^{-1}ABQ \in \mathcal{E}_{\mathcal{PR}}(X)$, then

$$\sigma_{e,\varepsilon}(A + B) \setminus \{0\} = (\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)) \setminus \{0\}.$$

Proof. (i) Let $\mu \notin \sigma^l_{e,\varepsilon}(A + B) \cup \{0\}$. Then, $\mu - A - B - D \in \Phi_l(X)$.

We assume there exists $Q_l \in \mathcal{I}nv^F_{\mu-A-B-D,l}(X)$. Thus, using (4.1), we have

$$\begin{aligned}(\mu - A)(\mu - B - D) =& A(B + D) + \mu(\mu - A - B - D)\\ =& AB\left(Q_l(\mu - A - B - D) + K\right) + \mu(\mu - A - B - D)\\ =& \left(ABQ_l + \mu I\right)(\mu - A - B - D) + ABK\\ =& \mu\left(\mu^{-1}ABQ_l + I\right)(\mu - A - B - D) + ABK.\end{aligned}$$

Obviously, $-\mu^{-1}ABQ_l \in \mathcal{E}_{PR}(X)$ then by Proposition 2.2 we infer that $\mu^{-1}ABQ_l + I \in \Phi(X)$. Therefore, by Lemma 2.3, we obtain $\left(\mu^{-1}ABQ_l + \mu I\right)(\mu - A - B - D) \in \Phi_l(X)$. Since $ABK \in \mathcal{K}(X)$ and by applying Lemma 2.2 we obtain

$$\mu\left(\mu^{-1}ABQ_l + I\right)(\mu - A - B - D) + ABK \in \Phi_l(X).$$

We conclude that

$$(\mu - A)(\mu - B - D) \in \Phi_l(X), \quad \text{for all } D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon.$$

Hence, by Lemma 2.4 we deduce that

$$(\mu - B - D) \in \Phi_l(X), \quad \text{for all } D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{4.6}$$

On the other hand, using the fact that $AB = BA$ and according to (4.2) we observe that

$$\begin{aligned}(\mu - B - D)(\mu - A) =& BA + \mu(\mu - A - B - D)\\ =& AB + \mu(\mu - A - B - D)\\ =& AB\left(Q_l(\mu - A - B - D) + K\right) + \mu(\mu - A - B - D)\\ =& \left(ABQ_l + \mu I\right)(\mu - A - B - D) + ABK\\ =& \mu\left(\mu^{-1}ABQ_l + I\right)(\mu - A - B - D) + ABK.\end{aligned}$$

Using the same reasoning we conclude that $(\mu - B - D)(\mu - A) \in \Phi_l(X)$. Therefore, by Lemma 2.4 we deduce that

$$(\mu - A) \in \Phi_l(X). \tag{4.7}$$

Finally, Equations (4.6) and (4.7) imply that $\mu \notin \left(\sigma_e^l(A) \cup \sigma_{e,\varepsilon}^l(B)\right) \cup \{0\}$. So, we obtain

$$\left(\sigma_e^l(A) \cup \sigma_{e,\varepsilon}^l(B)\right) \setminus \{0\} \subset \sigma_{e,\varepsilon}^l(A + B) \setminus \{0\}.$$

The other inclusion, which allows us to achieve equality, is given in [8, Theorem 4.3].

(ii) Let $\mu \notin \sigma_{e,\varepsilon}^r(A + B) \cup \{0\}$. Then, $\mu - A - B - D \in \Phi_r(X)$, for all $D \in \mathcal{L}(X)$ and $\|D\| < \varepsilon$. We assume there exists $Q_r \in \mathcal{I}nv^F_{\mu-A-B-D,r}(X)$. Thus,

$$\begin{aligned}(\mu - A)(\mu - B - D) =& AB + \mu(\mu - A - B - D)\\ =& \left((\mu - A - B - D)Q_r + K\right)AB + \mu(\mu - A - B - D)\\ =& (\mu - A - B - D)\mu\left(\mu^{-1}Q_rAB + I\right) + KAB.\end{aligned}$$

Evidently, $-\mu^{-1}Q_rAB \in \mathcal{E}_{\mathcal{PR}}(X)$ and by applying Proposition 2.2 we deduce that $\mu^{-1}Q_rAB + I \in \Phi(X)$. Since, KAB is compact, then by Lemma 2.2 we obtain

$$(\mu - A - B - D)\mu\left(\mu^{-1}Q_rAB + I\right) + KAB \in \Phi_l(X).$$

Consequently, we have $(\mu - A)(\mu - B - D) \in \Phi_r(X)$ and by Lemma 2.4 we infer that

$$(\mu - A) \in \Phi_r(X). \tag{4.8}$$

Further, we have $AB = BA$, so,

$$\begin{aligned}(\mu - B - D)(\mu - A) =& BA + \mu(\mu - A - B - D)\\ =& AB + \mu(\mu - A - B - D)\\ =& ((\mu - A - B - D)Q_r + K)\, AB + \mu(\mu - A - B - D)\\ =& (\mu - A - B - D)\mu\left(\mu^{-1}Q_rAB + I\right) + KAB.\end{aligned}$$

Using the same reasoning we conclude that $(\mu - B - D)(\mu - A) \in \Phi_r(X)$. Then, by Lemma 2.4 we deduce that

$$(\mu - B - D) \in \Phi_r(X), \quad \text{for all } D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{4.9}$$

Finally, Equations (4.8) and (4.9) imply that

$$\mu \notin \left(\sigma_e^r(A) \cup \sigma_{e,\varepsilon}^r(B)\right) \cup \{0\}.$$

So, we obtain

$$\left(\sigma_e^r(A) \cup \sigma_{e,\varepsilon}^r(B)\right) \setminus \{0\} \subset \sigma_{e,\varepsilon}^r(A + B) \setminus \{0\}.$$

The other inclusion, which allows us to achieve equality, is given in [8, Theorem 4.3].

(iii) Let $\mu \notin \sigma_{e,\varepsilon}(A + B) \cup \{0\}$. Then, $\mu - A - B - D \in \Phi(X)$ means that $\mu - A - B - D \in \Phi_l(X) \cap \Phi_r(X)$.

Now, by the hypothesis there exists $Q \in \mathcal{I}nv^F_{\mu-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\mu-A-B-D,r}(X)$, and by applying the results in statements (i) and (ii) we infer that $(\mu-A-B-D) \in \Phi_r(X)$ and $(\mu - A - B - D) \in \Phi_l(X)$, therefore $(\mu - A - B - D) \in \Phi(X)$. Also, using the hypothesis that $-\mu^{-1}QAB \in \mathcal{E}_{\mathcal{PR}}(X)$, $-\mu^{-1}ABQ \in \mathcal{E}_{\mathcal{PR}}(X)$ and $AB = BA$, we obtain the following two conditions $(\mu - A)(\mu - B - D) \in \Phi(X)$ and $(\mu - B - D)(\mu - A) \in \Phi(X)$. Therefore, following Theorem 2.1 we obtain $(\mu - A) \in \Phi(X)$ and $(\mu - B - D) \in \Phi(X)$ means that $\mu \notin (\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)) \cup \{0\}$. Then, we get the following inclusion

$$(\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)) \setminus \{0\} \subseteq \sigma_{e,\varepsilon}(A + B) \setminus \{0\}.$$

The other inclusion, which allows us to achieve equality, is given in [8, Theorem 4.3]. □

By the same reasoning as in the above theorem, we obtain the result stated in the following theorem.

Theorem 4.3. *Let A, $B \in \mathcal{L}(X)$ such that $AB = BA$ and $\mu \in \mathbb{C}^*$. For all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, the following statements hold.*

(i) *If there exists $Q_l \in \mathcal{I}nv^F_{\mu-A-B-D,l}(X$, such that $-\mu^{-1}ABQ_l \in \mathcal{E}_{\mathcal{PR}}(X)$, then*

$$\sigma_{w,\varepsilon}^l(A + B) \setminus \{0\} = \left(\sigma_{w,\varepsilon}^l(A) \cup \sigma_{w,\varepsilon}^l(B)\right) \setminus \{0\}.$$

(ii) *If there exists $Q_r \in \mathcal{I}nv^F_{\mu-A-B-D,r}(X)$ such that $-\mu^{-1}Q_rAB \in \mathcal{E}_{\mathcal{PR}}(X)$, then*

$$\sigma_{w,\varepsilon}^r(A + B) \setminus \{0\} = \left(\sigma_w^r(A) \cup \sigma_{w,\varepsilon}^r(B)\right) \setminus \{0\}.$$

(iii) If there exists $Q \in \mathcal{I}nv^F_{\mu-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\mu-A-B-D,r}(X)$ *such that* $-\mu^{-1}QAB \in \mathcal{E}_{\mathcal{PR}}(X)$ *and* $-\mu^{-1}ABQ \in \mathcal{E}_{\mathcal{PR}}(X)$*, then*

$$\sigma_{w,\varepsilon}(A+B) \setminus \{0\} = (\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)) \setminus \{0\}.$$

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