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THE APPLICATION OF A HYBRID QUANTUM-SPECTRAL-SUCCESSIVE LINEARIZATION METHOD FOR THE NONLINEAR EQUATIONS ARISING IN HEAT TRANSFER AND HEAT RADIATION

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Abstract. In this article, the vision of quantum computing is coupled with a pseudo-spectral method for two classes of heat equations. First, the cooling of a lumped system with variable specific heat and an unsteady nonlinear convective-radiative equation containing two small parameters are considered. The nonlinear equations are linearized using the successive linearization method. In an iterative scheme, the final state is computed using the quantum state at each iteration by a novel quantum algorithm. Two numerical illustrations are considered, and a comparison shows the efficiency of the method.

1. Introduction

Solving the nonlinear equations is very important, and many problems of heat transfer or heat radiation are nonlinear. As we know, numerical methods are very useful in solving these nonlinear problems. In the last decades, many numerical or semi-analytical methods have been developed for such problems. For example, Adomian decomposition method [4], non-perturbation methods [25], δ-expansion method [19], homotopy analysis method (HAM) [1, 2, 23], numerical simulation [21, 27, 32], spectral method [13], and many others.

Spectral methods and quasi-linearization techniques are powerful tools for tackling engineering problems, especially in solving differential equations, due to their efficiency and high numerical accuracy - advantages well documented in [18, 31] and numerous

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related studies. In contrast to traditional numerical approaches, spectral methods use a global framework, that provides superior precision, making them particularly effective for the treatment of nonlinear differential equations [26, 28].

Recent advances in quantum computing have led to algorithms that are able to extract information about solutions of differential equations far more efficiently than classical approaches. In this paper, a quantum pseudo-spectral method (QPSM) for the cooling of a lumped system with variable specific heat [1, 6, 14] and then an unsteady nonlinear convective-radiative equation containing two small parameters [2, 6, 15] can be considered. Recently, this method has also been used to solve the general Lane-Emden type equations [3]. For linear IVPs (initial value problems), QPSM is considered in [10], which is based on the quantum linear systems algorithm (QLSA), [16]. For a linear sparse of the system of d equations, QLSA can generate a quantum state proportional to the solution system in time poly(log$d$). The quantum algorithm with the pseudo-spectral method for time-dependent IVP and BVP described in [10] has the complexity poly(log$d$, log$(1/\epsilon)$). Newly, the non-linear quadratic ODEs are considered with the quantum method and Carleman linearization [20, 24] with complexity E^2q poly(log$E$; log$d$; log$1/\epsilon)/\epsilon$, where E is the evolution time and q measures the decay of the solution, and ϵ is the solution error.

While our work builds on existing research, alternative quantum approaches for solving ordinary differential equations (ODEs) have emerged. For example, [22] has proposed a quantum adaptation of the Euler method for simple nonlinear ODEs, that achieves logarithmic complexity in the system dimension - although it suffers from the exponential scaling of the evolution time, an inherent limitation for nonlinear ODEs. In addition to ODEs, quantum techniques have also been applied to partial differential equations (PDEs), including the following.

(a) ([5]) A continuous-variable quantum algorithm for initial value problems (IVPs) involving non-homogeneous linear PDEs.
(b) ([11]) A hybrid quantum-classical approach combining the Quantum Linear Systems Algorithm (QLSA) with finite element methods for Maxwell's equations.
(c) ([12]) A Hamiltonian simulation-based solver for wave equations using finite difference discretization.

First, we will consider the cooling of a lumped system with variable specific heat [6,14] with surface area A, volume V, specific heat C, density ρ, and initial temperature T_i. At the initial time $t = 0$, the system is exposed to a convective environment at temperature T_a with convective heat transfer coefficient ω. Suppose C satisfies temperature

$$C = C_a[1 + \beta(T - T_a)],$$

where C_a is the heat at T_a and β is a constant. The cooling equation with the initial condition is

$$\rho V C \frac{\mathrm{d}T}{\mathrm{d}t} + \omega A(T - T_a) = 0, \quad T(0) = T_i,$$

and with

$$u = \frac{T - T_a}{T_i - T_a}, \quad \tau = \frac{t(\omega A)}{\rho V C_a}, \quad \varepsilon = \beta(T - T_a).$$

Then, the following non-linear IVP is obtained for $\tau \in [0, S]$

$$(1 + \varepsilon u)\frac{\mathrm{d}u}{\mathrm{d}\tau} + u = 0, \quad u(0) = 1. \tag{1.1}$$

For doing QPSM, first, we have to linearize (1.1) by SLM (successive linearization method) because QPSM is efficient for linear models. Suppose

$$u(\tau) = U_i(\tau) + \sum_{k=0}^{i-1} u_k(\tau), \quad i = 1, 2, 3, \dots, \tag{1.2}$$

where $U_i(\cdot)$ are the unknown functions, and $u_k(\cdot)$ are the successive approximations solutions that obtained recursively by QPSM for solving the linear part of the equation obtained by substituting (1.2) in (1.1). Let the initial approximation, $u_0(\cdot)$, satisfies the initial condition in (1.1), for example here we put $u_0(\cdot) = 1$ for simplicity. Also, we assumed that, $\lim_{i\to+\infty} U_i = 0$.

By simple calculations and substituting (1.2) in (1.1), we have

$$U_i' + \varepsilon \left(U_i + \sum_{k=0}^{i-1} u_k \right) \left(U_i' + \sum_{k=0}^{i-1} u_k' \right) + U_i = -\sum_{k=0}^{i-1} \left(u_k' + u_k \right). \tag{1.3}$$

Starting from the initial approximation $u_0(\cdot) = 1$, the IVP obtained from the linearized form of (1.3) is solved to compute the later $u_i(\cdot)$. Hence, we should solve M time the following IVPs

$$(1 + \varepsilon a_{i-1})u_i' + (1 + \varepsilon a_{i-1}')u_i = \mu_{i-1}, \quad i \in [M] = \{1, 2, 3, \dots, M\}, \tag{1.4}$$

where

$$a_{i-1}(\cdot) = \sum_{k=0}^{i-1} u_k(\cdot),$$
$$\mu_{i-1}(\cdot) = -\Big((1 + \varepsilon a_{i-1}(\cdot))a_{i-1}'(\cdot) + a_{i-1}(\cdot) \Big),$$

with the initial condition $u_i(0) = 0$. Now we want to use the pseudo-spectral method by using the Chebyshev polynomials [17, 29]. By taking the truncated Chebyshev approximation of (1.4) we have

$$u_i(\cdot) = \sum_{l=0}^{n} c_{i,l} T_l(\cdot), \tag{1.5}$$

for any $n \in \mathbb{Z}^+$ and also for $\zeta \in [-1, 1]$, $T_l(\zeta) = \cos(l \arccos \zeta)$. Concerning the domain of (1.5), later we rescale the domain of (1.1) to $[-1, 1]$.

To solve the linear system derived from (1.4), the Chebyshev-Gauss-Lobatto quadrature nodes $\zeta_k = \cos(\frac{k\pi}{n})$ for $k \in [n+1]_0 = \{0, 1, \dots, n\}$ are used. For calculating

$c_{i,l}$, let $c'_{i,l}$ be defined as

$$\frac{\mathrm{d}u_i(\zeta)}{\mathrm{d}\zeta} = \sum_{l=0}^{n} c'_{i,l} T_l(\zeta),$$

with

$$c'_{i,l} = \sum_{j=0}^{n} [D_n]_{l,j} c_{i,j},$$

and the upper triangular matrix D_n is [30]

$$[D_n]_{l,j} = \frac{2j}{\sigma_l}, \quad j > l, l+j \text{ odd},$$

with

$$\sigma_l = \begin{cases} 1, & l \in [n], \\ 2, & l = 0. \end{cases}$$

Using (1.4), (1.5), we have for $k \in [n+1]_0$ and $i \in [M]$

$$\sum_{l=0}^{n} c'_{i,l} T_l(\zeta_k) = \Big(\mu_{i-1} - \sum_{l=0}^{n} (1 + \varepsilon a'_{i-1} c_{i,l}) T_l(\zeta_k)\Big) / (1 + \varepsilon a_{i-1}), \tag{1.6}$$

with imposing the initial conditions. The linear system (1.6) is solved with QLSA (quantum linear systems algorithm) [9]. Before rescaling, $[0, S]$ is divided into m subintervals to improve the accuracy.

$$[0, \Psi_1], [\Psi_1, \Psi_2], \dots, [\Psi_{m-1}, S],$$

with $\Psi_0 = 0,\ \Psi_m = S$. Each subinterval $[\Psi_q, \Psi_{q+1}]$ converted onto $[-1, 1]$ for $q \in [m]_0$, hence $\Delta_q = S/m = \Psi_{q+1} - \Psi_q$. Now, $\tau \in [\Psi_q, \Psi_{q+1}]$ converts to $\zeta \in [-1, 1]$ by

$$\zeta = K_q(\tau) = 1 - \frac{2(\tau - \Psi_q)}{\Delta_q}$$

and

$$\tau = IK_q(\zeta) = \Psi_q + \frac{\Delta_q(1 - \zeta)}{2}.$$

Hence, after rescaling, (1.4) converts to

$$z'_{i,q}(\zeta) = \Big(\mu_{i-1} - \Big(\frac{1}{D} + \varepsilon b'_{i-1,q}\Big) z_{i,q}(\zeta)\Big) / (1 + \varepsilon b_{i-1,q}), \tag{1.7}$$

where

$$b_{i-1,q}(\cdot) = \sum_{j=0}^{i-1} z_{j,q}(\cdot),$$

$$\eta_{i-1,q}(\cdot) = -\left((1 + \varepsilon b_{i-1,q}(\cdot)) b'_{i-1,q}(\cdot) + \frac{1}{D} b_{i-1,q}(\cdot)\right),$$

and $D = \frac{\mathrm{d}\zeta}{\mathrm{d}\tau}$ and $z_{i,q}(\zeta) = y_i(\tau) = y_i(IK_q(\zeta))$ for $\tau \in [\Psi_q, \Psi_{q+1}]$, $q \in [m]_0$, $i \in [M]$ and $z_{0,0} = 1$.

2. Quantum Pseudo-spectral Method

As discussed in the previous section, we should solve a linear system like

$$L_i|X_i\rangle = |B_i\rangle, \tag{2.1}$$

in each iteration by the QLSA presented in [10] from the linear equation (1.7). For a matrix L_i with a large condition number, κ_i, QLSA can selectively invert only the portion of $|B_i\rangle$ that lies within the well-conditioned subspace of L_i, i.e., the subspace spanned by eigenvectors corresponding to its largest eigenvalues. In this method, the quantum state $|X_i\rangle$ decomposes as $|X_i\rangle = |X_{i,bad}\rangle + |X_{i,good}\rangle$, where the first part is related to eigenvalues of L_i which are less than $\frac{1}{\kappa_i}$ and conversely for the second part. The second part serves as a flag, enabling the user to either estimate the ill-conditioned subspace's size or manage it using their preferred approach. If L_i is non-invertible and $\frac{1}{\kappa_i}$ is chosen smaller than the smallest nonzero eigenvalue of L_i, this method computes the pseudo-inverse of L_i. Preconditioning is another common classical technique for treating ill-conditioned matrices [8].

The vector

$$|X_i\rangle \in \mathbb{C}^{m+p+1} \otimes \mathbb{C}^{n+1}$$

describes the solution by

$$|X_i\rangle = \sum_{q=0}^{m-1}\sum_{l=0}^{n} c_{i,l}(\Psi_{q+1})|ql\rangle + \sum_{q=m}^{m+p}\sum_{l=0}^{n} x_i|ql\rangle, \tag{2.2}$$

where $c_{i,l}(\Psi_{q+1})$ is the Chebyshev series coefficient of $z_{i,q}(K_q(\Psi_{q+1}))$, x_i is the last state $z_{i,m-1}(K_q(\Psi_m))$, p is a padding trick [3, 7, 16]. We require $1 + \sum_{i=1}^{M} x_i$ as the output at S, because of $u_0(\cdot) = 1$.

The elements of L_i and B_i are computed according to the QPSM algorithm in Figure 1. It can be proved that for the linear equation (1.7), we have

$$\begin{aligned}|B_{i,q}\rangle =& 0|0\rangle + \sum_{l=1}^{n} \frac{\eta_{i-1,q}(\zeta_l)}{1+\varepsilon b_{i-1,q}(\zeta_l)}|l\rangle,\\ A_{i,q}(\cdot) =& -\frac{\frac{1}{D} + \varepsilon b'_{i-1,q}(\cdot)}{1+\varepsilon b_{i-1,q}(\cdot)},\end{aligned}$$

whereas before $\zeta_l = \cos(\frac{l\pi}{n})$.

3. Unsteady Nonlinear Convective-radiative Equation

We will consider a lumped system of combined convective-radiative heat transfers, the specific heat coefficient is linear with temperature [6, 14]. The cooling equation and the initial condition are as follows

$$\rho V C \frac{\mathrm{d}T}{\mathrm{d}t} + \omega A(T - T_a) + E\sigma A(T^4 - T_s^4) = 0, \quad T(0) = T_i,$$

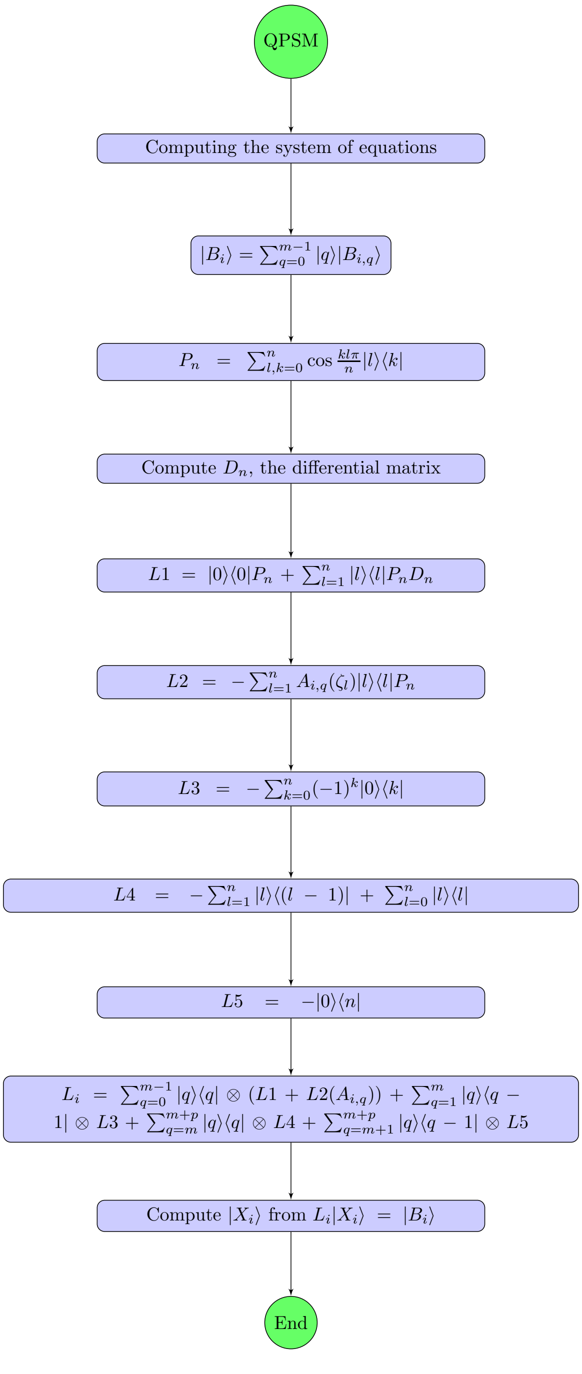

FIGURE 1. Algorithm of the QPSM.

which by using

$$u = \frac{T}{T_i}, \quad u_a = \frac{T_a}{T_i}, \quad \tau = \frac{t(\omega A)}{\rho V C_a}, \quad \varepsilon_1 = \beta T_i, \quad \varepsilon_2 = \frac{E\sigma T_i^3}{\omega}, \quad u_s = \frac{T_s}{T_i},$$

we have

$$(1 + \varepsilon_1(u - u_a))\frac{\mathrm{d}u}{\mathrm{d}\tau} + (u - u_a) + \varepsilon_2(u^4 - u_s^4) = 0, \quad u(0) = 1.$$

Now, a basic problem is how to guess ε_1 and ε_2. By QPSM, we can obtain a reasonable solution for every ε_1 and ε_2. But, by perturbation methods only for small values of ε_1 and ε_2, we can obtain reasonable solution [2]. For simplicity, we assume $u_a = u_s = 0$ and hence we have the following non-linear IVP for $\tau \in [0, S]$

$$(1 + \varepsilon_1 u)\frac{\mathrm{d}u}{\mathrm{d}\tau} + u + \varepsilon_2 u^4 = 0, \quad u(0) = 1. \tag{3.1}$$

In this case, as before, $u_0(\cdot) = 1$ and for $i \geq 1$, the other $u_i(\cdot)$ are computed by solving the linearized form of (3.1), i.e., the linear IVP

$$(1 + \varepsilon_1 a_{i-1})u_i' + (1 + \varepsilon_1 a_{i-1}' + 4\varepsilon_2 a_{i-1}^3)u_i = \mu_{i-1}, \quad i \in [M], \tag{3.2}$$

where

$$a_{i-1}(\cdot) = \sum_{l=0}^{i-1} u_l(\cdot),$$
$$\mu_{i-1}(\cdot) = -\Big((1 + \varepsilon_1 a_{i-1}(\cdot))a_{i-1}'(\cdot) + a_{i-1}(\cdot) + \varepsilon_2 a_{i-1}^4(\cdot)\Big),$$

with $u_i(0) = 0$. As before, by constructing m subintervals from $[0, S]$ and after rescaling, (3.2) converts to

$$z_{i,q}'(\zeta) = \Big(\mu_{i-1} - \Big(\frac{1}{D} + \varepsilon_1 b_{i-1,q}' + \frac{4}{D}\varepsilon_2 b_{i-1,q}^3\Big) z_{i,q}(\zeta)\Big)/(1 + \varepsilon_1 b_{i-1,q}), \tag{3.3}$$

where

$$b_{i-1,q}(\cdot) = \sum_{j=0}^{i-1} z_{j,q}(\cdot),$$
$$\eta_{i-1,q}(\cdot) = -\left((1 + \varepsilon_1 b_{i-1,q}(\cdot))b_{i-1,q}'(\cdot) + \frac{1}{D}b_{i-1,q}(\cdot) + \frac{\varepsilon_2}{D}b_{i-1,q}^4(\cdot)\right),$$

and $D = \frac{\mathrm{d}\zeta}{\mathrm{d}\tau}$ and $z_{i,q}(\zeta) = y_i(\tau) = y_i(IK_q(\zeta))$ for $\tau \in [\Psi_q, \Psi_{q+1}]$ and $q \in [m]_0$. For simplicity, we take $z_{0,0} = 1$ because of initial condition in (3.1). For implementing QPSM in this case we have

$$|B_{i,q}\rangle = 0|0\rangle + \sum_{l=1}^{n} \frac{\eta_{i-1,q}(\zeta_l)}{1 + \varepsilon_1 b_{i-1,q}(\zeta_l)}|l\rangle$$

and

$$A_{i,q}(\cdot) = -\frac{\frac{1}{D} + \varepsilon_1 b_{i-1,q}'(\cdot) + \frac{4\varepsilon_2}{D}b_{i-1,q}^4(\cdot)}{1 + \varepsilon_1 b_{i-1,q}(\cdot)}.$$

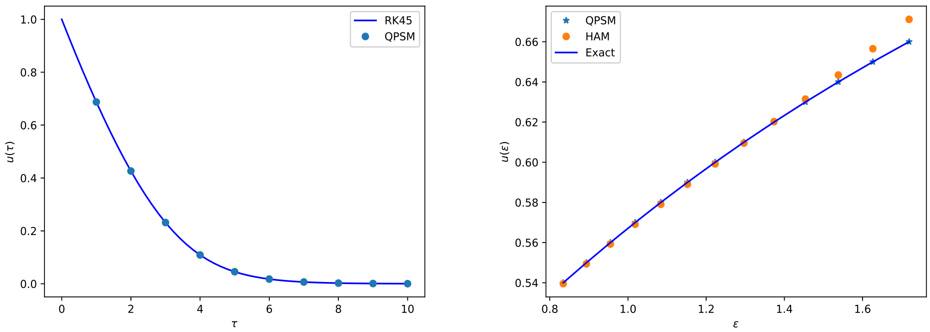

FIGURE 2. The results in Example 4.1 ($n = 10, m = 10, M = 5$).

4. Numerical Examples

Now, two examples of the cooling of a lumped system with variable specific heat (1.1), and unsteady nonlinear convective-radiative equation (3.1) are considered, respectively. In all examples, we put $p = 1$, and Python 3.12.4 is used for programming.

Example 4.1 ([1, 14]). Consider (1.1) with $\tau \in [0, 10]$, by separating the variables we find the exact implicit solution

$$\log u + \varepsilon(u - 1) + \tau = 0.$$

In Figure 2 (left side) the numerical simulation with the embedded Runge-Kutta formulae RK5(4) [13] and QPSM is compared. The exact solution for $\tau = 1$ is compared in Figure 2 (right side) for various ε in QPSM and HAM (with auxiliary homotopy parameter -0.8). We can see that for large ε the results of QPSM are reasonable. In this example, clearly, $u'(0) = \frac{-1}{1+\varepsilon}$ and $u''(0) = \frac{1}{(1+\varepsilon)^3}$. In Figure 3 (right and left sides) the values of $u'(0)$ and $u''(0)$ are compared for the QPSM and we can see a good coincidence.

Example 4.2 ([2, 15]). Consider (3.1) with $\tau \in [0, 10]$. Here we can see that

$$u'(0) = -\frac{1 + \varepsilon_2}{1 + \varepsilon_1}.$$

In Figure 4 (left side) the numerical simulation with the embedded Runge-Kutta formulae RK5(4) [13] and QPSM is compared. The exact solution for $\tau = 1$ is compared in Figure 2 (right side) for various $\varepsilon = \varepsilon_1 = \varepsilon_2$ in QPSM and HAM (with auxiliary homotopy parameter -0.7). We can see that for large ε the results of QPSM are quite reasonable. In figures 5 the values of $u'(0)$ is compared for $\varepsilon_2 = 2\varepsilon_1 = \varepsilon$ in the QPSM and we can see a good coincidence.

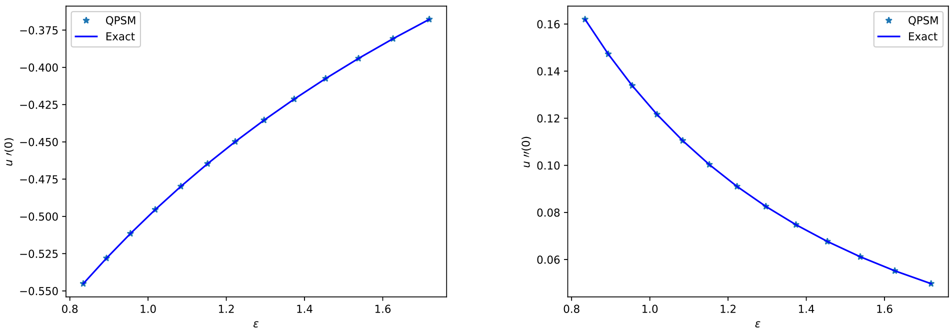

FIGURE 3. The results in Example 4.1 ($n = 10, m = 10, M = 5$).

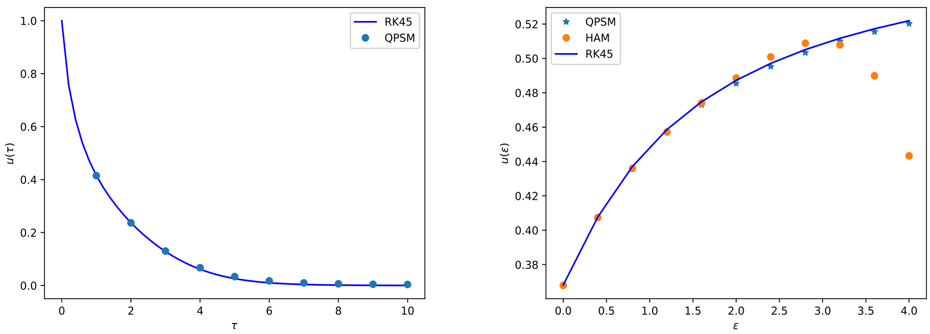

FIGURE 4. The results in Example 4.2 ($n = 10, m = 10, M = 5$).

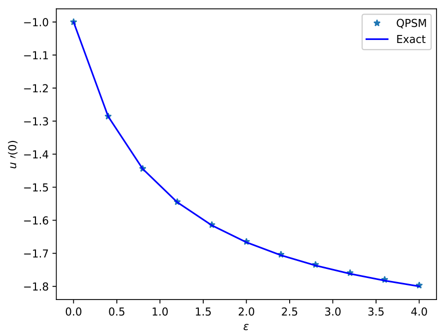

FIGURE 5. The results in Example 4.2 ($n = 10, m = 10, M = 5$).

5. Conclusions

In this manuscript, a novel method for solving the cooling of a lumped system with variable specific heat and unsteady nonlinear convective-radiative equation containing two small parameters is proposed. The latter problem has high nonlinearity. Using the successive linearization method, the nonlinear equations are linearized. Comparisons with homotopy analysis method and numerical simulation RK4(5) show the efficiency of the method.

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