Kragujevac Journal of Mathematics
Volume 47(2) (2023), Pages 281–296.

A STABILITY RESULT FOR A TIMOSHENKO SYSTEM WITH INFINITE HISTORY AND DISTRIBUTED DELAY TERM

ZINEB KHALILI[1] AND DJAMEL OUCHENANE[1]

Abstract. This manuscript is mainly focusing on a general stability of solution for one-dimensional Timoshenko system with infinite history and distributed delay term regardless also of the speeds of wave propagation. We prove our result by using the energy method combined with some properties of convex functions.

1. Introduction

In this paper, we consider the following Timoshenko system with infinite history and distributed delay term

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + \int_0^\infty g(s)\,\psi_{xx}(x,t-s)\,ds \\ +K(\varphi_x+\psi)(x,t) + \mu_1\psi_t(x,t) + \int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,ds = 0, \end{cases} \tag{1.1}$$

where $t \in (0,\infty)$ denotes the time variable and $x \in (0,1)$ is the space variable, the functions φ and ψ are respectively, the transverse displacement of the solid elastic material and the rotation angle, and ρ_1, ρ_2, μ_1, K are positive constants, $\mu_2 : [\tau_1,\tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds < \mu_1, \tag{1.2}$$

where τ_1 and τ_2 two real numbers satisfying $0 \le \tau_1 \le \tau_2$ and the relaxation function g satisfies the folowing assumptions.

(G1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

Key words and phrases. Timoshenko system, distributed delay, energy methode, infinite history.
2010 *Mathematics Subject Classification.* Primary: 35B40. Secondary: 74H40, 74H55, 93D20.
DOI
Received: April 07, 2020.
Accepted: August 24, 2020.

$$g(0) > 0, \quad b - \int_0^\infty g(s)\,ds = b - g_0 = L > 0.$$

(G2) There exists a positive constant ζ such that

$$g'(t) \leq -\zeta g(t), \quad \text{for all } t \geq 0. \tag{1.3}$$

System (1.1) is provided with the following initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \\ \psi_t(x,-t) = f_0(x,t) \text{ in } (0,1) \times (0,\tau_2), \end{cases}$$

and

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \quad \text{for all } t \geq 0,$$

where $x \in (0,1)$ and f_0 is the history function.

Let us first recall some result related to the problem we address. Said-Houari and Rahali [12] considered the following Timoshenko system with infinite history and a delay term in the internal feedback

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + \int_0^\infty g(s)\,\psi_{xx}(x,t-s)\,ds \\ +K(\varphi_x + \psi)(x,t) + \mu_1\psi_t(x,t) + \mu_2\psi_t(x,t-\tau) = 0. \end{cases} \tag{1.4}$$

They established the well-posedness of problem (1.4) and the exponential stability of solution. In the absence of the viscoelastic damping ($g \equiv 0$), problem (1.4) has been studied recently by Said-Houari and Laskri [11]. Under some assumption, they proved the well-posedness and established for $\mu_1 > \mu_2$ an exponential decay result for the case of equal-speed wave propagation, i.e.,

$$\frac{k}{\rho_1} = \frac{b}{\rho_2}.$$

Subsequently, the work in [11] has been extended to the case of time-varying delay of the form $\psi_t(x, t - \tau(t))$ by Kirane, Said-Houari and Anwar [6]. First, by using the variable norm technique of Kato and under some restriction on the parameters μ_1, μ_2 and on the delay function $\tau(t)$, the system has been shown to be well-posed. Second, under relationship between the weight of the delay term in the feedback, the weight of the term without delay and the wave speeds, an exponential decay result of the total energy has been proved.

In [6, 11], the authors have extended some works on the wave equation with delay to the Timoshenko system with delay. The stability of the wave equation with delay has become recently an active area of research and many authors have shown that delays can destabilize a system that is asymptotically stable in the absence of delays (see [2] for more details).

Kafini et al. [5] considered the following Timoshenko system of thermoelasticity of type III with delay

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x + \mu_1\varphi_t(x,t) + \mu_2\varphi_t(x,t-s) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, \\ \rho_3\theta_{tt} - k\theta_{xx} + \gamma\psi_{tx} - k\theta_{txx} = 0. \end{cases}$$

The authors established well-posedness and stability of the system for the cases of equal and nonequal speeds of wave propagation, they showed that the energy decays exponentially in the case of equal wave speeds in spite of the existence of the delay and in the opposite case it decays polynomially. Also, Kafini et al. [4] concerned with the following Timoshenko system of thermoelasticity of type III with distributive delay

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) + \gamma\theta_x(x,t) = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} - \kappa\theta_{txx} - \int_{\tau_1}^{\tau_2} g(s)\,\theta_{txx}(x,t-s)\,ds + \gamma\psi_{tx} = 0, \end{cases}$$

where $\tau_1 < \tau_2$ are non-negative constants. They proved an exponential decay in the case of equal wave speeds and a polynomial decay result in the case of nonequal wave speeds with smooth initial data. Very recently, Hao and Wang [3] considered the following Timoshenko-type system with distributed delay and past history

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x, \psi)_x + \beta\theta_{tx} = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) - \beta\theta_x + \int_0^\infty g(s)\,\psi_{xx}(x,t-s)\,ds + f(\psi) = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\varphi_{tx} - l\theta_{txx} + \gamma\psi_t + \int_{\tau_1}^{\tau_2} \mu(\zeta)\,\psi_t(x,t-\zeta)\,d\zeta = 0. \end{cases} \tag{1.5}$$

The authors proved well-posedness and stability of the system (1.5) for the cases of equal and nonequal speeds of wave propagation. Their results show that the damping effect is strong enough to uniformly stabilize the system even in the existence of time delay under suitable conditions.

Motivated by the works mentioned above, we investigate system (1.1) under suitable assumptions and show that even in the presence of the viscoelastic term ($g \neq 0$), we can establish a general energy decay regardless also of the speeds of wave propagation. To achieve our goals we make use the energy method combined with some properties of convex functions. The arguments of convexity were introduced by Lasiecka and Tataru [7] and used by Liu and Zuazua [8] and others.

2. Preliminaries

The main aim in this section is to present some materials needed in the proof of our result. We also state, without proof, a local existence result for problem (1.1). The proof can be established by using Faedo-Galerkin method as in [9]. Let us introduce the following new dependent variable

$$z(x,\rho,s,t) = \psi_t(x,t-s\rho), \quad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Then, we get the following system

$$\begin{cases} sz_t(x,\rho,s,t)+z_\rho(x,\rho,s,t)=0, \\ z(x,0,\tau,t)=\psi_t(x,t). \end{cases}$$

We then set an auxiliary variable as in [1]

$$\eta^t(x,s)=\psi(x,t)-\psi(x,t-s), \quad s\geq 0.$$

Then

$$\eta^t_t(x,s)+\eta^t_s(x,s)=\psi_t(x,t).$$

Hence, we can rewrite the problem (1.1) as

$$\begin{cases} \rho_1\varphi_{tt}(x,t)-K(\varphi_x+\psi)_x(x,t)=0, \\ \rho_2\psi_{tt}-b\psi_{xx}+K(\varphi_x+\psi)+\int_0^\infty g(s)\eta^t_{xx}(x,s)\,ds \\ +\mu_1\psi_t(x,t)+\int_{\tau_1}^{\tau_2}\mu_2(s)\psi_t(x,t-s)\,ds=0, \\ sz_t(x,\rho,s,t)+z_\rho(x,\rho,s,t)=0, \\ \eta^t_t(x,s)+\eta^t_s(x,s)=\psi_t(x,t), \end{cases} \tag{2.1}$$

where $x\in(0,1)$, $\rho\in(0,1)$ and $t>0$. System (2.1) subjected to the following initial conditions

$$\begin{cases} \left.\begin{array}{l}\varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x), \\ \psi(x,0)=\psi_0(x), \quad \psi_t(x,0)=\psi_1(x),\end{array}\right. \quad x\in(0,1), \\ z(x,\rho,s,0)=f_0(x,\rho s), \quad \text{in } (0,1)\times(0,1)\times(0,\tau_2), \\ \eta^t(x,0)=0, \quad \text{for all } t\geq 0, \\ \eta^0(x,s)=\eta_0(s)=0, \quad \text{for all } s\geq 0. \end{cases} \tag{2.2}$$

In addition, we consider the following boundary conditions

$$\begin{aligned} \varphi(0,t)=&\varphi(1,t)=\psi(0,t)=\psi(1,t)=0, \quad \text{for all } t\geq 0, \\ \eta^t(0,s)=&\eta^t(1,s)=0, \quad \text{for all } s\geq 0. \end{aligned} \tag{2.3}$$

We now define the energy space

$$\boldsymbol{H}:=\left[H_0^1(0,1)\times L^2(0,1)\right]^2\times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))\times L_g^2\left(\mathbb{R}^+,H_0^1(0,1)\right),$$

where $L_g^2(\mathbb{R}^+,H_0^1(0,1))$ denotes the Hilbert space of H_0^1-valued functions on $\mathbb{R}^+$.

3. Exponential Stability

The functional energy of the solution of problem (2.1)–(2.3) is given by

$$\begin{aligned} E(t)=&E\left(t,\varphi,\psi,z,\eta^t\right) \\ =&\frac{1}{2}\int_0^1\left(\rho_1\varphi_t^2+\rho_2\psi_t^2\right)dx+\frac{1}{2}\int_0^1\left\{K(\varphi_x+\psi)^2+b\psi_x^2\right\}dx \\ &+\frac{1}{2}\int_0^1\int_0^\infty g(s)\left|\eta^t_x(x,s)\right|^2 dsdx \\ &+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right|z^2(x,\rho,s,t)\,dsd\rho dx. \end{aligned} \tag{3.1}$$

We multiply $(2.1)_1$ by φ_t, $(2.1)_2$ by ψ_t and $(2.1)_3$ by $|\mu_2(s)|\, z$, integrating by parts over $(0,1)$, using Young and Cauchy-Schwarz's inequality we get

$$\begin{aligned}\frac{dE(t)}{dt} \leq & \frac{1}{2}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx \\ & - C\left\{\int_0^1 \psi_t \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t) + \int_0^1 \psi_t^2(x,t)\, dx\right\},\end{aligned} \tag{3.2}$$

where $C > 0$, which implies that the energy E is a non-increasing function with respect to t.

Our main stability result reads as follows.

Theorem 3.1. *Let $U_0 \in D(A)$. Assume that $\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds < \mu_1$ and*

$$\frac{K}{\rho_1} = \frac{b}{\rho_2}.$$

Then there exist two positive constants C and γ independent of t such that

$$E(t) \leq Ce^{-\gamma t}, \quad \text{for all } t > 0. \tag{3.3}$$

Remark 3.1. To derive the exponential decay of the solution, it is enough to construct a functional $L(t)$, equivalent to the energy $E(t)$, satisfying

$$\frac{dL(t)}{dt} \leq -\Lambda L(t), \quad \text{for all } t > 0,$$

where Λ is a positive constant. In order to obtain such a functional L, we need several lemmas.

Let us first define the following functional

$$I_1(t) := -\int_0^1 (\rho_1\varphi_t\varphi + \rho_2\psi_t\psi)\, dx - \frac{\mu_1}{2}\int_0^1 \psi^2 dx. \tag{3.4}$$

Then we have the following estimate.

Lemma 3.1. *Let $(\varphi, \psi, z, \eta^t)$ be the solution of (2.1)–(2.3), then for any $\varepsilon, \delta_1 > 0$, we have*

$$\begin{aligned}\frac{dI_1(t)}{dt} \leq & -\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx \\ & + \frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx + \frac{c\varepsilon_2}{2}\int_0^1 \psi^2 dx \\ & + (b+\delta_1)\int_0^1 \psi_x^2 dx + \frac{1}{2\varepsilon_2}\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(s)\,\psi_t^2(x,t-s)\, dsdx \\ & + K\int_0^1 (\varphi_x+\psi)^2 dx,\end{aligned} \tag{3.5}$$

where $c = 1/\pi^2$ is the Poincaré's constant.

Proof. Taking the derivative of (3.4), integrating by parts, we obtain

$$\frac{dI_1(t)}{dt} = -\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx - \int_0^1 \left(\rho_1\varphi_{tt}\varphi_t + \rho_2\psi_{tt}\psi_t\right) dx - \mu_1\int_0^1 \psi_t\psi dx. \tag{3.6}$$

Therefore, by using $(2.1)_1$, $(2.1)_2$, integration by parts, we obtain from (3.6)

$$\begin{aligned} \frac{dI_1(t)}{dt} = & -\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx + K\int_0^1 (\varphi_x+\psi)^2 dx + b\int_0^1 \psi_x^2 dx \\ & + \int_0^1 \psi \int_{\tau_1}^{\tau_2} \mu_2(s)\,\psi_t(x,t-s)\, dsdx \\ & + \int_0^1 \psi_x(x,t)\int_0^\infty g(s)\left|\eta_x^t(x,s)\right| dsdx. \end{aligned} \tag{3.7}$$

By exploiting Young and Poincaré's inequalities, we get for any $\varepsilon > 0$

$$\begin{aligned} & \int_0^1 \psi \int_{\tau_1}^{\tau_2} \mu_2(s)\,\psi_t(x,t-s)\, dsdx \\ \le & \frac{c\varepsilon_2}{2}\int_0^1 \psi^2 dx + \frac{1}{2\varepsilon_2}\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(s)\,\psi_t^2(x,t-s)\, dsdx. \end{aligned} \tag{3.8}$$

Moreover, Young, Hölder's inequalities and (1.3) imply that for any $\delta_1 > 0$

$$\begin{aligned} & \int_0^1 \psi_x(x,t)\int_0^\infty g(s)\left|\eta_x^t(x,s)\right| dsdx \\ \le & \delta_1\int_0^1 \psi_x^2(x,t)\, dx + \frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \end{aligned} \tag{3.9}$$

Inserting the estimates (3.8) and (3.9) into (3.7), then (3.5) is fulfilled. □

Now, let w be the solution of

$$-w_{xx} = \psi_x, \quad w(0) = w(1) = 0, \tag{3.10}$$

then

$$w(x,t) = -\int_0^x \psi(y,t)\, dy + x\left(\int_0^1 \psi(y,t)\, dy\right).$$

We have the following inequalities.

Lemma 3.2. *The solution of* (3.10) *satisfies*

$$\int_0^1 w_x^2 dx \le \int_0^1 \psi^2 dx$$

and

$$\int_0^1 w_t^2 dx \le \int_0^1 \psi_t^2 dx.$$

Proof. We multiply (3.10) by w, integrate by parts and use the Cauchy-Schwarz's inequality to obtain

$$\int_0^1 w_x^2 dx \le \int_0^1 \psi^2 dx.$$

Next, we differentiate (3.10) with respect to t and by the same procedure, we obtain

$$\int_0^1 w_t^2 dx \le \int_0^1 \psi_t^2 dx. \qquad \square$$

Let w be the solution of (3.10). We introduce the following functional

$$I_2(t) := \int_0^1 (\rho_2 \psi_t \psi + \rho_1 \varphi_t w)\, dx + \frac{\mu_1}{2}\int_0^1 \psi^2 dx. \tag{3.11}$$

Then, we have the following estimate.

Lemma 3.3. *Let* $(\varphi, \psi, z, \eta^t)$ *be the solution of* (2.1)–(2.3). *Then we have for any* $\varepsilon_3 > 0$,

$$\begin{aligned}\frac{dI_2(t)}{dt} \le & (\delta_1 - b)\int_0^1 \psi_x^2 dx + \rho_1\lambda_2 \int_0^1 \varphi_t^2 dx + \frac{c\varepsilon_3}{2}\int_0^1 \psi^2 dx \\ & + \left(\rho_2 + \frac{\rho_1}{4\lambda_2}\right)\int_0^1 \psi_t^2 dx + \left(\frac{\gamma\tau_0}{2\kappa\varepsilon_3} + \frac{\delta\gamma}{2\kappa\varepsilon_3}\right)\int_0^1 q^2 dx \\ & + \frac{1}{2\varepsilon_3}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\, ds dx + \frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx.\end{aligned} \tag{3.12}$$

Proof. By taking the derivative of (3.11), we conclude

$$\begin{aligned}\frac{dI_2(t)}{dt} = & - b\int_0^1 \psi_x^2 dx - K\int_0^1 \psi^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + K\int_0^1 w_x^2 dx \\ & + \rho_1 \int_0^1 \varphi_t w_t dx + \int_0^1 \psi_x(x,t)\int_0^\infty g(s)\,\eta_x^t(x,s)\, ds dx \\ & - \int_0^1 \psi \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t)\, ds dx.\end{aligned}$$

We apply Young and Poincaré's inequalities, we find

$$\int_0^1 \psi_x(x,t)\int_0^\infty g(s)\,\eta_x^t(x,s)\, ds dx \le \delta_1 \int_0^1 \psi_x^2(x,t) + \frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx,$$

and for any $\lambda_2 > 0$ we have

$$\rho_1 \int_0^1 \varphi_t \psi_t dx \le \rho_1\lambda_2 \int_0^1 \varphi_t^2 dx + \frac{\rho_1}{4\lambda_2}\int_0^1 \psi_t^2 dx. \qquad \square$$

Now, we define the functional I_3

$$I_3(t) := \rho_2 \int_0^1 \psi_t (\varphi_x + \psi)\, dx + \frac{\rho_1 b}{K}\int_0^1 \psi_x \varphi_t dx + \frac{\rho_1}{K}\int_0^1 \varphi_t \int_0^\infty g(s)\,\eta_x^t(x,s)\, ds dx. \tag{3.13}$$

Lemma 3.4. *Let* $(\varphi, \psi, z, \eta^t)$ *be the solution of* (2.1)–(2.3). *Assume that*

$$\frac{\rho_1}{K} = \frac{\rho_2}{b + g_0} = \frac{\rho_2}{b}. \tag{3.14}$$

Then, for any $\varepsilon_4 > 0$, *we have*

$$\frac{dI_3(t)}{dt} \le \left[\varphi_x\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\right)\right]_{x=0}^{x=1} - (K - 2\varepsilon_4)\int_0^1 (\varphi_x + \psi)^2 dx$$

$$+\left(\rho_2+\frac{\mu_1^2}{4\varepsilon_4}\right)\int_0^1 \psi_t^2 dx+\varepsilon_4\int_0^1 \varphi_t^2 dx+\frac{1}{2\varepsilon_4}\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\, z^2(x,1,s,t)\, dsdx \tag{3.15}$$
$$-g_0 C(\varepsilon_4)\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx.$$

Proof. Differentiating $I_3(t)$, we obtain

$$\begin{aligned}\frac{dI_3(t)}{dt}=&\rho_2\int_0^1 \psi_{tt}(\varphi_x+\psi)\,dx+\rho_2\int_0^1 \psi_t(\varphi_x+\psi)_t\,dx\\&+\frac{\rho_1 b}{K}\int_0^1 \psi_x\varphi_{tt}dx+\frac{\rho_1}{K}\int_0^1\varphi_t\int_0^\infty g(s)\,\eta_x^t(x,s)\,dsdx\\&+\frac{\rho_1 b}{K}\int_0^1 \psi_{xt}\varphi_t dx+\frac{\rho_1}{K}\int_0^1\varphi_{tt}\int_0^\infty g(s)\,\eta_{tx}^t(x,s)\,dsdx.\end{aligned}$$

Then, by using (2.1), we find

$$\begin{aligned}\frac{dI_3(t)}{dt}=&\rho_2\int_0^1(\varphi_x+\psi)\Big(b\psi_{xx}(x,t)-K(\varphi_x+\psi)(x,t)\\&-\mu_1\psi_t(x,t)-\int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,ds\Big)\,dx\\&+\int_0^1(\varphi_x+\psi)\int_0^\infty g(s)\,\eta_{xx}^t(x,s)\,dsdx+\rho_2\int_0^1\psi_t^2dx\\&+b\int_0^1(\varphi_x+\psi)_x\,\psi_x dx+\left(\frac{\rho_1 b}{K}-\rho_2\right)\int_0^1\psi_{tx}\varphi_t dx\\&+\frac{\rho_1}{K}\int_0^1\varphi_t\int_0^\infty g(s)\Big(\psi_{tx}(t,x)-\eta_{tx}^t(x,s)\Big)\,dsdx\\&+\frac{\rho_1}{K}\int_0^1(\varphi_x+\psi)_x\int_0^\infty g(s)\,\eta_x^t(x,s)\,dsdx.\end{aligned}$$

By (3.14), we obtain

$$\begin{aligned}\frac{dI_3(t)}{dt}=&-K\int_0^1(\varphi_x+\psi)^2dx-\mu_1\int_0^1(\varphi_x+\psi)\,\psi_t dx+\rho_2\int_0^1\psi_t^2dx\\&-\int_0^1(\varphi_x+\psi)\int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,dsdx\\&+\frac{\rho_1}{K}\int_0^1\varphi_t\int_0^\infty g'(s)\,\eta_x^t(x,s)\,dsdx\\&+\left[b\psi_x\varphi_x dx\right]_{x=0}^{x=1}+\left[\varphi_x(x,t)\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right].\end{aligned}\tag{3.16}$$

For any $\varepsilon_4>0$, Young's inequality leads to

$$\left|\mu_1\int_0^1(\varphi_x+\psi)\,\psi_t(x,t)\right|\le\varepsilon_4\int_0^1(\varphi_x+\psi)^2dx+\frac{\mu_1^2}{4\varepsilon_4}\int_0^1\psi_t^2dx \tag{3.17}$$

and

$$\left|\int_0^1(\varphi_x+\psi)\int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,dsdx\right|$$

$$\leq \frac{c\varepsilon_4}{2}\int_0^1 (\varphi_x+\psi)^2\,dx+\frac{1}{2\varepsilon_4}\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,z^2(x,1,s,t)\,dsdx \tag{3.18}$$

and

$$\begin{aligned}
&\left|\frac{\rho_1}{K}\int_0^1\varphi_t\int_0^\infty g'(s)\,\eta_x^t(x,s)\,dsdx\right| \\
\leq&\frac{\rho_1^2}{4K\varepsilon_4}\int_0^1\left(\int_0^\infty g'(s)\,\eta_x^t(x,s)\,ds\right)^2 dx+\varepsilon_4\int_0^1\varphi_t^2dx \\
\leq&-g(0)\,C(\varepsilon_4)\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx+\varepsilon_4\int_0^1\varphi_t^2dx.
\end{aligned} \tag{3.19}$$

Plugging (3.17), (3.18) and (3.19) into (3.16), then inequality (3.15) holds. □

Next, in order to handle the boundary terms appearing in (3.15) we use, as in [10], the function

$$q(x)=2-4x,\quad x\in(0,1).$$

So, we have the following result.

Lemma 3.5. *Let (φ,ψ,z,η^t) be the solution of* (2.1)*. Then we have that for a positive constant ε_6*

(3.20)

$$\begin{aligned}
&\left[\varphi_x\left(b\psi_x-\int_0^\infty g(s)\,\psi_x(t-s)\,ds\right)\right]_{x=0}^{x=1} \\
\leq&-\frac{\varepsilon_6}{K}\frac{d}{dt}\int_0^1\rho_1 q(x)\,\varphi_t\varphi_x dx+K^2\varepsilon_6\int_0^1(\varphi_x+\psi)^2\,dx \\
&-\frac{\rho_2}{4\varepsilon_6}\frac{d}{dt}\int_0^1 q(x)\,\psi_t\left(b\psi_x-\int_0^\infty g(s)\,\psi_x(t-s)\,ds\right)dx+3\varepsilon_6\int_0^1\varphi_x^2dx \\
&+\left(\varepsilon_6+\frac{b}{4\varepsilon_6}\left(4+\frac{3}{2\varepsilon_6^2}\right)\right)\int_0^1\psi_x^2dx+\frac{1}{4\varepsilon_6}\left(2\rho_2(b+g_0)+4\mu_1^2\varepsilon_6^2+\rho_2\varepsilon_6\right)\int_0^1\psi_t^2dx \\
&-\frac{\rho_2 g(0)\,C(\varepsilon_6)}{4\varepsilon_6}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx+\frac{2\rho_1\varepsilon_6}{K}\int_0^1\varphi_t^2dx \\
&+\frac{g_0}{4\varepsilon_6}\left(4+\frac{3}{2\varepsilon_6^2}\right)\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx \\
&+\frac{1}{2\varepsilon_4}\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,z^2(x,1,s,t)\,dsdx.
\end{aligned}$$

Proof. By using Young and Poincaré inequalities, we obtain for any $\varepsilon_6>0$

$$\begin{aligned}
&\left[\varphi_x\left(b\psi_x+\int_0^\infty g(s)\,\psi_x(t-s)\,ds\right)\right]_{x=0}^{x=1} \\
=&\varphi_x(1)\left(b\psi_x(1)+\int_0^\infty g(s)\,\psi_x(1,t-s)\,ds\right) \\
&-\varphi_x(0)\left(b\psi_x(0)+\int_0^\infty g(s)\,\psi_x(0,t-s)\,ds\right)
\end{aligned} \tag{3.21}$$

$$\le \frac{1}{4\varepsilon_6}\left[\left(b\psi_x(1)+\int_0^\infty g(s)\,\psi_x(1,t-s)\,ds\right)^2 + \left(b\psi_x(0)+\int_0^\infty g(s)\,\psi_x(0,t-s)\,ds\right)^2\right]+\varepsilon_6\left[\varphi_x(1)^2+\varphi_x(0)^2\right].$$

On the other hand, it is clear that

$$\begin{aligned}&\frac{d}{dt}\int_0^1 \rho_2 q(x)\,\psi_t\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\\ =&\int_0^1 \rho_2 q(x)\,\psi_{tt}\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\\ &+\int_0^1 \rho_2 q(x)\,\psi_t\left(b\psi_{tx}+\int_0^\infty g(s)\,\eta_{tx}^t(x,s)\,ds\right)dx.\end{aligned}$$

Now, using $(2.1)_2$, we find

$$\begin{aligned}&\frac{d}{dt}\int_0^1 \rho_2 q(x)\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\\ =&\int_0^1 q(x)\Big(b\psi_{xx}-k(\varphi_x+\psi)-\mu_1\psi_t\\ &-\int_{\tau_1}^{\tau_2}\mu_2(s)\,z(x,1,t)\,ds+\int_0^\infty g(s)\,\eta_{xx}^t(x,s)\,ds\Big)\\ &\times\left(b\psi_x-\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\\ &+\int_0^1 \rho_2 q(x)\,\psi_t\left(b\psi_{tx}+\int_0^\infty g(s)\,\eta_{tx}^t(x,s)\,ds\right)dx.\end{aligned}\tag{3.22}$$

By the fact that

$$\begin{aligned}&\int_0^1 q(x)\left(b\psi_{xx}+\int_0^\infty g(s)\,\eta_{xx}^t(x,s)\,ds\right)\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\\ =&-\frac{1}{2}\int_0^1 q'(x)\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)^2dx\\ &+\left[\frac{q(x)}{2}\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)^2\right]_{x=0}^{x=1}.\end{aligned}\tag{3.23}$$

The last term in (3.22) can be treated as follows

(3.24)

$$\begin{aligned}&\int_0^1 \rho_2 q(x)\,\psi_t\left(b\psi_{tx}+\int_0^\infty g(s)\,\eta_{tx}^t(x,s)\,ds\right)dx\\ =&\rho_2 b\int_0^1 q(x)\,\psi_t\psi_{tx}dx+\rho_2\int_0^1 q(x)\,\psi_t\int_0^\infty g(s)\,\eta_{tx}^t(x,s)\,dsdx\\ =&-\frac{\rho_2 b}{2}\int_0^1 q'(x)\,\psi_t^2dx+\rho_2\int_0^1 q(x)\,\psi_t\int_0^\infty g(s)\,\eta_{tx}^t(x,s)\,dsdx\\ =&-\frac{\rho_2 b}{2}\int_0^1 q'(x)\,\psi_t^2dx+\rho_2\int_0^1 q(x)\,\psi_t\int_0^\infty g(s)\left(\psi_t-\eta_s^t\right)_x dsdx\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\rho_2 b}{2}\int_0^1 q'(x)\,\psi_t^2 dx + \rho_2 g_0 \int_0^1 q(x)\,\psi_t \psi_{tx} dx - \rho_2 \int_0^1 q(x)\,\psi_t \int_0^\infty g(s)\,\eta_{sx}^t ds dx \\
&= -\frac{\rho_2 (b+g_0)}{2}\int_0^1 q'(x)\,\psi_t^2 dx + \rho_2 \int_0^1 q(x)\,\psi_t \int_0^\infty g'(s)\,\eta_x^t ds dx.
\end{aligned}
$$

Inserting (3.23) and (3.24) in (3.22), we arrive at

$$
\begin{aligned}
&\left(b\psi_x(0,t) + \int_0^\infty g(s)\,\eta_x^t(0,s)\,ds\right)^2 + \left(b\psi_x(1,t) + \int_0^\infty g(s)\,\eta_x^t(1,s)\,ds\right)^2 \\
&= -\frac{d}{dt}\int_0^1 \rho_2 q \psi_t \left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right) dx + 2\rho_2 (b+g_0)\int_0^1 \psi_t^2 dx \\
&- K\int_0^1 q(\varphi_x + \psi)\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right) dx \\
&+ \rho_2 \int_0^1 q\psi_t \int_0^\infty g'(s)\,\eta_x^t(x,s)\,ds dx \\
&- \mu_1 \int_0^1 q(x)\,\psi_t \left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right) dx \\
&+ 2\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)^2 dx \\
&- \int_0^1 q(x) \int_{\tau_1}^{\tau_2} \mu_2(s)\,\psi_t(x,t-s)\,ds \left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right) dx.
\end{aligned}
\tag{3.25}
$$

Now, we estimate terms in the RHS of (3.25) as follows.

First, using Minkowski and Young's inequalities, we have

$$
\begin{aligned}
&2\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)^2 dx \\
&\leq 4b^2 \int_0^1 \psi_x^2 dx + 4g_0 \int_0^1 \int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx.
\end{aligned}
\tag{3.26}
$$

Second, by Young's inequality and (3.26), we have for any $\lambda > 0$

$$
\begin{aligned}
&\left|K\int_0^1 q(x)(\varphi_x+\psi)\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right) dx\right| \\
&\leq 2K\left|\int_0^1 (\varphi_x+\psi)\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right) dx\right| \\
&\leq 4K^2\lambda \int_0^1 (\varphi_x+\psi)^2 dx + \frac{1}{4\lambda}\int_0^1 \left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)^2 dx \\
&\leq 4K^2\lambda \int_0^1 (\varphi_x+\psi)^2 dx + \frac{b^2}{2\lambda}\int_0^1 \psi_x^2 dx + \frac{g_0}{2\lambda}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&\left|\mu_1 \int_0^1 q(x)\,\psi_t \left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right) dx\right| \\
&\leq 4\mu_1\lambda \int_0^1 \psi_t^2 dx + \frac{b^2}{2\lambda}\int_0^1 \psi_x^2 dx + \frac{g_0}{2\lambda}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx
\end{aligned}
$$

and

$$\begin{aligned}&\left|-\int_0^1 q(x)\int_{\tau_1}^{\tau_2}\mu_2(s)\psi_t(x,t-s)\,ds\left(b\psi_x+\int_0^\infty g(s)\eta_x^t(x,s)\,ds\right)dx\right|\\ \leq& b\int_0^1 q(x)\psi_x\int_{\tau_1}^{\tau_2}\mu_2(s)\psi_t(x,t-s)\,dsdx\\ &+\int_0^1\left(q(x)\int_{\tau_1}^{\tau_2}\mu_2(s)\psi_t(x,t-s)\,ds\int_0^\infty g(s)\eta_x^t(x,s)\,ds\right)dx\\ \leq& 4\delta_0\lambda\int_{\tau_1}^{\tau_2}\mu_2(s)z^2(x,1,s,t)\,ds+\frac{b^2}{2\lambda}\int_0^1\psi_x^2dx+\frac{g_0}{2\lambda}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx.\end{aligned}$$

For any $\varepsilon_2>0$, we have

$$\begin{aligned}&\left|\rho_2\int_0^1 q\psi_t\int_0^\infty g'(s)\eta_x^t(x,s)\,dsdx\right|\\ \leq&\rho_2\varepsilon_2\int_0^1\psi_t^2dx-\rho_2 g(0)C(\varepsilon_2)\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2dsdx.\end{aligned}$$

Inserting all the above estimates into (3.25), we obtain

$$\begin{aligned}&\left(b\psi_x(0,t)+\int_0^\infty g(s)\eta_x^t(0,s)\,ds\right)^2+\left(b\psi_x(1,t)+\int_0^\infty g(s)\eta_x^t(1,s)\,ds\right)^2\\ \leq&-\frac{d}{dt}\int_0^1\rho_2 q\psi_t\left(b\psi_x+\int_0^\infty g(s)\eta_x^t(x,s)\,ds\right)dx\\ &+\left(2\rho_2(b+g_0)+4\mu_1^2\lambda+\rho_2\varepsilon_2\right)\int_0^1\psi_t^2dx\\ &+b^2\left(4+\frac{3}{2\lambda}\right)\int_0^1\psi_x^2dx+4K^2\lambda\int_0^1(\varphi_x+\psi)^2dx\\ &-\rho_2 g(0)C(\varepsilon_2)\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2dsdx\\ &+g_0\left(4+\frac{3}{2\lambda}\right)\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx+4\delta_0\lambda\int_{\tau_1}^{\tau_2}\mu_2(s)z^2(x,1,s,t)\,ds.\end{aligned}\tag{3.27}$$

On the other hand, we have

$$\begin{aligned}\left[\varphi_x^2(1)-\varphi_x^2(0)\right]\leq&-\frac{d}{dt}\frac{1}{k}\int_0^1\rho_1 q(x)\varphi_t\varphi_x dx\\ &+3\int_0^1\varphi_x^2dx+\int_0^1\psi_x^2dx+\frac{2\rho_1}{k}\int_0^1\varphi_t^2dx.\end{aligned}\tag{3.28}$$

Consequently, substituting (3.27) and (3.28) into (3.21), our desired estimate (3.20) holds. □

Now, we define the functional

$$I_4(t):=\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}se^{-s\rho}\left|\mu_2(s)\right|z^2(x,1,s,t)\,dsd\rho dx.\tag{3.29}$$

Then the following result holds.

Lemma 3.6. *Let $(\varphi, \psi, z, \eta^t)$ be the solution of (2.1)–(2.3). Then for $C_1 > 0$ we have*

$$\frac{dI_4(t)}{dt} \leq -C_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x,1,s,t)\, ds d\rho dx - C_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds dx + \mu_1 \int_0^1 \psi_t^2 dx, \tag{3.30}$$

where C_1 is a positive constant.

Proof. Differentiating (3.29) and using $z(x,0,s,t) = \psi_t$, $e^{-s} \leq e^{-s\rho}$, we get for all $\rho \in [0,1]$

$$\frac{dI_4(t)}{dt} \leq \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s} |\mu_2(s)| z^2(x,1,s,t)\, ds dx + \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_0^1 \psi_t^2 dx - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s} |\mu_2(s)| z^2(x,1,s,t)\, ds d\rho dx.$$

Since $s \to -e^{-s}$ is an increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$ for all $s \in [\tau_1, \tau_2]$. Finally, setting, $C_1 = -e^{-\tau_2}$ and recalling (1.2), we obtain (3.30). □

Proof of Theorem 3.1. We are now ready to define the Lyapunov functional $L(t)$ as follows

$$L(t) := NE(t) + \frac{1}{4} I_1(t) + N_2 I_2(t) + I_3(t) + \frac{\varepsilon_2}{K} \int_0^1 \rho_1 q \varphi_t \varphi_x dx + \frac{1}{4\varepsilon_2} \int_0^1 \rho_2 q(x) \psi_t \left(b\psi_x + \int_0^\infty g(s) \eta_x^t(x,s)\, ds \right) dx + N_4 I_4(t),$$

where N, N_2, N_4 are positive real numbers which will be chosen later.

Consequently, the estimates (3.2), (3.5), (3.12), (3.15), (3.20) and (3.30) together with (1.3) and the following inequality

$$\int_0^1 \varphi_x^2 dx \leq 2 \int_0^1 (\varphi_x + \psi)^2 dx + 2 \int_0^1 \psi_x^2 dx,$$

lead to

(3.31)

$$\begin{aligned} \frac{d}{dt} L(t) \leq & \Bigg\{ -MC - \frac{\rho_1}{4} + N_2 \left(\rho_2 + \frac{\rho_1}{4\lambda_2} \right) + \left(\rho_2 + \frac{\mu_1^2}{4\varepsilon_1} \right) \\ & + \frac{1}{4\varepsilon_2} \left(2\rho_2 (b + g_0) + 4\mu_1^2 \varepsilon_2^2 + \rho_2 \varepsilon_2 \right) + N_4 \mu_1 + \frac{1}{2\tau} \Bigg\} \int_0^1 \psi_t^2 dx \\ & + \left\{ \frac{1}{8\varepsilon_2} + \frac{N_2}{2\varepsilon_4} + \frac{1}{2\varepsilon_4} - C_1 N_4 \right\} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds dx \\ & + \left\{ -\frac{\rho_1}{4} + N_2 \rho_1 \lambda_2 + \frac{2\rho_1 \varepsilon_2}{K} + \varepsilon_1 \right\} \int_0^1 \varphi_t^2 dx \\ & + \left\{ -\left(\frac{3K}{4} - 2\varepsilon \right) + K^2 \varepsilon_2 + 6\varepsilon_2 + \frac{\varepsilon_4 c}{2} \right\} \int_0^1 (\varphi_x + \psi)^2 dx - I_3(t) \\ & + \Bigg\{ \frac{1}{4} (b + \delta_1) + N_2 (\delta_1 + \mu_2 C^* \lambda_2 - b) + 7\varepsilon_2 \end{aligned}$$

$$+\frac{b^2}{4\varepsilon_2}\left(4+\frac{3}{2\varepsilon_2^2}\right)\Big\}\int_0^1\psi_x^2dx+\left\{\left(\frac{c\varepsilon_2}{8}-\frac{cN_2\varepsilon_3}{2}\right)\right\}\int_0^1\psi^2dx$$
$$+\left\{\frac{g_0}{4\delta_1}\left(\frac{1}{4}+N_2\right)+\frac{g_0}{4\varepsilon_2}\left(4+\frac{2}{2\varepsilon_2^2}\right)\right.$$
$$\left.-\zeta\left(\frac{M}{2}-g_0C(\varepsilon_1)-\frac{\rho_2 g(0)C(\varepsilon_2)}{4\varepsilon_2}\right)\right\}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx$$
$$-C_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}se^{-s\rho}\left|\mu_2(s)\right|z^2(x,1,s,t)\,dsd\rho dx.$$

At this point, we have to choose our constants very carefully.

First, let us choose ε small enough such that

$$\varepsilon\le\frac{3K}{8}.$$

Then, we take $\varepsilon_2=\varepsilon_1$ and choose ε_2 small enough such that

$$\varepsilon_2\le\min\left\{\frac{K/8}{(K^2+6)},\frac{\rho_1/8}{(2\rho_1/K)+1}\right\}.$$

Then, we choose $\lambda_2=\delta_1$ and choose ε_2 small enough such that

$$\lambda_2\le\frac{b/2}{1+\mu_2C^*}.$$

Once all the above constants are fixed, we fix N_2 large enough such that

$$N_2\frac{b}{4}\ge\frac{1}{4}(b+\delta_1)+7\varepsilon_2+\frac{b}{4\varepsilon_2}\left(4+\frac{3}{2\varepsilon_2^2}\right).$$

After that, we pick λ_2 so small that

$$\lambda_2\le\frac{1}{32N_2}.$$

Finally, we choose M large enough so that, there exists a positive constant η_1, such that (3.31) becomes

$$\begin{aligned}\frac{d}{dt}L(t)\le&-\eta_1\int_0^1\left(\psi_t^2+\psi_x^2+\varphi_t^2+(\varphi_x+\psi)^2+\psi^2\right)dx\\&-\eta_1\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx\\&+\eta_1\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|z^2(x,1,s,t)\,dsdx\\&-\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}se^{-s\rho}\left|\mu_2(s)\right|z^2(x,1,s,t)\,dsd\rho dx,\end{aligned}$$

which implies by (3.1), that there exists also $\eta_2>0$, such that

$$\frac{d}{dt}L(t)\le-\eta_2E(t),\quad\text{for all } t\ge0. \tag{3.32}$$

In addition, we can choose M large enough so that

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t), \quad \text{for all } t \geq 0. \tag{3.33}$$

Combining (3.32) and (3.33), we conclude that there exists $\Lambda > 0$ such that

$$\frac{d}{dt} L(t) \leq -\Lambda L(t), \quad \text{for all } t \geq 0. \tag{3.34}$$

A simple integration of (3.34) leads to

$$L(t) \leq L(0)\, e^{-\Lambda t}, \quad \text{for all } t \geq 0. \tag{3.35}$$

Again, (3.33) and (3.35) yeilds the desired result (3.3). This completes the proof of Theorem 3.1. □

References

[1] C. M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Ration. Mech. Anal **37** (1970), 297–308. http://dx.doi.org/10.1007/BF00251609

[2] R. Datko, J. Lagnese and M. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations*, SIAM J. Control Optim. **24** (1986), 152–156. http://dx.doi.org/10.1109/CDC.1985.268529

[3] J. Hao and F. Wang, *Energy decay in a Timoshenko-type system for thermoelasticity of type III with distributed delay and past history*, Electron. J. Differential Equations **2018**(75) (2018), 1–27.

[4] M. Kafini, S. A. Messaoudi and M. I. Mustafa, *Energy decay result in a Timoshenko-type system of thermoelasticity of type III with distributive delay*, J. Math. Phys **54** (2013), Paper ID 101503. http://dx.doi.org/10.1063/1.4826102

[5] M. Kafini, S. A. Messaoudi, M. I. Mustafa and T. Apalara, *Well-posedness and stability results in a Timoshenko-type system of thermoelasticity of type III with delay*, Z. Angew. Math. Phys **66** (2015), 1499–1517. http://dx.doi.org/10.1007/s00033-014-0475-9

[6] M. Kirane, B. Said-Houari and M. N. Anwar, *Stability result for the Timoshenko system with a time-varying delay term in the internal feedbacks*, Comm. Pure Appl. Anal **10** (2011), 667–686. http://dx.doi.org/10.3934/cpaa.2011.10.667

[7] I. Lasiecka, D. Tataru et al., *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping*, Differential Integral Equations **6** (1993), 507–533.

[8] W.-J. Liu and E. Zuazua, *Decay rates for dissipative wave equations*, Ric. Mat. **48** (1999), 61–75.

[9] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM J. Control Optim. **45** (2006), 1561–1585, http://dx.doi.org/10.1137/060648891

[10] J. E. M. Rivera and R. Racke, *Mildly dissipative nonlinear Timoshenko systems-global existence and exponential stability*, J. Math. Anal. Appl. **276** (2002), 248–278. http://dx.doi.org/10.1016/j.amc.2010.08.021

[11] B. Said-Houari and Y. Laskri, *A stability result of a Timoshenko system with a delay term in the internal feedback*, Appl. Math. Comput. **217** (2010), 2857–2869. http://dx.doi.org/10.1016/S0022-247X(02)00436-5

[12] B. Said-Houari and R. Rahali, *A stability result for a Timoshenko system with past history and a delay term in the internal feedback*, Dynam. Systems Appl. **20** (2011), 327.

[1]Laboratory of Pure and Applied Mathematics,
University of Amar TELIDJI,
Laghouat, Algeria
Email address: z.Khalili@lagh-univ.dz
Email address: d.ouchenane@lagh-univ.dz