Kragujevac Journal of Mathematics
Volume 51(5) (2027), Pages 783–812.

HARMONIC ANALYSIS AND SOBOLEV SPACES ASSOCIATED WITH LINEAR CANONICAL BESSEL OPERATORS: SCHRÖDINGER EQUATIONS AND WELL-POSEDNESS CRITERIA

YOUSSEF BETTAIBI

Abstract. The article explores harmonic analysis and Sobolev spaces associated with the linear canonical Bessel operator $H^{\mathbf{m},\alpha}_{r,s}$. It addresses the free Schrödinger equation related to this operator and presents the properties of the solutions when the initial data belong to specific spaces, including $H^{\mathbf{m},\alpha}_{s}$. The paper establishes Strichartz-type estimates for these Schrödinger equations. Subsequently, it defines spaces for solutions of the nonlinear Schrödinger equation associated with the Bessel operator, given by

$$i\partial_t u + \Delta_\alpha^{\mathbf{m}^{-1}} u = |u|^p u, \quad u\big|_{t=0} = g \in H^{\mathbf{m},\alpha}_{s}.$$

The article concludes by demonstrating the local well-posedness of solutions under the subcritical condition $p \in \left(0, \frac{2}{n+|\alpha|}\right]$ and the global well-posedness under the critical condition $p = \frac{2}{n+|\alpha|}$ for the L^2_α-nonlinear Schrödinger equations associated with the Bessel operator.

1. Introduction

This scientific study is a rigorous examination of Schrödinger equations related to linear canonical Bessel operator. The research significantly improves the linear and nonlinear features of the concerned mathematical models.

To delve into the Schrödinger equations associated with the linear canonical Bessel operator on $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_k > 0 \text{ for } k = 1, \dots, n\}$, we examine the

Key words and phrases. Harmonic analysis, Schrödinger-Bessel equations, Linear canonical Bessel transform, Strichartz-type Schrödinger-Bessel estimates, Sobolev spaces.

2020 *Mathematics Subject Classification.* Primary: 35Q55, 35Q41, 42B37. Secondary: 42B35, 42B30, 42B10.

DOI

Received: January 09, 2025.

Accepted: December 31, 2025.

following system:

$$(S_p) \qquad \begin{cases} i\partial_t u(t,x) + \Delta_\alpha^{\mathbf{m}^{-1}} u(t,x) = F(u(t,x)), & (t,x) \in I \times \mathbb{R}_+^n, \\ u_{|t=0} = g, \end{cases}$$

where $n \geq 2$, u is a complex-valued function defined on $I \times \mathbb{R}_+^n$, $F \in C(\mathbb{C}, \mathbb{C})$ satisfying

$$F(z) = |z|^p z, \quad p > 0, \tag{1.1}$$

and $\Delta_\alpha^{\mathbf{m}}$ is the linear canonical Bessel operator of index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\mathbf{m} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ such that $b \neq 0$, is defined by:

$$\begin{aligned} \Delta_\alpha^{\mathbf{m}} &= \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} + \left(\frac{2\alpha_k + 1}{x_k} - 2i\frac{d}{b}x_k \right) \frac{\partial}{\partial x_k} - \left(\frac{d^2}{b^2} x_k^2 + 2i\left(\alpha_k + 1\right) \frac{d}{b} \right) \\ &= \sum_{k=1}^n L_{\alpha_k}^{\mathbf{m}}, \quad \alpha_k > -\frac{1}{2}, \end{aligned}$$

where $L_{\alpha_k}^{\mathbf{m}}$ is the linear canonical Bessel operator of index $\alpha_k \in \left(-\frac{1}{2}, +\infty\right)$ defined by

$$L_{\alpha_k}^{\mathbf{m}} = \frac{\partial^2 u}{\partial x_k^2} + \left(\frac{2\alpha_k + 1}{x_k} - 2i\frac{d}{b}x_k \right) \frac{\partial}{\partial x_k} - \left(\frac{d^2}{b^2} x_k^2 + 2i\left(\alpha_k + 1\right) \frac{d}{b} \right).$$

The natural functional setting for these operators is the weighted Hilbert space

$$L_\alpha^2\left(\mathbb{R}_+^n\right) := L^2\left(\mathbb{R}_+^n, \mu_{\alpha,n}(x)\right),$$

where $\mu_{\alpha,n}$ is the measure defined on $\mathbb{R}_+^n$ by

$$d\mu_{\alpha,n}(x) = \prod_{k=1}^n \frac{x_k^{2\alpha_k+1}}{\sqrt{2\pi}2^{\alpha_k}\Gamma(\alpha_k+1)} dx,$$

and dx the Lebesgue measure on $\mathbb{R}^n$, with $\langle \cdot, \cdot \rangle_\alpha$ the scalar product on the Hilbert space $L_\alpha^2(\mathbb{R}_+^n)$, given by

$$\langle f, g \rangle_\alpha = \int_{\mathbb{R}_+^n} f(x)\overline{g(x)} d\mu_{\alpha,n}(x).$$

As operator in $L_\alpha^2\left(\mathbb{R}_+^n\right)$, the linear canonical Bessel operator $\Delta_\alpha^{\mathbf{m}}$ is first defined as minimal operators with dense domain $C_{*,c}^\infty\left(\mathbb{R}^n\right)$ of C^∞-functions with compact support in $\mathbb{R}^n$, even with respect to all variables.

The operator $L_\nu^{\mathbf{m}}$ is related to the Bessel operator L_ν by

$$L_\nu^{\mathbf{m}} = e^{\frac{i}{2}\cdot\frac{d}{b}r^2} \circ L_\nu \circ e^{-\frac{i}{2}\cdot\frac{d}{b}r^2},$$

with the Bessel operator defined in [4] by

$$L_\nu = \frac{d^2}{dr^2} + \frac{2\nu+1}{r} \cdot \frac{d}{dr} = r^{-2\nu-1} \frac{d}{dr}\left(r^{2\nu+1} \frac{d}{dr} \right).$$

Using the above unitary equivalence, the self-adjointness properties of the linear canonical Bessel operator $L_\nu^{\mathbf{m}}$ reduce to those of the classical Bessel operator L_ν

acting in $L^2(\mathbb{R}_+, r^{2\nu+1}dr)$. More precisely, as shown in [9, 11], the inverse-square Schrödinger operator $B_\alpha = -\frac{d^2}{dx^2} + \frac{\beta^2-\frac{1}{4}}{x^2}$ in $L^2(\mathbb{R}_+, dx)$ which is unitarily equivalent to $-L_\nu$ with the identification $\beta = \nu^2$, is essentially self-adjoint on $C_c^\infty(\mathbb{R}_+)$ if and only if $\nu \geq 1$. For $-\frac{1}{2} < \nu < 1$, the endpoint 0 is of limit-circle type and the operator B_β (and equivalently L_ν) admits a family of self-adjoint extensions; in this regime, the Friedrichs extension of the associated quadratic form provides a canonical self-adjoint realization. Since the quadratic phase transformation defining $L_\nu^{\mathbf{m}}$ is unitary, the same classification of self-adjointness and self-adjoint extensions applies to $L_\nu^{\mathbf{m}}$.

Although the differential expression for $\Delta_\alpha^{\mathbf{m}}$ explicitly involves only parameters b and d, the operator is defined for the full matrix $\mathbf{m} \in SL(2,\mathbb{R})$ to maintain consistency with the linear canonical transform group structure. The parameters a and c appear in the eigenfunctions, inverse transforms, and ensure proper composition properties. In the case when $\alpha = (-\frac{1}{2}, \ldots, -\frac{1}{2}, \alpha_n)$ and $\mathbf{m} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, it becomes the n-dimensional Weinstein operator of index α_n, which was examined by in [28].

In [12, 21, 30], the authors conducted a harmonic analysis related to the canonical Bessel operator $L_\alpha^{\mathbf{m}}$. This analysis established multiple properties of the canonical Fourier Bessel transform, encompassing a Riemann-Lebesgue lemma, inversion formula, operational formulas, Plancherel theorem, Babenko inequality, and various uncertainty inequalities. The linear canonical Fourier Bessel transform represents a broadening of the linear canonical transform (LCT) and belongs to a category of linear integral transforms characterized by a matrix parameter denoted as $\mathbf{m} \in SL(2,\mathbb{R})$ [6, 20, 38]. This category includes well-known transforms such as the Fourier transform, fractional Fourier transform, and the Fresnel transform, among others, and holds significance in various domains of physics and mathematics [6, 31, 38].

Considerable efforts have been invested in extending the LCT theory to incorporate other integral transforms, such as the Hankel transform, Dunkl transform [10, 15], and Fourier Bessel transform [36]. In a groundbreaking work [37], Wolf expanded the concept of LCT to the Hankel setting, establishing a class of linear integral transforms with a kernel featuring a Bessel function and matrix parameter $\mathbf{m} \in SL(2,\mathbb{R})$. Notable instances of this transformation include the Hankel transform and fractional Hankel transform [23].

Furthermore, the Dunkl linear canonical transform (DLCT) was introduced in [34] as a generalization of the LCT within the Dunkl transform framework [10]. DLCT encompasses various well-known transforms, such as the Dunkl transform [10, 15], fractional Dunkl transform [16, 17], and canonical Fourier Bessel transform [12, 34].

In this article, we establish the Strichartz-type Schrödinger estimates associated with the linear canonical Bessel operator, the Strichartz estimates are useful for establishing the existence of solutions for semilinear Schrödinger. Strichartz estimates were first proved by Strichartz [35] for solutions of Schrödinger and wave equations on $\mathbb{R}^n$. They were generalized to non-endpoint admissible pairs (q, r) by Ginibre and Velo [19] and by Lindblad and Sogge [24]. The end point estimates were proved by Keel and Tao [22], and the Strichartz-type Schrödinger estimates associated with several operators were generated, including the Dunkl operator in [27].

Furthermore, this work delves into the harmonic aspects of the linear canonical Bessel operator, exploring Sobolev-type spaces associated with it. For $s \in \mathbb{R}$ and $1 \leq r \leq +\infty$, the Sobolev space $H^{\mathbf{m},\alpha}_{r,s}$ is defined, with the notation $H^{\mathbf{m},\alpha}_{2,s}$ abbreviated to $H^{\mathbf{m},\alpha}_{s}$, providing a rigorous functional framework for analyzing the solutions of the Schrödinger equation with initial condition $g \in H^{\mathbf{m},\alpha}_{s}$. The study of such spaces follows the line of research developed in [2,25,26,29,33], where Sobolev-type structures associated with various special operators, such as Bessel, Jacobi, Gyrator, and Weinstein operators, are investigated. These works highlight how operator-adapted Sobolev spaces capture the intrinsic geometry and spectral behavior of the corresponding differential operators. In the present context, the criticality of the power p is examined within $H^{\mathbf{m},\alpha}_{s}$, and Definition 4.1 distinguishes between critical and subcritical cases according to specific relations between s and p. This refined analysis deepens the understanding of the nonlinear interaction governed by the linear canonical Bessel operator. We will apply these results in the special case $s = 0$ (i.e., when the initial data $g \in L^2_\alpha(\mathbb{R}^n)$) to determine and study the qualitative properties of solutions to system (S_p).

The paper is organized as follows. In Section 2, we recall the main results about the harmonic analysis associated with the linear canonical Bessel operator. Furthermore, we define and study the Sobolev spaces associated with the linear canonical Bessel operator $H^{\mathbf{m},\alpha}_{r,s}$. We introduce in Section 3 the free Schrödinger equation associated with the linear canonical Bessel operator. In the same section, we prove that the problem (S) has a unique solution if the initial data g belong to $\mathcal{S}'_*(\mathbb{R}^n)$, and we present properties of solution when the initial data g belong respectively to the spaces $\mathcal{S}_*(\mathbb{R}^n)$ and $H^{\mathbf{m},\alpha}_{r,s}$. Finally in thes section we establish the Strichartz-type Schrödinger estimates associated with the linear canonical Bessel operator. In Section 4, we define appropriate spaces for solutions of the nonlinear Schrödinger equations associated with the canonical Bessel operator (S_p) and critical power p when the initial data g belongs to $H^{\mathbf{m},\alpha}_{s}$, and we have successfully established the local well-posedness of the solution for the nonlinear Schrödinger equation (S_p) in the scenario where $s = 0$, p is critical, and under critical, with the additional condition that the initial data g belongs to $L^2_\alpha(\mathbb{R}^n_+)$. Moreover, we present a compelling demonstration that when the norm $\|g\|_{L^2_\alpha}$ is sufficiently small, and p is critical, the equation (S_p) exhibits global well-posedness.

2. Preliminaries

2.1. Harmonic analysis associated with the linear canonical Bessel operator.

Notations. In what follows, we need the following notations:

- $||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$,

• $\mathcal{C}_*(\mathbb{R}^n)$, the space of continuous functions on $\mathbb{R}^n$, even with respect to all variables,
• $\mathcal{E}_*(\mathbb{R}^n)$, the space of C^∞-functions on $\mathbb{R}^n$, even with respect to all variables,
• $\mathcal{S}_*(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$, even with respect to all variables,
• $\mathcal{D}_*(\mathbb{R}^n)$, the space of C^∞-functions on $\mathbb{R}^n$ which are of compact supports, even with respect to all variables,
• $\mathcal{H}_*(\mathbb{C}^n)$, the space of entire functions on $\mathbb{C}^n$, even with respect to all variables, rapidly decreasing and of exponential type,
• $\mathcal{S}'_*(\mathbb{R}^n) := \Big\{T : \mathcal{S}_*(\mathbb{R}^n) \to \mathbb{C} \mid T \text{ is a continuous linear functional}\Big\}$, where $\langle T, \phi\rangle$ denotes the action of T on $\phi \in \mathcal{S}_*(\mathbb{R}^n)$,
• $L^p_\alpha(\mathbb{R}^n_+)$, $1 \le p \le +\infty$, the space of measurable functions on $\mathbb{R}^n_+$ such that

$$\|f\|_{L^p_\alpha} = \left[\int_{\mathbb{R}^n_+} |f(x)|^p d\mu_{\alpha,n}(x)\right]^{\frac{1}{p}} < +\infty, \quad \text{if } 1 \le p < +\infty,$$
$$\|f\|_{L^\infty_\alpha} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n_+} |f(x)| < +\infty,$$

• throughout this paper, $\mathbf{m} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an arbitrary matrix in $SL(2,\mathbb{R})$ such that $b \neq 0$.

In this section, we shall collect some results and definitions from the theory of the harmonic analysis associated with the linear canonical Bessel operator on $\mathbb{R}_+$, developed [12]. The linear canonical transform play an important role in many fields of optics, radar system analysis, medium system analysis, filter design, phase retrieval, pattern recognition and many others. It was introduced a number of years ago in the mathematics literature but appears to have remained largely unknown to the signal processing community, to which it may, however, be potentially useful.

Proposition 2.1 ([12])**.** *Let* $\mathbf{m} \in SL(2,\mathbb{R})$.
(1) $\Delta^{\mathbf{m}}_\alpha$ *leaves* $\mathcal{S}_*(\mathbb{R}^n)$ *invariant.*
(2) *For every* $f, g \in \mathcal{S}_*(\mathbb{R}^n)$, $\langle \Delta^{\mathbf{m}}_\alpha f, g\rangle_\alpha = \langle f, \Delta^{\mathbf{m}}_\alpha g\rangle_\alpha$.

Remark 2.1. (1) The identity in Proposition 2.1 (2) remains valid for $f \in \mathcal{H}_*(\mathbb{C}^n)$ and $g \in \mathcal{D}_*(\mathbb{R}^n)$, by a standard density argument.

(2) The operator $\Delta^{\mathbf{m}}_\alpha$, initially defined on the dense domain $C^\infty_{*,c}(\mathbb{R}^n) \subset L^2_\alpha\left(\mathbb{R}^n_+\right)$, is symmetric. Proposition 2.1 shows that $\Delta^{\mathbf{m}}_\alpha$ leaves the Schwartz space $\mathcal{S}_*(\mathbb{R}^n)$ invariant and is symmetric on this domain. If $\alpha_k \ge 1$ for all $k = 1, \dots, n$, then each one-dimensional operator $L^{\mathbf{m}}_{\alpha_k}$ is essentially self-adjoint. By tensorization and strong commutativity, the multidimensional operator $\Delta^{\mathbf{m}}_\alpha$ is therefore essentially self-adjoint on $C^\infty_{*,c}(\mathbb{R}^n)$. In this case, we denote by $\overline{\Delta^{\mathbf{m}}_\alpha}$ its unique selfadjoint extension. In the general case $\alpha_k > -\frac{1}{2}$, the operator $\Delta^{\mathbf{m}}_\alpha$ admits several self-adjoint extensions. More precisely, when $-\frac{1}{2} < \alpha_k < 1$, the corresponding one-dimensional operators are not essentially self-adjoint and possess a family of self-adjoint extensions (see [9, 11]).

Throughout this work, we select the Friedrichs extension of $\Delta_\alpha^{\mathbf{m}}$, which will be denoted by the same symbol.

Let us begin with the following result, which provides the eigenfunction $\Psi_{\alpha,n}^{\mathbf{m}}(\lambda,\cdot)$ associated with the linear canonical Bessel operator $\Delta_\alpha^{\mathbf{m}}$.

Proposition 2.2 ([12])**.** *For all $z=(z_1,z_2,\dots,z_n)\in\mathbb{C}^n$, the system*

$$\begin{cases} L_{\alpha_k}^{\mathbf{m}} u\left(\cdot,z_k\right) = -\frac{z_k^2}{b^2} u\left(\cdot,z_k\right), & \text{for } 1\le k\le n,\\ u\left(0,z\right)=e^{\frac{ia}{2b}\|z\|^2}, & \\ \frac{\partial u}{\partial\lambda_k}(0,z)=0, & \text{for } \ 1\le k\le n,\end{cases}$$

has a unique solution $\Psi_{\alpha,n}^{\mathbf{m}}(\lambda,\cdot)$ given by

$$\Psi_{\alpha,n}^{\mathbf{m}}\left(\lambda,z\right)=e^{\frac{i}{2}\left(\frac{d}{b}\|\lambda\|^2+\frac{a}{b}\|z\|^2\right)}\prod_{k=1}^{n} j_{\alpha_k}\left(\frac{\lambda_k z_k}{b}\right),\quad \text{for all } \lambda\in\mathbb{C}^n,$$

where j_{α_k} is the normalized Bessel function of index α_k, defined by

$$j_{\alpha_k}(\xi)=\Gamma(\alpha_k+1)\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\Gamma(n+\alpha_k+1)}\left(\frac{\xi}{2}\right)^{2n},\quad \text{for all } \xi\in\mathbb{C}.$$

In particular for $x,y\in\mathbb{R}_+^n$, $|\Psi_{\alpha,n}^{\mathbf{m}}(x,y)|\le 1$.

Remark 2.2. (1) For all $\lambda\in\mathbb{R}^n$, $\Psi_{\alpha,n}^{\mathbf{m}}(\lambda,\cdot)\in\mathcal{H}_*(\mathbb{C}^n)$.

(2) The function $\Psi_{\alpha,n}^{\mathbf{m}}(\lambda,\cdot)$ satisfies

$$\Delta_\alpha^{\mathbf{m}}\Psi_{\alpha,n}^{\mathbf{m}}(\lambda,\cdot)=-\frac{\|\lambda\|^2}{b^2}\,\Psi_{\alpha,n}^{\mathbf{m}}(\lambda,\cdot),$$

and is therefore an *eigenfunction* of the operator $\Delta_\alpha^{\mathbf{m}}$ corresponding to the eigenvalue $-\frac{\|\lambda\|^2}{b^2}$.

Definition 2.1. The linear canonical Bessel transform is given for $f\in L_\alpha^1(\mathbb{R}_+^n)$ by

$$\mathcal{F}_\alpha^{\mathbf{m}}(f)(\lambda)=\frac{1}{(ib)^{|\alpha|+n}}\int_{\mathbb{R}_+^n} f(x)\Psi_{\alpha,n}^{\mathbf{m}}(\lambda,x)d\mu_{\alpha,n}(x),\quad \text{for all } \lambda\in\mathbb{R}_+^n,$$

where$(ib)^{|\alpha|+n}=|b|^{|\alpha|+n}e^{i\frac{\pi}{2}(|\alpha|)\operatorname{sgn}(b)}$, and $|\alpha|=\alpha_1+\cdots+\alpha_n$ thus $|\alpha|$ may be negative.

Using the properties of the classical Fourier transform on $\mathbb{R}^n$ and of the Bessel transform, one can easily see the following relation, which will play an important role in the sequel.

Some basic properties of the transform $\mathcal{F}_\alpha^{\mathbf{m}}$ are summarized in the following results. For the proofs, we refer to [12].

Proposition 2.3. (1) *For all $f\in L_\alpha^1(\mathbb{R}_+^n)$, we have*

$$\|\mathcal{F}_\alpha^{\mathbf{m}}(f)\|_{L_\alpha^\infty}\le\frac{1}{|b|^{|\alpha|+n}}\|f\|_{L_\alpha^1}.$$

(2) *For all* $f \in L^1_\alpha(\mathbb{R}^n_+)$ *with* $\mathcal{F}^{\mathbf{m}}_\alpha f \in L^1_\alpha(\mathbb{R}^n_+)$,

$$\left(\mathcal{F}^{\mathbf{m}}_\alpha \circ \mathcal{F}^{\mathbf{m}^{-1}}_\alpha\right) f = \left(\mathcal{F}^{\mathbf{m}^{-1}}_\alpha \circ \mathcal{F}^{\mathbf{m}}_\alpha\right) f = f, \quad a.e. \tag{2.1}$$

(3) *For all* $f, g \in L^1_\alpha(\mathbb{R}^n_+)$, *we have*

$$\int_{\mathbb{R}^n_+} \mathcal{F}^{\mathbf{m}}_\alpha (f)(x)\overline{g(x)} d\mu_{\alpha,n}(x) = \int_{\mathbb{R}^n_+} f(x) \overline{\mathcal{F}^{\mathbf{m}^{-1}}_\alpha (g)(x)} d\mu_{\alpha,n}(x).$$

Proposition 2.4. *For all* $f \in L^1_\alpha(\mathbb{R}^n_+)$ *such that* $\Delta^{\mathbf{m}^{-1}}_\alpha f \in L^1_\alpha(\mathbb{R}^n_+)$, *we have*

$$\mathcal{F}^{\mathbf{m}}_\alpha(\Delta^{\mathbf{m}^{-1}}_\alpha f)(x) = \frac{-\|x\|^2}{b^2}\mathcal{F}^{\mathbf{m}}_\alpha(f)(x),$$

$$\Delta^{\mathbf{m}}_\alpha \mathcal{F}^{\mathbf{m}}_\alpha(f)(x) = -\mathcal{F}^{\mathbf{m}}_\alpha\left(\frac{\|\cdot\|^2}{b^2}f\right)(x).$$

The following result shows that the linear canonical Bessel transform verifies Plancherel's theorem.

Theorem 2.1. (1) *For all* $f \in \mathcal{S}_*(\mathbb{R}^n)$, *we have* $\|\mathcal{F}^{\mathbf{m}}_\alpha f\|_{L^2_\alpha} = \|f\|_{L^2_\alpha}$.

(2) *If* $f \in L^1_\alpha(\mathbb{R}^n_+) \cap L^2_\alpha(\mathbb{R}^n_+)$, *then* $\mathcal{F}^{\mathbf{m}}_\alpha f \in L^2_\alpha(\mathbb{R}^n_+)$ *and* $\|\mathcal{F}^{\mathbf{m}}_\alpha f\|_{L^2_\alpha} = \|f\|_{L^2_\alpha}$.

(3) *The linear canonical Bessel transform has a unique extension to an isometric isomorphism of* $L^2_\alpha(\mathbb{R}^n_+)$.

(4) *For all f and* $g \in L^2_\alpha(\mathbb{R}^n_+)$, *we have*

$$\langle \mathcal{F}^{\mathbf{m}}_\alpha f, g\rangle_\alpha = \left\langle f, \mathcal{F}^{\mathbf{m}^{-1}}_\alpha g\right\rangle_\alpha.$$

Example 2.1. Let $\lambda \in \mathbb{R}^*$ and $g \in \mathcal{S}_*(\mathbb{R}^n)$ therefore $g_{\lambda,\mathbf{m}} \in \mathcal{S}_*(\mathbb{R}^n)$ where

$$g_{\lambda,\mathbf{m}}(x) = e^{-i\frac{a}{2b}\left(1-\lambda^2\right)\|x\|^2} g(\lambda x),$$

and we have

$$\mathcal{F}^{\mathbf{m}}_\alpha (g_{\lambda,\mathbf{m}})(x) = \lambda^{-2(|\alpha|+n)} e^{i\frac{d}{2b}\left(1-\frac{1}{\lambda^2}\right)\|x\|^2} \mathcal{F}^{\mathbf{m}}_\alpha (g)\left(\frac{x}{\lambda}\right), \quad \text{for all } x \in \mathbb{R}.$$

Definition 2.2. The translation operator $T^{\mathbf{m}}_{x,\alpha}$, $x \in \mathbb{R}^n_+$, associated with the operator $\Delta^{\mathbf{m}}_\alpha$, is defined on $\mathcal{C}_*(\mathbb{R}^n)$, for all $y \in \mathbb{R}^n_+$, by

$$\begin{aligned} T^{\mathbf{m}}_{x,\alpha} f(y) = &\left(\frac{2}{\sqrt{\pi}}\right)^n \left(\prod_{k=1}^n \frac{\Gamma(\alpha_k+1)}{\Gamma\left(\alpha_k+\frac{1}{2}\right)}\right) \int_{(0,\pi)^n} \left(\prod_{k=1}^n e^{-i\frac{d}{b}x_k y_k \cos\theta_k}\right) \\ &\times f\left(\sqrt{x_1^2+y_1^2+2x_1y_1\cos\theta_1}, \dots, \sqrt{x_n^2+y_n^2+2x_ny_n\cos\theta_n}\right) \\ &\times (\sin\theta_1)^{2\alpha_1} \cdots (\sin\theta_n)^{2\alpha_n} \, d\theta_1 \cdots d\theta_n. \end{aligned}$$

The following proposition summarizes some properties of the linear canonical Bessel translation operator.

Proposition 2.5 ([34]). (1) *For $f \in \mathcal{C}_*(\mathbb{R}^n)$, we have*

$$T^{\mathbf{m}}_{x,\alpha} f(y) = T^{\mathbf{m}}_{y,\alpha} f(x) \text{ and } T^{\mathbf{m}}_{0,\alpha} f = f, \quad \text{for all } x, y \in \mathbb{R}^n_+.$$

(2) *For all $f \in \mathcal{E}_*(\mathbb{R}^n)$ and $y \in \mathbb{R}^n_+$, the function $x \mapsto T^{\mathbf{m}}_{x,\alpha} f(y)$ belongs to $\mathcal{E}_*(\mathbb{R}^n)$.*

(3) *Let $f \in L^p_\alpha(\mathbb{R}^n_+)$, $1 \leq p \leq +\infty$ and $x \in \mathbb{R}^n_+$. Then, $T^{\mathbf{m}}_{x,\alpha} f$ belongs to $L^p_\alpha(\mathbb{R}^n_+)$ and we have $\|T^{\mathbf{m}}_{x,\alpha} f\|_{L^p_\alpha} \leq \|f\|_{L^p_\alpha}$.*

(4) *The function $\Psi^{\mathbf{m}}_{\alpha,n}(\cdot, \lambda)$, $\lambda \in \mathbb{C}^d$, satisfies on $\mathbb{R}^n_+$ the following product formula:*

$$T^{\mathbf{m}}_{x,\alpha} \left[\Psi^{\mathbf{m}}_{\alpha,n}(\cdot, \lambda)\right](y) = e^{-\frac{i}{2}\cdot\frac{a}{b}||\lambda||^2} \Psi^{\mathbf{m}}_{\alpha,n}(x, \lambda)\, \Psi^{\mathbf{m}}_{\alpha,n}(y, \lambda), \quad \text{for all } x, y \in \mathbb{R}^n_+.$$

(5) *Let $f \in L^p_\alpha(\mathbb{R}^n_+)$, $p = 1$ or 2, and $x \in \mathbb{R}^n_+$, we have*

$$\mathcal{F}^{\mathbf{m}}_\alpha \left(T^{\mathbf{m}^{-1}}_{x,\alpha} f\right)(y) = e^{-\frac{i}{2}\cdot\frac{d}{b}||y||^2} \Psi^{\mathbf{m}^{-1}}_{\alpha,n}(x, y)\, \mathcal{F}^{\mathbf{m}}_\alpha(f)(y), \quad \text{for all } y \in \mathbb{R}^n_+.$$

(6) *We have*

$$\Delta^{\mathbf{m}}_\alpha \circ T^{\mathbf{m}}_{x,\alpha} = T^{\mathbf{m}}_{x,\alpha} \circ \Delta^{\mathbf{m}}_\alpha, \quad \text{for all } x \in \mathbb{R}^n_+.$$

Definition 2.3. The linear canonical Bessel convolution product of $f, g \in \mathcal{C}_*(\mathbb{R}^n)$ is given by:

$$f \underset{\alpha,\mathbf{m}}{*} g(x) = \int_{\mathbb{R}^n_+} e^{-i\frac{d}{b}||y||^2} T^{\mathbf{m}}_{x,\alpha} f(y)\, g(y)\, d\mu_{\alpha,n}(y), \quad \text{for all } x \in \mathbb{R}^n_+.$$

Proposition 2.6 (Young's inequality [34]). *i) Let $p, q, r \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then, for all $f \in L^p_\alpha(\mathbb{R}^n_+)$ and $g \in L^q_\alpha(\mathbb{R}^n_+)$, the function $f \underset{\alpha,\mathbf{m}}{*} g \in L^r_\alpha(\mathbb{R}^n_+)$ and we have*

$$\|f \underset{\alpha,\mathbf{m}}{*} g\|_{L^r_\alpha} \leq \|f\|_{L^p_\alpha} \|g\|_{\alpha,q}.$$

ii) Let $f \in L^1_\alpha(\mathbb{R}^n_+)$ and $g \in L^p_\alpha(\mathbb{R}^n_+)$, $p \in [1, 2]$. For all $\lambda \in \mathbb{R}^n$, we have

$$\mathcal{F}^{\mathbf{m}}_\alpha \left(f \underset{\alpha,\mathbf{m}^{-1}}{*} g\right)(\lambda) = (ib)^{|\alpha|} e^{-\frac{i}{2}\cdot\frac{d}{b}\|\lambda\|^2} \mathcal{F}^{\mathbf{m}}_\alpha(f)(\lambda)\ \mathcal{F}^{\mathbf{m}}_\alpha(g)(\lambda). \tag{2.2}$$

Definition 2.4. (1) The linear canonical Bessel transform of a distribution $S \in \mathcal{S}'_*(\mathbb{R}^n)$ is defined by

$$\langle \mathcal{F}^{\mathbf{m}}_\alpha(S), \phi \rangle = \left\langle S, \mathcal{F}^{\mathbf{m}^{-1}}_\alpha(\phi) \right\rangle, \quad \text{for all } \phi \in \mathcal{S}_*(\mathbb{R}^n).$$

(2) Let τ be in $\mathcal{S}'_*(\mathbb{R}^n)$. We define linear canonical Bessel operator $\Delta^{\mathbf{m}}_\alpha \tau$, by

$$\langle \Delta^{\mathbf{m}}_\alpha \tau, \phi \rangle = \langle \tau, \Delta^{\mathbf{m}}_\alpha \phi \rangle, \quad \text{for all } \phi \in \mathcal{S}_*(\mathbb{R}^n).$$

Remark 2.3. As an immediate consequence of the two previous theorems, we have the transform $\mathcal{F}^{\mathbf{m}}_\alpha$ is a topological isomorphism from $\mathcal{S}'_*(\mathbb{R}^n)$ onto itself.

Example 2.2. Let $P^{\mathbf{m}^{-1}}_t$, $t > 0$, be the function defined by

$$P^{\mathbf{m}^{-1}}_t(x) = \frac{1}{(2t)^{|\alpha|+n}} e^{-(\frac{i}{2}\cdot\frac{a}{b} + \frac{1}{4t})\|x\|^2}, \quad \text{for all } x \in \mathbb{R}^n.$$

(1) The linear canonical Bessel transform $\mathcal{F}_\alpha^{\mathbf{m}}$ of $P_t^{\mathbf{m}^{-1}}$ is given by:

$$\mathcal{F}_\alpha^{\mathbf{m}}\left(P_t^{\mathbf{m}^{-1}}\right)(\lambda)=\frac{1}{(ib)^{|\alpha|+n}}e^{-(\frac{t}{b^2}-i\frac{d}{b})\|\lambda\|^2},\quad \text{for all } \lambda\in\mathbb{R}_+^n.$$

Let us observe that for $z\in\mathbb{C}$, with positive real part, the two functions

$$z\mapsto\mathcal{F}_\alpha^{\mathbf{m}}\left(P_z^{\mathbf{m}^{-1}}\right)\quad\text{ and }\quad z\mapsto\frac{1}{(ib)^{|\alpha|+n}}e^{-(\frac{z}{b^2}-i\frac{d}{b})\|\cdot\|^2}$$

are holomorphic on the domain $\operatorname{Re} z>0$. As they coincide on the real axis, they coincide in the whole domain.

Now, if t positive, considering a sequence of z_n with positive real part which tends to it, we get, as the linear canonical Bessel transform is continuous on tempered distributions that

$$\mathcal{F}_\alpha^{\mathbf{m}}\left(P_{it}^{\mathbf{m}^{-1}}\right)(\lambda)=\frac{1}{(ib)^{|\alpha|+n}}e^{-i(\frac{t}{b^2}-\frac{d}{b})\|\lambda\|^2},\quad \text{for all } \lambda\in\mathbb{R}^n,$$

then we have the following result

$$e^{-i\frac{t}{b^2}\|\lambda\|^2}=(ib)^{|\alpha|+n}e^{-i\frac{d}{b}\|\lambda\|^2}\mathcal{F}_\alpha^{\mathbf{m}}\left(P_{it}^{\mathbf{m}^{-1}}\right)(\lambda),\quad \text{for all } \lambda\in\mathbb{R}^n.$$

(2) The linear canonical Bessel translation $T_{x,\alpha}^{\mathbf{m}^{-1}}$ of $P_{it}^{\mathbf{m}^{-1}}$, $t\in\mathbb{R}^*$, is given by

$$T_{x,\alpha}^{\alpha,\mathbf{m}^{-1}}(P_{it}^{\mathbf{m}^{-1}})(y)=\frac{1}{(2it)^{|\alpha|+n}}e^{i\left(\frac{1}{4t}-\frac{a}{2b}\right)\left(\|x\|^2+\|y\|^2\right)}\prod_{k=1}^{n}j_{\alpha_k}\left(\frac{x_ky_k}{2t}\right),\quad \text{for all } x,y\in\mathbb{R}_+^n,$$

the ensuing inequality can be succinctly as follows

$$\left|T_{x,\alpha}^{\alpha,\mathbf{m}^{-1}}(P_{it}^{\mathbf{m}^{-1}})(y)\right|\leq\frac{1}{(2|t|)^{|\alpha|+n}},\quad \text{for all } x,y\in\mathbb{R}_+^n. \tag{2.3}$$

2.2. **Sobolev spaces associated with the linear canonical Bessel operator.** For $s\in\mathbb{R}$, $1\leq r\leq+\infty$, we define the Sobolev space associated with the linear canonical Bessel operator by

$$H_{r,s}^{\mathbf{m},\alpha}=\left\{u\in\mathcal{S}'_*(\mathbb{R}^n):\|u\|_{H_{r,s}^{\mathbf{m},\alpha}}\equiv\left\|\mathcal{F}_\alpha^{\mathbf{m}^{-1}}\left(\left(1+\|\lambda\|^2\right)^s\mathcal{F}_\alpha^{\mathbf{m}}(u)\right)\right\|_{L_\alpha^r}<+\infty\right\}.$$

We will use the notation $H_{2,s}^{\mathbf{m},\alpha}$ by $H_s^{\mathbf{m},\alpha}$.

Remark 2.4. In accordance with Plancherel theorem, the space $H_s^{\mathbf{m},\alpha}$ can be defined utilizing the following norm:

$$\|u\|_{H_s^{\mathbf{m},\alpha}}=\left\|\left(1+\|\lambda\|^2\right)^s\mathcal{F}_\alpha^{\mathbf{m}}(u)\right\|_{L_\alpha^2}.$$

There is another version of Sobolev spaces defined using the linear canonical Bessel operator, we then define for k be a non-negative integer and $1\leq r\leq+\infty$

$$W_{r,k}^{\mathbf{m},\alpha}=\left\{u\in\mathcal{S}'_*(\mathbb{R}^n):\|u\|_{W_{r,k}^{\mathbf{m},\alpha}}\equiv\sum_{j=0}^{k}\left\|\left(\Delta_\alpha^{\mathbf{m}^{-1}}\right)^j(u)\right\|_{L_\alpha^r}<+\infty\right\}.$$

Proposition 2.7. *Let $s = k$ be a non-negative integer. Then, $H_k^{\mathbf{m},\alpha} = W_{2,k}^{\mathbf{m},\alpha}$, that is to say*

$$u \in H_k^{\mathbf{m},\alpha} \Leftrightarrow (\Delta_\alpha^{\mathbf{m}^{-1}})^j u \in L_\alpha^2\left(\mathbb{R}_+^n\right), \quad \textit{for all } j \in \mathbb{N}, j \leq k, .$$

Proof. Let k be a non-negative integer and $u \in H_k^{\mathbf{m}^{-1},\alpha}$, we have

$$\begin{aligned}
\|u\|^2_{H_k^{\mathbf{m},\alpha}} &= \left\| \left(1 + \|\lambda\|^2\right)^k \mathcal{F}_\alpha^{\mathbf{m}}(u) \right\|^2_{L_\alpha^2} \\
&= \int_{\mathbb{R}_+^n} \left(1 + \|\lambda\|^2\right)^{2k} \left|\mathcal{F}_\alpha^{\mathbf{m}}(u)(\lambda)\right|^2 d\mu_{\alpha,n}(\lambda) \\
&= \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \int_{\mathbb{R}_+^n} \left| \|\lambda\|^{2j} \mathcal{F}_\alpha^{\mathbf{m}}(u)(\lambda) \right|^2 d\mu_{\alpha,n}(\lambda) \\
&= \sum_{j=0}^{k} b^{2k} \begin{pmatrix} k \\ j \end{pmatrix} \int_{\mathbb{R}_+^n} \left| \left((\Delta_\alpha^{\mathbf{m}^{-1}})^j u\right)(\lambda) \right|^2 d\mu_{\alpha,n}(\lambda).
\end{aligned}$$

So, $u \in H_k^{\mathbf{m},\alpha}$ if and only if $\left\|(\Delta_\alpha^{\mathbf{m}^{-1}})^j u\right\|_{L_\alpha^2} < +\infty$ for all $j \leq k$. ☐

Proposition 2.8. (1) *Let $1 \leq r < +\infty$. The space $H_{r,s}^{\mathbf{m},\alpha}$ provided with the norm $\|\cdot\|_{H_{r,s}^{\mathbf{m},\alpha}}$ is a Banach space.*

(2) *We have $H_0^{\mathbf{m},\alpha} = L_\alpha^2\left(\mathbb{R}_+^n\right)$.*

(3) *Let s_1, s_2 in $\mathbb{R}$ such that $s_1 \geq s_2$. Then, $H_{s_1}^{\mathbf{m},\alpha} \hookrightarrow H_{s_2}^{\mathbf{m},\alpha}$.*

(4) *Let $s \in \mathbb{R}$ such that $s \geq 0$. Then,*

$$c_2 \left\| \|x\|^{2s} \mathcal{F}_\alpha^{\mathrm{m}}(\cdot) \right\|_{L_\alpha^2} \leq \|\cdot\|_{H_s^{\mathbf{m},\alpha}} \leq c_1 \left(\|\cdot\|_{L_\alpha^2} + \left\| \|x\|^{2s} \mathcal{F}_\alpha^{\mathrm{m}}(\cdot) \right\|_{L_\alpha^2} \right).$$

Proof. (1) It is clear that the space $L_\alpha^r\left(\mathbb{R}_+^n\right)$ is complete and since $\mathcal{F}_\alpha^{\mathbf{m}^{-1}}\left((1 + \|\lambda\|^2)^s \mathcal{F}_\alpha^{\mathbf{m}}\right)$ is an isomorphism from $\mathcal{S}'_*(\mathbb{R}^n)$ onto itself, $H_{r,s}^{\mathbf{m},\alpha}$ is then a Banach space.

The results (2) and (3) follow immediately from the definition of the Sobolev space associated with the linear canonical Bessel operator.

(4) We use for all $s \geq 0$ there exists c_1 and c_2 such that

$$c_2 \|x\|^{4s} \leq \left(1 + \|x\|^2\right)^{2s} \leq c_1 \left(1 + \|x\|^{4s}\right).$$

☐

3. The Schrödinger Equation Associated with the Canonical Bessel Operator

3.1. **The free Schrödinger equation associated with the linear canonical Bessel operator.**

Notation. For $u \in \mathcal{S}'\Big(\mathbb{R}; \mathcal{S}'_*(\mathbb{R}^n)\Big)$, its action on $\psi \in \mathcal{S}\Big(\mathbb{R}; \mathcal{S}_*(\mathbb{R}^n)\Big)$ is given by

$$\langle u, \psi \rangle_{\mathcal{S}'(\mathbb{R}; \mathcal{S}'_*(\mathbb{R}^n)), \mathcal{S}(\mathbb{R}; \mathcal{S}_*(\mathbb{R}^n))} = \int_{\mathbb{R}} \langle u_t, \psi(t, \cdot) \rangle \, dt,$$

where $\langle u_t, \psi(t,\cdot)\rangle$ denotes the action of the distribution $u_t \in \mathcal{S}'_*(\mathbb{R}^n)$ on $\psi(t,\cdot) \in \mathcal{S}_*(\mathbb{R}^n)$.

We consider the free Schrödinger equation associated with the linear canonical Bessel operator

$$(S) \qquad \begin{cases} i\partial_t u + \Delta_\alpha^{\mathbf{m}^{-1}} u = 0, \\ u_{|t=0} = g. \end{cases}$$

Theorem 3.1. *If g is an element of $\mathcal{S}'_*(\mathbb{R}^n)$, then there exists a unique solution u of the system (S) in $\mathcal{E}(\mathbb{R}; \mathcal{S}'_*(\mathbb{R}^n))$.*

Proof. First, we establish the existence. For $t \in \mathbb{R}$, we define

$$u_t = \mathcal{F}_\alpha^{\mathbf{m^{-1}}} \left(e^{-\frac{it}{b^2}\|\cdot\|^2} \mathcal{F}_\alpha^{\mathbf{m}} g \right).$$

Since $g \in \mathcal{S}'_*(\mathbb{R}^n)$, $(\xi \to e^{-\frac{it}{b}\|\xi\|^2}) \in L^\infty_\alpha(\mathbb{R}^n_+) \cap \mathcal{E}_*(\mathbb{R}^n)$ and the transform $\mathcal{F}_\alpha^{\mathbf{m}}$ is an isomorphism from $\mathcal{S}'_*(\mathbb{R}^n)$ onto itself. Thus, $u_t \in \mathcal{S}'_*(\mathbb{R}^n)$. For $\varphi \in \mathcal{S}_*(\mathbb{R}^n)$, we have

$$\langle u_t, \varphi\rangle = \left\langle \mathcal{F}_\alpha^{\mathbf{m}}(g), e^{-\frac{it}{b}\|\cdot\|^2} \mathcal{F}_\alpha^{\mathbf{m}}(\varphi) \right\rangle.$$

This implies that $u_t \in \mathcal{E}(\mathbb{R}; \mathcal{S}'_*(\mathbb{R}^n))$ and $\mathcal{F}_\alpha^{\mathbf{m}}(u_t) \in \mathcal{E}(\mathbb{R}; \mathcal{S}'_*(\mathbb{R}^n))$. Recall that u is defined by

$$\langle u, \psi\rangle_{\mathcal{S}'(\mathbb{R};\mathcal{S}'_*(\mathbb{R}^n)),\mathcal{S}(\mathbb{R};\mathcal{S}_*(\mathbb{R}^n))} = \int_{\mathbb{R}} \langle u_t, \psi(t,\cdot)\rangle\, dt, \quad \psi \in \mathcal{S}(\mathbb{R}; \mathcal{S}_*(\mathbb{R}^n)).$$

Therefore, for $\psi \in \mathcal{S}(\mathbb{R}; \mathcal{S}_*(\mathbb{R}^n))$, we have

$$\begin{aligned}
&\left\langle (i\partial_t + \Delta_\alpha^{\mathbf{m}^{-1}})u, \psi \right\rangle_{\mathcal{S}'(\mathbb{R};\mathcal{S}'_*(\mathbb{R}^n)),\mathcal{S}(\mathbb{R};\mathcal{S}_*(\mathbb{R}^n))} \\
=& -\left\langle u, (i\partial_t - \Delta_\alpha^{\mathbf{m}^{-1}})\psi \right\rangle_{\mathcal{S}'(\mathbb{R};\mathcal{S}'_*(\mathbb{R}^n)),\mathcal{S}(\mathbb{R};\mathcal{S}_*(\mathbb{R}^n))} \\
=& -i\int_{\mathbb{R}} \left\langle u_t, (\partial_t + i\Delta_\alpha^{\mathbf{m}^{-1}})\psi(t,\cdot) \right\rangle dt \\
=& -\int_{\mathbb{R}} \left\langle (\mathcal{F}_\alpha^{\mathbf{m}})^{-1} \left(e^{-\frac{it}{b^2}\|\cdot\|^2} \mathcal{F}_\alpha^{\mathbf{m}} g \right), \partial_t\psi(t,\cdot) + i\Delta_\alpha^{\mathbf{m}^{-1}}\psi(t,\cdot) \right\rangle dt \\
=& -i\int_{\mathbb{R}} \left\langle e^{-\frac{it}{b^2}\|\cdot\|^2} \mathcal{F}_\alpha^{\mathbf{m}} g, (\partial_t - i\frac{\|\cdot\|^2}{b^2}) \mathcal{F}_\alpha^{\mathbf{m}}(\psi(t,\cdot)) \right\rangle dt \\
=& -i\int_{\mathbb{R}} \left\langle \mathcal{F}_\alpha^{\mathbf{m}} g, \partial_t (e^{-\frac{it}{b^2}\|\cdot\|^2} \mathcal{F}_\alpha^{\mathbf{m}}(\psi(t,\cdot))) \right\rangle dt \\
=& -i\int_{\mathbb{R}} \partial_t \left\langle \mathcal{F}_\alpha^{\mathbf{m}} g, e^{-\frac{it}{b^2}\|\cdot\|^2} \mathcal{F}_\alpha^{\mathbf{m}}(\psi(t,\cdot)) \right\rangle dt \\
=&0.
\end{aligned}$$

Thus, u is a solution of equation (S).

Now, we establish uniqueness, or equivalently, we prove that $u \equiv 0$ is the unique solution of the following linear problem

$$\begin{cases} i\partial_t u(t,x) + \Delta_\alpha^{\mathbf{m}^{-1}} u(t,x) = 0, \\ u(0,x) = 0. \end{cases}$$

For $\psi \in \mathcal{S}(\mathbb{R}; \mathcal{S}_*(\mathbb{R}^n))$, we have

$$\begin{aligned} 0 &= \left\langle (i\partial_t + \Delta_\alpha^{\mathbf{m}^{-1}})u, \psi \right\rangle_{\mathcal{S}'(\mathbb{R};\mathcal{S}'_*(\mathbb{R}^n)),\mathcal{S}(\mathbb{R};\mathcal{S}_*(\mathbb{R}^n))} \\ &= -i\int_{\mathbb{R}} \left\langle u_t, \partial_t\psi(t,\cdot) + i\Delta_\alpha^{\mathbf{m}^{-1}}\psi(t,\cdot)\right\rangle dt \\ &= -i\int_{\mathbb{R}} \partial_t \langle u_t, \psi\rangle\, dt + i\int_{\mathbb{R}} \left[\langle \partial_t u_t, \psi\rangle - i\left\langle u_t, \Delta_\alpha^{\mathbf{m}^{-1}}\psi(t,\cdot)\right\rangle\right] dt. \end{aligned}$$

Since

$$\partial_t \langle u_t, \psi\rangle = \langle \partial_t u_t, \psi\rangle + \langle u_t, \partial_t\psi\rangle,$$

and as $\psi \in \mathcal{S}(\mathbb{R}; \mathcal{S}_*(\mathbb{R}^n))$, we have

$$\int_{\mathbb{R}} \partial_t \langle u_t, \psi(t,\cdot)\rangle\, dt = 0.$$

Consequently, we have

$$\int_{\mathbb{R}} \left[\langle \partial_t u_t, \psi(t,\cdot)\rangle - i\left\langle u_t, \Delta_\alpha^{\mathbf{m}^{-1}}\psi(t,\cdot)\right\rangle\right] dt = 0.$$

By using the fact that $\mathcal{F}_\alpha^{\mathbf{m}}(\partial_t u_t) = \partial_t(\mathcal{F}_\alpha^{\mathbf{m}} u_t)$, we deduce that for all $\psi \in \mathcal{S}(\mathbb{R}; \mathcal{S}_*(\mathbb{R}^n))$,

$$\int_{\mathbb{R}} \left[\left\langle \partial_t(\mathcal{F}_\alpha^{\mathbf{m}} u_t), (\mathcal{F}_\alpha^{\mathbf{m}})^{-1}\psi(t,\cdot)\right\rangle + i\left\langle \mathcal{F}_\alpha^{\mathbf{m}} u_t, \|\cdot\|^2 (\mathcal{F}_\alpha^{\mathbf{m}})^{-1}\psi(t,\cdot)\right\rangle\right] dt = 0.$$

We choose ψ in the form $(\mathcal{F}_\alpha^{\mathbf{m}})^{-1}\psi(t,\xi) = e^{it\|\xi\|^2}\varphi(\xi)\chi(t)$, where $\varphi \in \mathcal{S}_*(\mathbb{R}^n)$ and $\chi \in \mathcal{S}(\mathbb{R})$. This leads to

$$\int_{\mathbb{R}} \left[\left\langle \partial_t(\mathcal{F}_\alpha^{\mathbf{m}} u_t), e^{\frac{it}{b^2}\|\cdot\|^2}\varphi\right\rangle + i\left\langle \mathcal{F}_\alpha^{\mathbf{m}} u_t, \|\cdot\|^2 e^{\frac{it}{b^2}\|\cdot\|^2}\varphi\right\rangle\right] \chi(t) dt = 0.$$

This equality holds for all $\chi \in \mathcal{S}_*(\mathbb{R})$, thus we have

$$\left\langle \partial_t(\mathcal{F}_\alpha^{\mathbf{m}} u_t), e^{\frac{it}{b^2}\|\cdot\|^2}\varphi\right\rangle + i\left\langle \mathcal{F}_\alpha^{\mathbf{m}} u_t, \|\cdot\|^2 e^{\frac{it}{b^2}\|\cdot\|^2}\varphi\right\rangle = 0.$$

Since

$$\partial_t \left\langle \mathcal{F}_\alpha^{\mathbf{m}} u_t, e^{\frac{it}{b^2}\|\cdot\|^2}\varphi\right\rangle = \left\langle \partial_t(\mathcal{F}_\alpha^{\mathbf{m}} u_t), e^{\frac{it}{b^2}\|\cdot\|^2}\varphi\right\rangle + i\left\langle \mathcal{F}_\alpha^{\mathbf{m}} u_t, \|\cdot\|^2 e^{\frac{it}{b^2}\|\cdot\|^2}\varphi\right\rangle,$$

for all $\varphi \in \mathcal{S}_*(\mathbb{R}^n)$, the function $t \to \left\langle \mathcal{F}_\alpha^{\mathbf{m}} u_t, e^{\frac{it}{b^2}\|\cdot\|^2}\varphi\right\rangle$ is constant.

Taking into account the fact that $g = 0$ for all $t \in \mathbb{R}$, we obtain

$$\left\langle \mathcal{F}_\alpha^{\mathbf{m}} u_t, e^{\frac{it}{b^2}\|\cdot\|^2}\varphi\right\rangle = \left\langle \mathcal{F}_\alpha^{\mathbf{m}} u_t, e^{\frac{it}{b^2}\|\cdot\|^2}\varphi\right\rangle\Big|_{t=0} = \langle \mathcal{F}_\alpha^{\mathbf{m}} g, \varphi\rangle.$$

This holds true for all $\varphi \in \mathcal{S}_*(\mathbb{R}^n)$, hence $u = 0$. ☐

Remark 3.1. Let g be an element of $\mathcal{S}_*(\mathbb{R}^n)$. The solution of the system given by Theorem 3.1 belongs to $\mathcal{E}(\mathbb{R}; \mathcal{S}_*(\mathbb{R}^n))$, and it has the following form:

$$u(t,x) = \mathcal{F}_\alpha^{\mathbf{m}^{-1}} \left(e^{-\frac{it}{b^2}\|\cdot\|^2} \mathcal{F}_\alpha^{\mathbf{m}} g \right)(x).$$

Definition 3.1. We define the operator, for $t \in \mathbb{R}$:

$$e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} : \mathcal{S}_*(\mathbb{R}^n) \to \mathcal{S}_*(\mathbb{R}^n), \quad g \mapsto u(t,\cdot),$$

where u is the solution of the Schrödinger-linear canonical Bessel equation with the initial condition g. This operator is called the propagator.

Remark 3.2. We utilize the outcomes from earlier to derive the subsequent results.

(1) Building on the results provided in Examples 2.2 and the formulations outlined in equations (2.1) and (2.2), it can be deduced that

$$e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} g = \mathcal{F}_\alpha^{\mathbf{m}^{-1}} \left(e^{-\frac{it}{b^2}\|\cdot\|^2} \mathcal{F}_\alpha^{\mathbf{m}} g \right) \tag{3.1}$$

$$= \left(P_{it}^{\mathbf{m}^{-1}} \underset{\alpha,\mathbf{m}^{-1}}{*} g \right). \tag{3.2}$$

(2) We observe that

$$e^{-i(t+s)\Delta_\alpha^{\mathbf{m}^{-1}}} = e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} e^{-is\Delta_\alpha^{\mathbf{m}^{-1}}}, \quad \left(e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} \right)^* = e^{it\Delta_\alpha^{\mathbf{m}^{-1}}}, \quad \text{for all } t, s \in \mathbb{R}.$$

(3) Examining the outcomes presented in equations (2.3) and (3.2), we deduce the continuity of the propagator $e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}}$ from $L^1_\alpha(\mathbb{R}^n_+)$ to $L^\infty_\alpha(\mathbb{R}^n_+)$, along with the subsequent inequality:

$$\left\| e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} g \right\|_{L^\infty_\alpha} \leq \frac{C}{|t|^{|\alpha|+n}} \|g\|_{L^1_\alpha}. \tag{3.3}$$

(4) By combining the Plancherel formula and equation (3.1), we obtain

$$\left\| e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} g \right\|_{L^2_\alpha} = \|g\|_{L^2_\alpha} \quad \text{and} \quad \left\| e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} g \right\|_{H_s^{\mathbf{m},\alpha}} = \|g\|_{H_s^{\mathbf{m},\alpha}}, \tag{3.4}$$

where $s \in \mathbb{R}$

Proposition 3.1. *If $r \in [2, +\infty]$ and $t \neq 0$, then $e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}}$ maps $L^{r'}_\alpha \left(\mathbb{R}^n_+\right)$ continuously to $L^r_\alpha \left(\mathbb{R}^n_+\right)$ and*

$$\left\| e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} g \right\|_{L^r_\alpha} \leq \frac{C}{|t|^{\alpha(r)}} \|g\|_{\alpha,r'}, \tag{3.5}$$

where $\alpha(r) = (|\alpha| + n)(1 - \frac{2}{r})$ and $\frac{1}{r} + \frac{1}{r'} = 1$.

Proof. It follows from (3.3) and (3.4) that

$$\left\| e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} g \right\|_{L^\infty_\alpha} \leq \frac{C}{|t|^{|\alpha|+n}} \|g\|_{L^1_\alpha} \quad \text{and} \quad \left\| e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} g \right\|_{L^2_\alpha} = \|g\|_{L^2_\alpha}.$$

Applying the Riesz-Thorin theorem to obtain the inequality. □

Corollary 3.1. *If $r \in [2,+\infty]$, $s \in \mathbb{R}$ and $t \neq 0$, then there exists a constant C such that*

$$\left\| e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} g \right\|_{H^{\mathbf{m},\alpha}_{r,s}} \leq \frac{C}{|t|^{\alpha(r)}} \|g\|_{H^{\mathbf{m},\alpha}_{r',s}}.$$

Proof. Using if $t \neq 0$ we have

$$e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} \left[\mathcal{F}_\alpha^{\mathbf{m}^{-1}} \left(\left(1+\|\lambda\|^2\right)^s \mathcal{F}_\alpha^{\mathbf{m}}(g) \right) \right] = \mathcal{F}_\alpha^{\mathbf{m}^{-1}} \left[\left(\left(1+\|\lambda\|^2\right)^s \mathcal{F}_\alpha^{\mathbf{m}} \left(e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} g \right) \right) \right].$$

Now, according to (3.5), we have finished the proof. □

3.2. **Dispersion phenomena.** Here we strictly analyze generalized dispersion phenomena in the context of the Schrödinger equation using sophisticated mathematical methods like interpolation techniques. Our generalization is founded upon earlier research presented in [22]. For this we use the following notations.

Notations. For an interval I of $\mathbb{R}$ and a Banach space X, we adopt the following notations.

• $L^q(I;X)$ Banach space of (classes of) measurable functions $u : I \to X$ such that $\|u\|_{L^q(I;X)} < +\infty$, with

$$\|u\|_{L^q(I;X)} = \left(\int_I \|u(t,\cdot)\|_X^q dt \right)^{\frac{1}{q}}, \quad \text{if } 1 \leq q < +\infty,$$
$$\|u\|_{L^\infty(I;X)} = \operatorname*{ess\,sup}_{t\in I} \|u(t,\cdot)\|_X.$$

• $C(\bar{I};X)$ the space of continuous functions from $\bar{I}$ to X. Note that when I is bounded, then equipped with the norm $\|u\|_{L^\infty(I;X)}$, $C(\bar{I};X)$ is a Banach space.

• $C_c(I, \mathcal{S}(\mathbb{R}^n))$ is the space of continuous functions from I into $\mathcal{S}(\mathbb{R}^n)$ compactly supported in I, equipped with the topology of uniform convergence on the compact subintervals of I.

Definition 3.2. We say that the exponent pair (q,r) is $|\alpha|$-admissible if $q,r \geq 2$, $(q,r,|\alpha|+n) \neq (2,+\infty,1)$ and

$$\frac{2}{q} = \alpha(r) := (|\alpha|+n)\left(1 - \frac{2}{r}\right).$$

The $TT^\star$-criterion, attributed to Keel-Tao, is a well-known result in the field of harmonic analysis and partial differential equations. It is frequently employed to establish the well-posedness of specific dispersive or wave equations, one can prove the following result using some of the ideas and steps outlined in [1, 22].

Theorem 3.2 ($TT^\star$-criterion [22])**.** *Let $(U(t))_{t\in\mathbb{R}}$ be a bounded family of continuous operators of $L^2_\alpha(\mathbb{R}^n_+)$ on $L^2_\alpha(\mathbb{R}^n_+)$ such that*

$$\|U(t)U^*(t')\,f\|_{L^\infty_\alpha} \leq \frac{C}{|t-t'|^{|\alpha|+n}} \|f\|_{L^1_\alpha}.$$

Then, the estimates

$$\|U(t)g\|_{L^q\left(\mathbb{R};L^r_\alpha\left(\mathbb{R}^n_+\right)\right)} \leq C\,\|g\|_{L^2_\alpha} \tag{3.6}$$

and

$$\left\|\int_{\mathbb{R}} U^*(t)f(t,\cdot)dt\right\|_{L^2_\alpha} \leq C\|f\|_{L^{q'}\left(\mathbb{R};L^{r'}_\alpha\left(\mathbb{R}^n_+\right)\right)}, \tag{3.7}$$

hold for any $|\alpha|$-admissible exponent (q,r), where q',r' are the conjugate exponents of q and r and U^* is the adjoint operator of U.

Moreover, for any $|\alpha|$-admissible exponent pairs (q,r) and (q_1,r_1), we have

$$\left\|\int_{\mathbb{R}} U(t)U^*\left(t'\right)f\left(t',\cdot\right)dt'\right\|_{L^q\left(\mathbb{R};L^r_\alpha\left(\mathbb{R}^n_+\right)\right)} \leq C\|f\|_{L^{q_1'}\left(\mathbb{R};L^{r_1'}_\alpha\left(\mathbb{R}^n_+\right)\right)}. \tag{3.8}$$

Furthermore, if

$$\|U(s)U^*(t)f\|_{L^\infty_\alpha} \leq \frac{C}{(1+|t-s|)^{|\alpha|+n}}\|f\|_{L^1_\alpha},$$

then (3.6), (3.7) *and* (3.8) *hold for all $|\alpha|$-admissible (q,r) and (q_1,r_1).*

3.3. **Strichartz-type Schrödinger estimates associated with the linear canonical Bessel operator.** The Strichartz estimates that we will present in the following also involve the time variable and will be very important when studying the uniqueness of a solution of the nonlinear Schrödinger equation. We consider in the rest of this article the Schrödinger equation associated with the linear canonical Bessel operator given by:

$$\begin{cases} i\partial_t u(t,x)+\Delta_\alpha^{\mathbf{m}^{-1}}u(t,x)=F(t,x), \\ u(0,x)=g(x). \end{cases} \tag{3.9}$$

Moreover, under the same conditions, Duhamel's formula implies

$$u(t,x)=e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}}g(x)-i\int_0^t e^{-i(t-s)\Delta_\alpha^{\mathbf{m}^{-1}}}F(s,x)ds, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^n_+.$$

Theorem 3.3. *Suppose that $n+|\alpha|\geq\frac{1}{2}$, and let (q,r) and (q_1,r_1) be $|\alpha|$-admissible pairs. If u is a solution of the problem* (3.9) *with initial data g and source term F for $0<T<+\infty$, then the following estimate holds*

$$\|u\|_{L^q\left([0,T];L^r_\alpha\left(\mathbb{R}^n_+\right)\right)}+\|u\|_{C\left([0,T];L^2_\alpha\left(\mathbb{R}^n_+\right)\right)} \leq C\left(\|g\|_{L^2_\alpha}+\|F\|_{L^{q_1'}\left([0,T];L^{r_1'}_\alpha\left(\mathbb{R}^n_+\right)\right)}\right),$$

where C is a positive constant.

Conversely, if the above estimate holds for all g, F, and T, then the pairs (q,r) and (q_1,r_1) must be $|\alpha|$-admissible.

Proof. Let us begin by proving the sufficient condition. Assume that (q,r) satisfy the conditions of the theorem, and let u be a solution of (3.9). This follows from the equation (S) and Remark 3.1, according to the density of $\mathcal{S}_*(\mathbb{R}^n)$ in $L^p_\alpha(\mathbb{R}^n_+)$ for

$1 \leq p < +\infty$. Therefore, $e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}}$ satisfies the conditions of Theorem 3.2. Using the equation (3.6), we have

$$\|u\|_{L^q\left([0,T];L^r_\alpha\left(\mathbb{R}^n_+\right)\right)} \leq C\left(\|g\|_{L^2_\alpha} + \|F\|_{L^{q_1'}\left([0,T];L^{r_1'}_\alpha\left(\mathbb{R}^n_+\right)\right)}\right),$$

where r_1' and q_1' are successive conjugates of r_1 and q_1.

Let Φ_α be the operator defined as

$$\Phi_\alpha(F)(t,x) := \int_0^t e^{-i(t-s)\Delta_\alpha^{\mathbf{m}^{-1}}} F(s,x)\,ds.$$

We now turn our attention to the question of continuity in L^2_α. The continuity of $e^{-i(\cdot)\Delta_\alpha^{\mathbf{m}^{-1}}} g$ is established through the Plancherel formula, yielding the inequality:

$$\left\|e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} g\right\|_{C\left([0,T];L^2_\alpha\left(\mathbb{R}^n_+\right)\right)} \leq \|g\|_{L^2_\alpha}.$$

Furthermore, for every $|\alpha|$-admissible pair (q_1, r_1), we obtain

$$\begin{aligned}
\|\Phi_\alpha F(t)\|^2_{L^2_\alpha} &= \left\langle \int_0^t e^{-i(t-s)\Delta_\alpha^{\mathbf{m}^{-1}}} F(s)ds, \int_0^t e^{-i(t-s)\Delta_\alpha^{\mathbf{m}^{-1}}} F(\sigma)d\sigma \right\rangle_\alpha \\
&= \int_0^t \int_0^t \left\langle e^{-i(t-s)\Delta_\alpha^{\mathbf{m}^{-1}}} F(s), e^{-i(t-\sigma)\Delta_\alpha^{\mathbf{m}^{-1}}} F(\sigma)\right\rangle_\alpha d\sigma ds \\
&\leq \int_0^t \int_{\mathbb{R}} \left\langle F(s), e^{-i(s-\sigma)\Delta_\alpha^{\mathbf{m}^{-1}}} F(\sigma)\right\rangle_\alpha d\sigma ds \\
&= \int_0^t \left\langle F(s), \int_{\mathbb{R}} e^{-i(s-\sigma)\Delta_\alpha^{\mathbf{m}^{-1}}} F(\sigma)d\sigma\right\rangle_\alpha ds,
\end{aligned}$$

we leverage the intrinsic property $\left(e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}}\right)^\star = e^{it\Delta_\alpha^{\mathbf{m}^{-1}}}$. By employing Hölder's inequality sequentially in both spatial and temporal dimensions, and invoking the pertinent equation (3.7), we derive the following significant result:

$$\begin{aligned}
\|\Phi_\alpha F(t)\|^2_{L^2_\alpha} &\leq \|F\|_{L^{q_1'}\left([0,T];L^{r_1'}_\alpha\left(\mathbb{R}^n_+\right)\right)} \left\| \int_{\mathbb{R}} e^{-i(\cdot-\sigma)\Delta_\alpha^{\mathbf{m}^{-1}}} F(\sigma)d\sigma \right\|_{L^{q_1}\left([0,T];L^{r_1}_\alpha\left(\mathbb{R}^n_+\right)\right)} \\
&\leq C\|F\|^2_{L^{q_1'}\left([0,T];L^{r_1'}_\alpha\left(\mathbb{R}^n_+\right)\right)}.
\end{aligned}$$

To establish the continuity of the quantity $\Phi_\alpha(F)$ in the space $L^2_\alpha(\mathbb{R}^n_+)$, we leverage the following identity:

$$\Phi_\alpha(F)(t+\varepsilon) = e^{-i\varepsilon\Delta_\alpha^{\mathbf{m}^{-1}}}\left[\Phi_\alpha(F)(t) + \Phi_\alpha\left(1_{[t,t+\varepsilon]}F\right)(t)\right],$$

where $e^{-i\varepsilon\Delta_\alpha^{\mathbf{m}^{-1}}}$ is an operator acting on $L^2_\alpha(\mathbb{R}^n_+)$. The continuity of $e^{-i\varepsilon\Delta_\alpha^{\mathbf{m}^{-1}}}$ as an operator on this space plays a crucial role in establishing the desired continuity of $\Phi_\alpha(F)$.

Moreover, we exploit the property that

$$\left\|1_{[t,t+\varepsilon]}F\right\|_{L^{q_1'}([0,T];L_\alpha^{r_1'}(\mathbb{R}_+^n))} \to 0, \quad \text{as } \varepsilon \to 0.$$

This convergence result ensures that the term involving $1_{[t,t+\varepsilon]}F$ becomes negligible as ε approaches zero. Consequently, it allows us to conclude that $\Phi_\alpha(F)$ is continuous in $L_\alpha^2(\mathbb{R}_+^n)$.

In light of these considerations, we successfully complete the proof of the continuity of $\Phi_\alpha(F)$ in the specified function space. With this, the culmination of our analysis concludes the proof of the presented theorem. ☐

In direct continuity with our preceding theorem, we present the consequential Corollary 3.2. This corollary seamlessly applies and extends the theoretical constructs established earlier, providing insightful results within the context of intervals in $\mathbb{R}$. By leveraging $|\alpha|$-admissible pairs, Corollary 3.2 establishes the existence of a constant C, offering concise bounds for the operator $e^{-i(\cdot)\Delta_\alpha^{\mathbf{m}^{-1}}}$ and the quantity $\Phi_\alpha(F)$ in specific function spaces.

Corollary 3.2. *Let I be an interval of $\mathbb{R}$. If (q,r) and (q_1,r_1) are $|\alpha|$-admissible pairs, then there exists a constant C such that*

$$\left\|e^{-i(\cdot)\Delta_\alpha^{\mathbf{m}^{-1}}}g\right\|_{L^q\left(\mathbb{R};L_\alpha^r\left(\mathbb{R}_+^n\right)\right)} \leq C\|g\|_{L_\alpha^2} \tag{3.10}$$

and

$$\|\Phi_\alpha(F)\|_{L^q\left(I;L_\alpha^r\left(\mathbb{R}_+^n\right)\right)} \leq C\|F\|_{L^{q_1'}\left(I;L_\alpha^{r_1'}\left(\mathbb{R}_+^n\right)\right)}.$$

The applicability of the estimate established in Corollary 3.2 extends systematically to Sobolev spaces associated with the linear canonical Bessel operator.

Corollary 3.3. *Given any $s\in\mathbb{R}$ and $k\in\mathbb{N}$, the following properties hold.*

(1) *If (q,r) is an $|\alpha|$-admissible pair, then there exists a constant C such that*

$$\left\|e^{-i(\cdot)\Delta_\alpha^{\mathbf{m}^{-1}}}g\right\|_{L^q\left(\mathbb{R};W_{r,k}^{\mathbf{m},\alpha}\right)} \leq C\|g\|_{H_k^{\mathbf{m},\alpha}}, \quad \textit{for all } g\in H_k^{\mathbf{m},\alpha}.$$

(2) *If (q,r) is an $|\alpha|$-admissible pair, then there exists a constant C such that*

$$\left\|e^{-i(\cdot)\Delta_\alpha^{\mathbf{m}^{-1}}}g\right\|_{L^q\left(\mathbb{R};H_{r,s}^{\mathbf{m},\alpha}\right)} \leq C\|g\|_{H_s^{\mathbf{m},\alpha}}, \quad \textit{for all } g\in H_s^{\mathbf{m},\alpha}.$$

(3) *Let I be an interval of $\mathbb{R}$. If (q,r) and (q_1,r_1) are $|\alpha|$-admissible pairs, then there exists a constant C such that*

$$\|\Phi_\alpha(F)\|_{L^q\left(I;W_{r,k}^{\mathbf{m},\alpha}\right)} \leq C\|F\|_{L^{q_1'}\left(I,W_{r_1',k}^{\mathbf{m},\alpha}\right)}, \quad \textit{for all } F\in L^{q_1'}\left(I,W_{r_1',k}^{\mathbf{m},\alpha}\right).$$

(4) *Let I be an interval of $\mathbb{R}$. If (q,r) and (q_1,r_1) are $|\alpha|$-admissible pairs, then there exists a constant C such that*

$$\|\Phi_\alpha(F)\|_{L^q\left(I;H^{\mathbf{m},\alpha}_{r,s}\right)} \leq C\|F\|_{L^{q_1'}\left(I,H^{\mathbf{m},\alpha}_{r_1',s}\right)}, \quad \text{for all } F \in L^{q_1'}\left(I, H^{\mathbf{m},\alpha}_{r_1',s}\right).$$

Proof. For brevity, we focus our attention on establishing the validity of estimates (1) and (2), with the proofs of (3) and (4) mirroring a similar logical structure.

In (1) and (2), the substitution of g by $(\Delta^{\mathbf{m}^{-1}}_\alpha)^j g$, where $j \leq k$ and g by $\mathcal{F}^{\mathbf{m}^{-1}}_\alpha\left((1+\|\lambda\|^2)^s \mathcal{F}^{\mathbf{m}}_\alpha(g)\right)$ in (3.10). The two transformations are based on the two relationships defined by

$$\Delta^{\mathbf{m}^{-1}}_\alpha e^{-it\Delta^{\mathbf{m}^{-1}}_\alpha} g = e^{-it\Delta^{\mathbf{m}^{-1}}_\alpha} \Delta^{\mathbf{m}^{-1}}_\alpha g$$

and

$$e^{-i(\cdot)\Delta^{\mathbf{m}^{-1}}_\alpha}\left[\mathcal{F}^{\mathbf{m}^{-1}}_\alpha\left(\left(1+\|\lambda\|^2\right)^s \mathcal{F}^{\mathbf{m}}_\alpha(g)\right)\right] = \mathcal{F}^{\mathbf{m}^{-1}}_\alpha\left[\left(\left(1+\|\lambda\|^2\right)^s \mathcal{F}^{\mathbf{m}}_\alpha\left(e^{-i(\cdot)\Delta^{\mathbf{m}^{-1}}_\alpha} g\right)\right)\right].$$

It is imperative to underscore the utilization of Proposition 2.7 to conclude the proof outlined in (1), along with the application of Plancherel's theorem in (2). Through these methodological steps, the desired result is successfully derived. □

Remark 3.3. Leveraging the insights provided by Corollary 3.3, the estimation derived in Theorem 3.3 can be extended to Sobolev spaces associated with the linear canonical Bessel operator. The ensuing outcome is expressed as follows: Let (q,r) and (q_1,r_1) be $|\alpha|$-admissible pairs. If u is a solution of the problem (3.9) with initial data g and source term F for $0 < T < +\infty$, then the following estimate holds:

$$\|u\|_{L^q\left([0,T];H^{\mathbf{m},\alpha}_{r,s}\right)} + \|u\|_{C\left([0,T];H^{\mathbf{m},\alpha}_{s}\right)} \leq C\left(\|g\|_{H^{\mathbf{m},\alpha}_s} + \|F\|_{L^{q_1'}\left([0,T];H^{\mathbf{m},\alpha}_{r_1',s}\right)}\right),$$

where C is a positive constant.

Conversely, if the above estimate holds for all g, F, and T, then the pairs (q,r) and (q_1,r_1) must be $|\alpha|$-admissible.

4. Solutions for Nonlinear Schrödinger Equations Associated with the Canonical Bessel Operator

On the basis of the results obtained in Remarks 3.3 we define, for $s \geq 0$ and I an interval, the space

$$X^{\mathbf{m},\alpha}_s(I) = \left\{u : \|u\|_{L^q\left(I;H^{\mathbf{m},\alpha}_{r,s}\right)} + \|u\|_{C\left(I;H^{\mathbf{m},\alpha}_{s}\right)} < +\infty \text{ for } 0 \leq \frac{2}{q} = \alpha(r) < 1\right\}. \tag{4.1}$$

The observation in Remarks 3.3 implies that

$$g \in H^{\mathbf{m},\alpha}_s \Rightarrow e^{-i(\cdot)\Delta^{\mathbf{m}^{-1}}_\alpha} g \in X^{\mathbf{m},\alpha}_s(\mathbb{R}).$$

If I is not compact, we define $X^{\mathbf{m},\alpha}_{s,\mathrm{loc}}(I)$ by replacing L^q with L^q_{loc} in (4.1).

Remark 4.1. The spaces $X_s^{\mathbf{m},\alpha}(I)$ are not naturally Banach spaces because the range of r is semi-open (for $|\alpha|+n\geq\frac{1}{2}$). To obtain Banach spaces, we restrict this range by considering $0\leq\alpha(r)\leq\alpha(r_0)<1$. We define, with $\frac{2}{q_0}=\alpha(r_0)$, the spaces

$$\begin{aligned}X_{s,r_0}^{\mathbf{m},\alpha}(I)&=\left\{u:\|u\|_{L^q\left(I;H_{r,s}^{\mathbf{m},\alpha}\right)}+\|u\|_{C\left(I;H_s^{\mathbf{m},\alpha}\right)}<+\infty\text{ for }0\leq\frac{2}{q}=\alpha(r)\leq\alpha(r_0)<1\right\}\\&=C\left(I;H_s^{\mathbf{m},\alpha}\right)\cap L^{q_0}\left(I;H_{r_0,s}^{\mathbf{m},\alpha}\right),\end{aligned}$$

which are Banach spaces with the natural norms. The same modification is made for the local spaces in time.

Theorem 4.1. *Let $\beta\in\mathbb{R}$ and $\lambda\in]0,+\infty[$. Define the transformation*

$$u_{\lambda,\mathbf{m}}(x,t)=\lambda^{\beta}e^{-i\frac{a}{2b}\left(1-\lambda^2\right)\|x\|^2}u\left(\lambda x,\lambda^2t\right). \tag{4.2}$$

(1) *For $\beta=\frac{2}{p}$, the equation (S_p) is invariant under the transformation.*
(2) *For $\beta=|\alpha|+n-2s$, the quantity*

$$\left\|\|x\|^{2s}\mathcal{F}_\alpha^{\mathrm{m}}(g)\right\|_{L_\alpha^2} \tag{4.3}$$

is invariant under this transformation of the initial condition of the equation (S_p).

Proof. (1) Let u be a solution of (S_p), we have

$$\left(i\partial_t+\Delta_\alpha^{\mathbf{m}^{-1}}\right)u_{\lambda,\mathbf{m}}(x,t)=\lambda^{2-\beta p}\left|u_{\lambda,\mathbf{m}}(x,t)\right|^p u_{\lambda,\mathbf{m}}(x,t).$$

Since the equation (S_p) is invariant under the transformation (4.2), we obtain the result.

(2) Let $s\geq 0$ and g the initial condition of the equation (S_p). We start with $g\in\mathcal{S}_*(\mathbb{R}^n)$ therefore $g_{\lambda,\mathbf{m}}\in\mathcal{S}_*(\mathbb{R}^n)$ and

$$\left\|\|x\|^{2s}\mathcal{F}_\alpha^{\mathrm{m}}(g_{\lambda,\mathbf{m}})\right\|_{L_\alpha^2}^2=\int_{\mathbb{R}_+^n}\|x\|^{4s}\left|\mathcal{F}_\alpha^{\mathbf{m}}\left(g_{\lambda,\mathbf{m}}\right)(x)\right|^2d\mu_{\alpha,n}(x).$$

We use equation (2.1), we have

$$\begin{aligned}\left\|\|x\|^{2s}\mathcal{F}_\alpha^{\mathrm{m}}(g_{\lambda,\mathbf{m}})\right\|_{L_\alpha^2}^2&=\lambda^{-4(|\alpha|+n)+2\beta}\int_{\mathbb{R}_+^n}\|x\|^{4s}\left|\mathcal{F}_\alpha^{\mathbf{m}}(g)\left(\frac{x}{\lambda}\right)\right|^2d\mu_{\alpha,n}(x)\\&=\lambda^{-2(|\alpha|+n)+2\beta+4s}\int_{\mathbb{R}_+^n}\|x\|^{4s}\left|\mathcal{F}_\alpha^{\mathbf{m}}(g)(x)\right|^2d\mu_{\alpha,n}(x)\\&=\lambda^{-2(|\alpha|+n)+2\beta+4s}\left\|\|x\|^{2s}\mathcal{F}_\alpha^{\mathrm{m}}(g)\right\|_{L_\alpha^2}^2.\end{aligned}$$

Since the quantity (4.3) is invariant under the transformation (4.2) of the initial condition of the equation (S_p), we obtain $\beta=|\alpha|+n-2s$. Now according to the density of $\mathcal{S}_*(\mathbb{R}^n)$ in $H_s^{\mathbf{m},\alpha}$, we find the same result for $g\in H_s^{\mathbf{m},\alpha}$. □

The situation where the two values of β in Theorem 4.1 considered are equal is important enough to call for a definition establishes different categories for the criticality of the power p at the level of $H_s^{\mathbf{m},\alpha}$.

Definition 4.1. Let $s \in \left[0, \frac{n+|\alpha|}{2}\right)$ and $p \in [0, +\infty[$, the power p is termed:
- critical if $p = \frac{2}{n+|\alpha|-2s}$,
- subcritical if $p < \frac{2}{n+|\alpha|-2s}$.

4.1. **L^2_α-Local well-posedness for nonlinear Schrödinger equations associated with the linear canonical Bessel operator.** Our result in this subsection concerns the local well-posedness of (S_p) in $L^2_\alpha\left(\mathbb{R}^n_+\right)$ in both subcritical and critical cases, i.e., $s = 0$ and $p \in \left(0, \frac{2}{|\alpha|+n}\right]$.

Theorem 4.2. *If $p \in \left(0, \frac{2}{|\alpha|+n}\right]$, then for every $g \in L^2_\alpha\left(\mathbb{R}^n_+\right)$, there exist $T^*, T_* \in (0, +\infty]$ and a unique maximal solution u of (S_p) belonging to*

$$C\left((-T_*, T^*)\,; L^2_\alpha\left(\mathbb{R}^n_+\right)\right) \cap L^q_{loc}\left((-T_*, T^*)\,; L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right),$$

for every $|\alpha|$-admissible pair $(q, p+2)$.

Proof. We proceed in two steps.

Step 1: local existence. For the existence, we use a fixed point argument.

• If $p \in \left(0, \frac{2}{|\alpha|+n}\right)$, fix $T > 0, M > \|g\|_{\alpha,2}$ and set

$$X_{M,T} := \Big\{u \in L^q\left((-T,T); L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right) \cap C\left((-T,T); L^2_\alpha\left(\mathbb{R}^n_+\right)\right)$$
$$: \|u\|_{L^\infty\left((-T,T);L^2_\alpha\left(\mathbb{R}^n_+\right)\right)} + \|u\|_{L^q\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)} \le M\Big\},$$

where $(q, p+2)$ is $|\alpha|$-admissible pair. Note that by Theorem 3.3 and Corollary 3.2, this space is empty. Moreover, we can verify that $X_{M,T}$ is a complete metric space when it is equipped with the distance:

$$d(u,v) = \|u-v\|_{L^\infty\left((-T,T);L^2_\alpha\left(\mathbb{R}^n_+\right)\right)} + \|u-v\|_{L^q\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)}.$$

To simplify, put

$$\|v\|_{X_{M,T}} = \|v\|_{L^\infty\left((-T,T);L^2_\alpha\left(\mathbb{R}^n_+\right)\right)} + \|v\|_{L^q\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)},$$

if $v \in X_{M,T}$. Take $g \in L^2_\alpha\left(\mathbb{R}^n_+\right)$. We wish find some conditions on T and M such that the operator $\mathcal{H}_\alpha$, given by

$$\mathcal{H}_\alpha(u)(t,\cdot) := e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} g(\cdot) - i\int_0^t e^{-i(t-s)\Delta_\alpha^{\mathbf{m}^{-1}}} F(u(s,\cdot))ds$$

is a strict contraction on $X_{M,T}$.

By the nonlinearity assumption (1.1) and Theorem 3.3, it follows that

$$\|\mathcal{H}_\alpha(u)\|_{X_{M,T}} \le C\left(\left\|e^{-i(\cdot)\Delta_\alpha^{\mathbf{m}^{-1}}} g\right\|_{X_{M,T}} + \|u\|^{p+1}_{L^{(p+1)q_1'}\left((-T,T);L_\alpha^{(p+1)r_1'}\left(\mathbb{R}^n_+\right)\right)}\right),$$

with (q_1, r_1) is a $|\alpha|$-admissible pair.

On the other hand, we use the condition $p \in \left(0, \frac{2}{|\alpha|+n}\right)$, we take $r_1 = p+2$ and $(q_1 = q, p+2)$ a $|\alpha|$ admissible pair such that $q > p+2$. Then, applying Corollary 3.2 and Hölder's inequality in time, we obtain

$$\|\mathcal{H}_\alpha(u)\|_{X_{M,T}} \le C\|g\|_{L^2_\alpha} + CT^{\frac{q-p-2}{q}} \|u\|^{p+1}_{L^q\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)}. \tag{4.4}$$

Hence, for every $u \in X_{M,T}$ one has

$$\|\mathcal{H}_\alpha(u)\|_{X_{M,T}} \le C\|g\|_{L^2_\alpha} + CT^{\frac{q-p-2}{q}} M^{p+1}.$$

Choosing $M = 2C\|g\|_{L^2_\alpha}$, we see that if T is sufficiently small (depending on $\|g\|_{L^2_\alpha}$), then $\mathcal{H}_\alpha(u) \in X_{M,T}$ for all $u \in X_{M,T}$. Moreover, arguing as above, we obtain

$$d\left(\mathcal{H}_\alpha(u), \mathcal{H}_\alpha(v)\right) \le CT^{\frac{q-p-2}{q}} M^p d(u,v),$$

for all $u, v \in X_{M,T}$. Thus, $\mathcal{H}_\alpha$ is a contraction in $X_{M,T}$ provided T is small enough, more precisely if $T < \left(\frac{1}{2CM^p}\right)^{\frac{q}{q-p-2}}$. Hence, $\mathcal{H}_\alpha$ has a fixed point u, which is the unique solution of (S_p) in $X_{M,T}$, and there exist $T^*, T_* \in (0, +\infty]$ such that u belongs to

$$C\left(\left(-T_*, T^*\right); L^2_\alpha\left(\mathbb{R}^n_+\right)\right) \cap L^q_{loc}\left(\left(-T_*, T^*\right); L^r_\alpha\left(\mathbb{R}^n_+\right)\right),$$

for the $|\alpha|$-admissible pair $(q, p+2)$, with

$$T^* = \sup\{T > 0, \text{ there exists a solution of } (S_p) \text{ on [0, T]}\},$$

$$T_* = \sup\{T > 0, \text{ there exists a solution of } (S_p) \text{ on [-T, 0]}\}.$$

Moreover, from Theorem 3.3 and by the argument we use to prove (4.4), we have

$$u \in L^{q_1}_{loc}\left(\left(-T_*, T^*\right); L^{r_1}_\alpha\left(\mathbb{R}^n_+\right)\right),$$

for every $|\alpha|$-admissible pair (q_1, r_1).

- If $p = \frac{2}{|\alpha|+n}$, let $g \in L^2_\alpha\left(\mathbb{R}^n_+\right)$. Since $e^{-i(\cdot)\Delta^{\mathbf{m}^{-1}}_\alpha} g \in L^{p+2}\left(\mathbb{R}; L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)$, by Corollary 3.2, we have

$$\left\|e^{-i(\cdot)\Delta^{\mathbf{m}^{-1}}_\alpha} g\right\|_{L^{p+2}\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)} \to 0, \quad \text{as } T \downarrow 0. \tag{4.5}$$

Therefore for $\epsilon > 0$, there is $T > 0$, such that

$$\left\|e^{-i(\cdot)\Delta^{\mathbf{m}^{-1}}_\alpha} g\right\|_{L^{p+2}\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)} < \epsilon. \tag{4.6}$$

Let us consider the set

$$Y_\epsilon := \left\{u \in L^{p+2}\left((-T,T); L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right) : \|u\|_{L^{p+2}\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)} \le 2\epsilon\right\}.$$

We can verify that Y_ϵ is a complete metric space when it is equipped with the distance

$$d(u,v) = \|u - v\|_{L^{p+2}\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)}.$$

As above, by Theorem 3.3, the following estimate holds

$$\begin{aligned}&\|\mathcal{H}_\alpha(u)\|_{L^{p+2}\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)}\\ \leq& C\left(\left\|e^{-i(\cdot)\Delta_\alpha^{\mathbf{m}^{-1}}}g\right\|_{L^{p+2}\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)}+\|u\|^{p+1}_{L^{(p+2)}\left((-T,T);L^{(p+2)}_\alpha\left(\mathbb{R}^n_+\right)\right)}\right).\end{aligned}$$

Hence, for every $u\in Y_\epsilon$,

$$\|\mathcal{H}_\alpha(u)\|_{L^{p+2}\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)}\leq C\left\|e^{-i(\cdot)\Delta_\alpha^{\mathbf{m}^{-1}}}g\right\|_{L^{p+2}\left((-T,T);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)}+C\epsilon^{p+1}.$$

From relations (4.5) and (4.6), we see that if T is small enough, then we can choose M such that $\mathcal{H}_\alpha(u)$ belongs to Y_ϵ for all $u\in Y_\epsilon$. As above we prove also that $\mathcal{H}_\alpha$ is a contraction on the space Y_ϵ provided T is sufficiently small. Thus $\mathcal{H}_\alpha$ has a fixed point u, which is the unique solution of (S_p) in Y_ϵ. Moreover, from Theorem 3.3, there exists $T^*,T_*\in(0,+\infty]$ such that

$$u\in C\left((-T_*,T^*);L^2_\alpha\left(\mathbb{R}^n_+\right)\right)\cap L^{q_1}_{loc}\left((-T_*,T^*);L^{r_1}_\alpha\left(\mathbb{R}^n_+\right)\right),$$

for every $|\alpha|$-admissible pair (q_1,r_1).

Step 2: uniqueness. Recognizing the intrinsically local nature of uniqueness is the first step in our study. As a result, our task reduces to establishing uniqueness on possibly small time intervals. This distinction is important because it enables us to focus our analysis on specific temporal regions.

To illustrate this, we consider positive times, noting that the treatment of negative times mirrors the same rationale. Let

$$u_1,u_2\in C\left([0,T];L^2_\alpha\left(\mathbb{R}^n_+\right)\right),$$

be two solutions of the equation (S_p). We proceed under the assumption that $u_1(t)\neq u_2(t)$ holds for some $t\in[0,T]$. Let

$$t_0=\inf\left\{t\in[0,T],u_1(t)\neq u_2(t)\right\}.$$

Since both u_1 and u_2 are continuous into $L^2_\alpha\left(\mathbb{R}^n_+\right)$, this definition makes sense and $u_1(t_0)=u_2(t_0)=\chi$. Moreover, the curves $U_1(t)=u_1(t+t_0)$ and $U_2(t)=u_2(t+t_0)$ both satisfy the equation $w=e^{-i(\cdot)\Delta_\alpha^{\mathbf{m}^{-1}}}\chi+\Phi_\alpha(F(w))$ on $[0,T-t_0]$. As above we apply Theorem 3.3 and the argument of proof of (4.4), to obtain that for all $t\in[t_0,T]$

$$\begin{aligned}&\|u_1-u_2\|_{L^q\left((t_0,t);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)}\\ \leq& C\left(t-t_0\right)^{\frac{2-(|\alpha|+n)p}{2}}\sum_{i=1}^{2}\|u_i\|^p_{L^q\left((t_0,t);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)}\|u_1-u_2\|_{L^q\left((t_0,t);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)},\end{aligned}$$

where $\left(q = \frac{2(p+2)}{p(|\alpha|+n)}, p+2\right)$ is a $|\alpha|$-admissible pair. For $t > t_0$, but sufficiently close to t_0, it follows that

$$C\,(t-t_0)^{\frac{2-(|\alpha|+n)p}{2}} \sum_{i=1}^{2} \|u_i\|^p_{L^q\left((t_0,t);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)} < 1,$$

and so that

$$\|u_1 - u_2\|_{L^q\left((t_0,t);L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)} = 0.$$

This contradicts the choice of t_0, and thus proves that $u_1(t) = u_2(t)$ for all $t \in [0,T]$. □

Proposition 4.1. *Assume that F is as in Theorem 4.2, $g \in L^2_\alpha\left(\mathbb{R}^n_+\right)$ and u is the maximal solution of (S_p), then we have the following.*

i) *If $p \in \left(0, \frac{2}{|\alpha|+n}\right)$ and $T^* < +\infty$ (respectively, $T_* < +\infty$), then*

$$\|u(t)\|_{L^2_\alpha} \to +\infty, \quad \text{as } t \uparrow T^* (\text{respectively, as } t \downarrow -T_*).$$

ii) *If $p = \frac{2}{|\alpha|+n}$ and $T^* < +\infty$ (respectively, $T_* < +\infty$), then*

$$\|u\|_{L^q\left((0,T^*);L^r_\alpha\left(\mathbb{R}^n_+\right)\right)} = +\infty \qquad \left(\text{respectively, } \|u\|_{L^q\left((-T_*,0);L^r_\alpha\left(\mathbb{R}^n_+\right)\right)} = +\infty\right),$$

for every $|\alpha|$-admissible pair (q,r), with $r \geq p+2$.

Proof. i) If $p \in \left(0, \frac{2}{|\alpha|+n}\right)$, it follows from Step 1 of the proof of Theorem 4.2 and the uniqueness property that

$$T^* - t \geq \left(\frac{1}{4C^2\|u(t,\cdot)\|^p_{L^2_\alpha}}\right)^{\frac{q}{q-p-2}}.$$

Suppose now that $T^* < +\infty$, then

$$\|u(t)\|_{L^2_\alpha} \geq \left(\frac{1}{4C^2\,(T^*-t)^{\frac{q-p-2}{q}}}\right)^{\frac{1}{p}}.$$

As $q > p+2$, we obtain $\|u(t)\|_{L^2_\alpha} \to \infty$ as $t \uparrow T^*$. One shows by the same argument that if $T_* < +\infty$, then $\|u(t)\|_{L^2_\alpha} \to +\infty$ as $t \downarrow -T_*$.

ii) If $p = \frac{2}{|\alpha|+n}$, we demonstrate the explosion alternative by contradiction. Suppose that $T^* < +\infty$ and $\|u\|_{L^{(p+2)}\left((0,T^*);L^{(p+2)}_\alpha\left(\mathbb{R}^n_+\right)\right)} < +\infty$. Let $0 \leq t \leq t+\tau < T^*$. It follows that

$$e^{-i\tau\Delta_\alpha^{\mathbf{m}-1}} u(t,\cdot) = u(t+\tau,\cdot) - i\int_0^\tau e^{-i(\tau-s)\Delta_\alpha^{\mathbf{m}-1}} F(u(t+s,\cdot))\,ds.$$

By Theorem 3.3, we deduce that there exists $C \geq 0$ such that

$$\left\| e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} u(t) \right\|_{L^{(p+2)}\left((0,T^*-t);L_\alpha^{(p+2)}\left(\mathbb{R}_+^n\right)\right)} \leq \|u\|_{L^{(p+2)}\left((t,T^*);L_\alpha^{(p+2)}\left(\mathbb{R}_+^n\right)\right)} + C\|u\|^{p+1}_{L^{(p+2)}\left((t,T^*);L_\alpha^{(p+2)}\left(\mathbb{R}_+^n\right)\right)}.$$

Consequently,

$$\lim_{t\uparrow T^*} \left\| e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} u(t) \right\|_{L^{(p+2)}\left((0,T^*-t);L_\alpha^{(p+2)}\left(\mathbb{R}_+^n\right)\right)} = 0. \tag{4.7}$$

On the other hand, the dual homogeneous Strichartz estimate (3.7) implies that there exists a positive constant $M(u)$ such that

$$\|u(t)\|_{L^2_\alpha} \leq M(u), \quad \text{for all } t \in [0, T^*[. \tag{4.8}$$

By combining the inequalities (4.7)–(4.8) with the dual homogeneous Strichartz estimate (3.7) and Theorem 3.3, we obtain

$$\lim_{t_1,t_2\uparrow T^*} \|u(t_1) - u(t_2)\|_{L^2_\alpha} = 0.$$

As L^2_α is complete, there exists an element u^* in L^2_α such that

$$\lim_{t\uparrow T^*} \|u(t) - u^*\|_{L^2_\alpha} = 0.$$

We consider the following system

$$(S'_p) \qquad \begin{cases} i\partial_t v(t,x) + \Delta_\alpha^{\mathbf{m}^{-1}} v(t,x) = F(v(t,x)), & (t,x) \in I \times \mathbb{R}_+^n, \\ v_{|t=0} = u^* \in L^2_\alpha\left(\mathbb{R}_+^n\right). \end{cases}$$

According to Theorem 4.2, there exist $S_*, S^* \in]0, +\infty]$ and a unique solution v on $]-S_*, S^*[$ of the system (S'_p). We define

$$U(t) = \begin{cases} u(t), & \text{if } t \in]-T_*, T^*[, \\ v(t - T^*), & \text{if } t \in [T^*, T^* + S^*[. \end{cases}$$

It is clear that U is a solution of (S'_p) that extends u over the interval $]-T_*, T^*+S^*[$. This contradicts the fact that T^* is a maximal time for a solution of (S_p), hence the absurdity. This shows that $\|u\|_{L^{(p+2)}\left((0,T^*);L_\alpha^{(p+2)}\left(\mathbb{R}_+^n\right)\right)} = +\infty$.

Let (q,r) be a $|\alpha|$-admissible pair with $r > p+2$. It follows from Hölder's inequality that for all $T < T^*$,

$$\|u\|_{L^{(p+2)}\left((0,T);L_\alpha^{(p+2)}\left(\mathbb{R}_+^n\right)\right)} \leq \|u\|^\lambda_{L^\infty\left((0,T);L^2_\alpha\left(\mathbb{R}_+^n\right)\right)} \|u\|^{1-\lambda}_{L^q\left((0,T);L^r_\alpha\left(\mathbb{R}_+^n\right)\right)},$$

where $\lambda = \frac{2(r-p-2)}{(p+2)(r-2)}$. Using the dual homogeneous Strichartz estimate (3.7) and Theorem 3.3, we can estimate $\|u\|^\lambda_{L^\infty\left((0,T);L^2_\alpha\left(\mathbb{R}_+^n\right)\right)}$. This allows us to deduce that

$$\|u\|_{L^q\left((0,T^*);L^r_\alpha\left(\mathbb{R}_+^n\right)\right)} = +\infty.$$

By the same argument, it can be shown that if $T_* < +\infty$, then

$$\|u\|_{L^q\left((-T_*,0);L^r_\alpha\left(\mathbb{R}^n_+\right)\right)} = +\infty.$$

□

4.2. L^2_α**-Global well-posedness for nonlinear Schrödinger equations associated with the linear canonical Bessel operator.** Our result in this subsection concerns the global well-posedness of (S_p) in $L^2_\alpha\left(\mathbb{R}^n_+\right)$ in both subcritical and critical cases, i.e., $p = \frac{2}{|\alpha|+n}$. In the proof of local existence with $p = \frac{2}{|\alpha|+n}$, we used T is small enough, we can change this condition by $\|g\|_{L^2_\alpha}$ small enough. This is what we are going to prove in the following corollary.

Corollary 4.1. *If $p = \frac{2}{|\alpha|+n}$, there exists $\delta > 0$ such that if $g \in L^2_\alpha\left(\mathbb{R}^n_+\right)$ satisfies $\|g\|_{L^2_\alpha} \leq \delta$, (i.e., $\|g\|_{L^2_\alpha}$ small enough), for $T \in (0, +\infty[$ there exist a unique solution u of (S_p) belonging to*

$$C\left((-T,T)\,; L^2_\alpha\left(\mathbb{R}^n_+\right)\right) \cap L^q\left((-T,T)\,; L^r_\alpha\left(\mathbb{R}^n_+\right)\right),$$

for every $|\alpha|$-admissible pair (q, r).

Proof. For $T > 0$, by Theorem 3.3, the following Strichartz estimate holds

$$\|\mathcal{H}_\alpha(u)\|_{X_T} \leq C\left(\|g\|_{L^2_\alpha} + \|F(u)\|_{L^{q'}\left([-T,T];L^{r'}_\alpha\left(\mathbb{R}^n_+\right)\right)}\right)$$

with $(q = p+2,\ r = p+2)$ a $|\alpha|$-admissible and

$$X_T = C\left([-T,T]; L^2_\alpha\left(\mathbb{R}^n_+\right)\right) \cap L^{p+2}\left([-T,T]; L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right),$$

the Banach space with norm

$$\|v\|_{X_T} = \|v\|_{L^{p+2}\left([-T,T];L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)} + \|v\|_{C\left([-T,T];L^2_\alpha\left(\mathbb{R}^n_+\right)\right)}.$$

By our nonlinearity assumption (1.1) and Hölder inequality, we have

$$\|\mathcal{H}_\alpha(u)\|_{X_T} \leq C\left(\|g\|_{L^2_\alpha} + \|u\|^{p+1}_{L^{p+2}\left([-T,T];L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)}\right). \tag{4.9}$$

As above, we prove that $\mathcal{H}_\alpha$ maps the Banach space X_T into itself, and moreover the ball $X_{M,T}$ into itself, provided M and $\|g\|_{L^2_\alpha}$ are small enough, where

$$X_{M,T} = \{u \in X_T : \|u\|_{X_T} \leq M\}.$$

Let δ small enough, and also that $\|g\|_{L^2_\alpha} < \delta$. By (4.9), we can see that

$$\|\mathcal{H}_\alpha(u)\|_{X_T} \leq C\delta + CM^{p+1} < M,$$

provided δ is such that $2C\delta = M$ and $CM^p \leq \frac{1}{2}$, and we assume now that $u_i \in X_{M,T}$, $i = 1, 2$. We have also

$$\begin{aligned}
&\|\mathcal{H}_\alpha(u_1) - \mathcal{H}_\alpha(u_2)\|_{X_T} \leq C\| F(u_1) - F(u_2)\|_{L^{q'}\left([-T,T];L^{r'}_\alpha\left(\mathbb{R}^n_+\right)\right)} \\
\leq &C\|u_1 - u_2\|_{L^{p+2}\left([-T,T];L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)} \\
&\times\left(\|u_1\|^p_{L^{p+2}\left([-T,T];L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)} + \|u_2\|^p_{L^{p+2}\left([-T,T];L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)}\right) \\
\leq &\|u_1 - u_2\|_{X_{M,T}} 2CM^p \leq \frac{1}{2}\|u_1 - u_2\|_{X_{M,T}},
\end{aligned}$$

provided M is so small that $2CM^p \leq \frac{1}{2}$. Thus, if initial data are small enough, i.e., $\|g\|_{L^2_\alpha} < \delta$, then the map $\mathcal{H}_\alpha$ is a contraction which implies that there exists a unique solution $u(t,x)$ of the Cauchy problem (S_p) such that $u(t,x) \in L^{p+2}\Big([-T,T];$ $L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\Big)$. As observed above one can see that this is the unique solution in $u(t,x) \in C\left([-T,T], L^2_\alpha\left(\mathbb{R}^n_+\right)\right)$, i.e., $u \in X_T$, with small initial data in L^2_α. □

Theorem 4.3. *If $p = \frac{2}{|\alpha|+n}$, there exists $\delta > 0$ such that if $g \in L^2_\alpha\left(\mathbb{R}^n_+\right)$ satisfying $\|g\|_{L^2_\alpha} \leq \delta$, (i.e., $\|g\|_{L^2_\alpha}$ small enough), then the corresponding maximal L^2_α-solution is global, i.e., $T^* = T_* = +\infty$. Moreover, u belongs to $L^{p+2}_{loc}\left(\mathbb{R}; L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)$.*

Proof. As Corollary 4.1, we assume that initial data are small enough, i.e., $\|g\|_{L^2_\alpha} < \delta$. For $k \in \mathbb{N}$, there is a unique solution $u_k \in X_k$. Similarly, there is a unique solution $u_{k+1} \in X_{k+1}$, but $u_k = u_{k+1}$ for all t in $[-k,k]$. Thus, $u = \lim_{k\to+\infty} u_k$, is a unique solution of (S_p) in $C\left(\mathbb{R}; L^2_\alpha\left(\mathbb{R}^n_+\right)\right) \cap L^{p+2}_{loc}\left(\mathbb{R}; L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)$, which achieved the proof of the global existence. □

4.3. **Scattering for nonlinear Schrödinger equations associated with the linear canonical Bessel operator.** In this subsection, let $t_0 \in \overline{\mathbb{R}}$ we consider the nonlinear Schrödinger equations associated with the linear canonical Bessel operator

$$(S_{p,t_0}) \qquad \begin{cases} i\partial_t u(t,x) + \Delta^{\mathbf{m}^{-1}}_\alpha u(t,x) &= F(u(t,x)), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n_+, \\ \left(e^{-it\Delta^{\mathbf{m}^{-1}}_\alpha} u(t,x)\right)_{|t=t_0} &= g(x), \qquad\quad x \in \mathbb{R}^n_+, \end{cases}$$

with F satisfying (1.1). We use the convention: if $t_0 = +\infty$ (resp. $t_0 = -\infty$), then we denote $g = u_+$ (resp. $g = u$), and solving (S_{p,t_0}) means that we construct wave operators.

In all these cases, we seek mild solutions to (S_{p,t_0}), that is, we solve

$$u(t,\cdot) = e^{-it\Delta^{\mathbf{m}^{-1}}_\alpha} g(\cdot) - i\int_{t_0}^t e^{-i(t-s)\Delta^{\mathbf{m}^{-1}}_\alpha} F(u(s,\cdot))ds.$$

Actually, using our general methods [18], it is possible to prove scattering for small $L^2_\alpha\left(\mathbb{R}^n_+\right)$ data. Thus we can prove the following.

Proposition 4.2. *Let* $p = \frac{2}{|\alpha|+n}$ *and* $t_0 \in \overline{\mathbb{R}}$. *There exists* δ_0 *such that if* $g \in L^2_\alpha\left(\mathbb{R}^n_+\right)$ *with* $\|g\|_{L^2_\alpha} < \delta_0$, *then the unique global solution* u *has the scattering property: there exists* $u_\pm \in L^2_\alpha\left(\mathbb{R}^n_+\right)$ *such that*

$$\left\| u(t) - e^{-it\Delta_\alpha^{\mathbf{m}^{-1}}} u_\pm \right\|_{L^2_\alpha} \to 0, \quad \text{as } t \to \pm\infty.$$

If $t_0 = +\infty$ *(resp.* $t_0 = -\infty$*), then* $u_+ = g$ *(resp.* $u_- = g$*).*

Proof. According to the proof of Theorem 4.2, for $p = \frac{2}{|\alpha|+n}$ and small initial data in L^2_α, there exists a unique solution u of (S_{p,t_0}) such that

$$u \in C\left(\mathbb{R}; L^2_\alpha\left(\mathbb{R}^n_+\right)\right) \cap L^{p+2}\left(\mathbb{R}; L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right).$$

Scattering then follows from the Cauchy criterion. Indeed, using Theorem 3.3 , for $t_1 \leq t_2$, we have

$$\begin{aligned} \left\| e^{it_2\Delta_\alpha^{\mathbf{m}^{-1}}} u(t_2,\cdot) - e^{it_2\Delta_\alpha^{\mathbf{m}^{-1}}} u(t_1,\cdot) \right\|_{L^2_\alpha\left(\mathbb{R}^n_+\right)} &= \left\| \int_{t_1}^{t_2} e^{is\Delta_\alpha^{\mathbf{m}^{-1}}} F(u(s,\cdot)) ds \right\|_{L^2_\alpha\left(\mathbb{R}^n_+\right)} \\ &\leq C\|u\|^{p+1}_{L^{p+2}\left([t_1,t_2]; L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)}. \end{aligned}$$

Notice that the right-hand side goes to zero when $t_1, t_2 \to \pm\infty$ since $u \in L^{p+2}\left(\mathbb{R}; L^{p+2}_\alpha\left(\mathbb{R}^n_+\right)\right)$. The result follows easily, since the group $e^{-i(\cdot)\Delta_\alpha^{\mathbf{m}^{-1}}}$ is unitary on $L^2_\alpha\left(\mathbb{R}^n_+\right)$. ☐

Conclusion and Prospects

In this paper, we have extensively investigated nonlinear Schrödinger equations related to the linear canonical Bessel operator under the initial data condition $g \in L^2_\alpha(\mathbb{R}^n)$, with a specific focus on the subcritical condition and where the subcritical condition has been treated with special care. Looking ahead, the investigation offers new directions for exploration, the beginning of a thorough study of nonlinear Schrödinger equations related to the linear canonical Bessel operator. This task will be accomplished with proper consideration of the Sobolev space $H^{\mathbf{m},\alpha}_s$ in (4.1), encompassing subcritical and critical elements as created by the demands in 4.1. However, this area of research needs special consideration to some inherent challenges. Perhaps the most crucial task is finding a relationship between $\|u^{p+1}\|_{H^{\mathbf{m},\alpha}_{r,s}}$, $\|u\|_{H^{\mathbf{m},\alpha}_{r,s}}$, and $\|u\|_{\alpha,q}$, for the classical setting. Adapting to the interdependent interplay between these norms is quite a challenge, demanding subtle focus and creative methods in order to untie the interplay of the complexities at hand. In summary, although the research offers a basis for future research, overcoming obstacles like the above-defined relation shall prove crucial in advancing our understanding of nonlinear Schrödinger equations related to the linear canonical Bessel operator as well as its generalizability to other mathematical contexts.

References

[1] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer, Berlin, 2011. https://doi.org/10.1007/978-3-642-16830-7

[2] N. Ben Salem and A. Dachraoui, *Sobolev type spaces associated with Jacobi differential operators*, Integral Transforms Spec. Funct. **18**(3) (2007), 163–184. https://doi.org/10.1080/10652460008819253

[3] J. J. Betancor, A. J. Castro and A. Nowak, *Calderón-Zygmund operators in the Bessel setting*, Monatsh. Math. **167**(3–4) (2012), 375–403. https://doi.org/10.1007/s00605-011-0348-7

[4] F. Bouzeffour, *The fractional multi-dimensional Bessel transform and its role in anisotropic Riesz-Bessel potential*, Integral Transforms Spec. Funct. **35**(6) (2024), 1–11. https://doi.org/10.1080/10652469.2024.2381148

[5] F. Bouzeffour and M. Garayev, *On the fractional Bessel operator*, Integral Transforms Spec. Funct. **33**(3) (2021), 230–246. https://doi.org/10.1080/10652469.2021.1925268

[6] A. Bultheel and H. Martínez-Sulbaran, *Recent developments in the theory of the fractional Fourier and linear canonical transforms*, Bull. Belg. Math. Soc. Simon Stevin **13**(5) (2007), 971–1005. https://doi.org/10.36045/bbms/1170347822

[7] Y. Bettaibi, *Positivity of global solutions for a singular nonlinear heat equation associated with the Bessel operator*, Positivity **29** (2025), Article ID 57. https://doi.org/10.1007/s11117-025-01149-5

[8] S. A. Collins, *Lens-system diffraction integral written in terms of matrix optics*, J. Opt. Soc. Am. **60**(9) (1970), 1168–1177. https://doi.org/10.1364/JOSA.60.001168

[9] L. Bruneau, J. Dereziński and V. Georgescu, *Homogeneous Schrödinger operators on the half-line*, Ann. Henri Poincaré **12**(3) (2011), 547–590. *https://doi.org/10.1007/s00023-011-0078-3*

[10] M. F. E. de Jeu, *The Dunkl transform*, Invent. Math. **113**(1) (1993), 147–162. https://doi.org/10.1007/BF01244305

[11] J. Dereziński and S. Richard, *On Schrödinger operators with inverse square potentials on the half-line*, Ann. Henri Poincaré **18**(3) (2017), 869–928. https://doi.org/10.1007/s00023-016-0520-7

[12] L. Dhaouadi, J. Sahbani and A. Fitouhi, *Harmonic analysis associated with the canonical Fourier-Bessel transform*, Integral Transforms Spec. Funct. **32**(4) (2021), 290–315. https://doi.org/10.1080/10652469.2020.1823977

[13] D. V. Dinh, *Strichartz estimates for the fractional Schrödinger and wave equations on compact manifolds without boundary*, J. Differential Equations **263**(12) (2017), 8804–8837. https://doi.org/10.1016/j.jde.2017.08.045

[14] V. D. Dinh, *Global Strichartz estimates for the fractional Schrödinger equations on asymptotically Euclidean manifolds*, J. Funct. Anal. **275**(8) (2018), 1943–2014. https://doi.org/10.1016/j.jfa.2018.07.006

[15] C. F. Dunkl, *Hankel transforms associated to finite reflection groups*, in: *Hypergeometric Functions on Domains of Positivity*, Springer, Dordrecht, 1992, 123–138.

[16] S. Ghazouani and F. Bouzeffour, *Heisenberg uncertainty principle for a fractional power of the Dunkl transform on the real line*, J. Comput. Appl. Math. **294** (2016), 151–176. https://doi.org/10.1016/j.cam.2015.06.013

[17] S. Ghazouani and F. Bouzeffour, *A fractional power for Dunkl transform*, Bull. Math. Anal. Appl. **6**(3) (2014), 1–30.

[18] J. Ginibre and G. Velo, *On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case*, J. Funct. Anal. **32**(1) (1979), 1–32. https://doi.org/10.1016/0022-1236(79)90076-4

[19] J. Ginibre and G. Velo, *Generalized Strichartz inequalities for the wave equation*, J. Funct. Anal. **133**(1) (1995), 50–68. https://doi.org/10.1006/jfan.1995.1119

[20] J. J. Healy, M. A. Kutay, H. M. Ozaktas and J. T. Sheridan (Eds.), *Linear Canonical Transforms: Theory and Applications*, Springer, New York, 2016.

[21] K. Hleili and M. Hleili, *Quantitative uncertainty principles for the canonical Fourier–Bessel transform*, Kragujevac J. Math. **49**(4) (2025), 567–581. https://doi.org/10.46793/KgJMat2504.567H

[22] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120**(5) (1998), 955–980.

[23] F. H. Kerr, *A fractional power theory for Hankel transforms in* $L^2(\mathbb{R}^+)$, J. Math. Anal. Appl. **158**(1) (1991), 114–123.

[24] H. Lindblad and C. D. Sogge, *On existence and scattering with minimal regularity for semilinear wave equations*, J. Funct. Anal. **130**(2) (1995), 357–426. https://doi.org/10.1006/jfan.1995.1075

[25] K. Mahato and S. Arya, *Gyrator potential operator and* L^p*-Sobolev spaces involving the Gyrator transform*, J. Pseudo-Differ. Oper. Appl. **11** (2020), 987–1004. https://doi.org/10.1007/s11868-020-00329-1

[26] K. Mahato and S. Arya, *Pseudo-differential operators associated with the Gyrator transform on Sobolev spaces*, Appl. Math. Comput. **390** (2021), 125676. https://doi.org/10.1002/mma.10585

[27] H. Mejjaoli, *Strichartz estimates for the Dunkl wave equation and applications*, J. Math. Anal. Appl. **346**(1) (2008), 41–54. https://doi.org/10.1016/j.jmaa.2008.05.004

[28] H. B. Mohamed and Y. Bettaibi, *Pseudo-differential operators in the generalized Weinstein setting*, Rend. Circ. Mat. Palermo, II. Ser. **72** (2023), 3345–3361. https://doi.org/10.1007/s12215-022-00827-7

[29] H. B. Mohamed and Y. Bettaibi, *New Sobolev-Weinstein spaces and applications*, Math. Methods Appl. Sci. **45** (2022), 12345–12368. https://doi.org/10.3770/j.issn:2095-2651.2022.03.004

[30] H. B. Mohamed and A. Saoudi, *Linear canonical Fourier-Bessel wavelet transform: properties and inequalities*, Integral Transforms Spec. Funct. **35**(4) (2024), 270–290. https://doi.org/10.1080/10652469.2024.2317724

[31] M. Moshinsky and C. Quesne, *Linear canonical transformations and their unitary representations*, J. Math. Phys. **12**(8) (1971), 1772–1780. https://doi.org/10.1063/1.1665805

[32] A. Nowak and K. Stempak, *Riesz transforms for multi-dimensional Laguerre function expansions*, Adv. Math. **215** (2) (2007), 642–678. https://doi.org/10.1016/j.aim.2007.04.010

[33] R. S. Pathak and P. K. Pandey, *Sobolev type spaces associated with Bessel operators*, J. Math. Anal. Appl. **215**(1) (1997), 95–111. https://doi.org/10.1006/jmaa.1997.5620

[34] S. Ghazouani and J. Sahbani, *The heat semigroups and uncertainty principles related to canonical Fourier-Bessel transform*, J. Pseudo-Differ. Oper. Appl. **15** (2024), Article ID 36. https://doi.org/10.1007/s11868-024-00608-z

[35] R. S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44**(3) (1977), 705–714. https://doi.org/10.1215/S0012-7094-77-04430-1

[36] K. Trimeche, *Generalized Harmonic Analysis and Wavelet Packets*, CRC Press, Boca Raton, 2001. https://doi.org/10.1201/9781482283174

[37] K. B. Wolf, *Canonical transforms. II. Complex radial transforms*, J. Math. Phys. **15**(12) (1974), 2102–2111. https://doi.org/10.1063/1.1666590

[38] K. B. Wolf, *Integral Transforms in Science and Engineering*, Springer, New York, 2013. https://doi.org/10.1007/978-1-4757-0872-1

Laboratory of Mathematics and Applications LR17ES11
Faculty of Sciences of Gabes,
University of Gabes, Tunisia
Email address: youssef.bettaibi@yahoo.com
ORCID iD: https://orcid.org/0000-0002-1986-6985