Kragujevac Journal of Mathematics
Volume 51(9) (2027), Pages 1599–1609.

μ-PRIME SUBMODULES

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Abstract. We introduce and study μ-prime ideals and μ-prime submodules defined by an endomorphism μ compatible with the module structure. We obtain several characterizations of these notions and investigate their behavior under fundamental constructions, including localization, homomorphisms, quotients, and direct products. A μ-version of the prime avoidance lemma is established. We further introduce μ-multiplicative systems and the μ-radical, and prove a representation theorem for the latter in terms of μ-prime submodules. These results extend classical properties of prime ideals and prime submodules to an endomorphism-based setting.

1. Introduction

Throughout this article, all rings are assumed to be commutative with identity, and all modules are unitary. Unless otherwise specified, R denotes a ring and M an R-module. We use $N \leq M$ for a submodule N of M, with $N \lneq M$ when N is proper. For a nonempty subset $X \subseteq R$, let $(N : X) = \{r \in R \mid rX \subseteq N\}$. For $m \in M$, we write $(N : m)$ for $(N : \{m\})$. The set of zero divisors of M is denoted by $\mathrm{Zd}_R(M) = \{r \in R \mid rm = 0 \text{ for some } 0 \neq m \in M\}$. Also, we use $\mathrm{Spec}(M)$ and $\mathrm{Spec}(R)$ for the set of prime submodules of M and prime ideals of R, respectively.

In commutative algebra, prime ideals are fundamental objects and have motivated various generalizations to the setting of modules. One of the basic such notions is that of a prime submodule [4], and several other prime-like submodule notions have subsequently been introduced, including classical prime submodules [3] and weakly prime submodules [2]. More recently, endomorphisms have also been incorporated into the study of prime-like structures. For instance, endo-prime modules [5], endo-prime

Key words and phrases. μ-prime ideal, μ-prime submodule, endomorphism, μ-radical.
2020 *Mathematics Subject Classification.* Primary: 13C13. Secondary: 13C99, 13A15.
DOI
Received: August 08, 2026.
Accepted: October 06, 2026.

ideals [8], and weakly endo-prime ideals [6] have been introduced and investigated. Motivated by this line of research, in this paper we propose a different way of incorporating endomorphisms into the classical primality condition for both ideals and submodules.

Throughout the paper, we consider an additive endomorphism $\mu_R : (R, +) \to (R, +)$ and an R-module endomorphism $\mu_M : M \to M$. For simplicity, both maps are denoted by μ whenever no confusion can arise. We assume the compatibility condition $\mu_M(am) = \mu_R(a)m$ for all $a \in R$ and $m \in M$. In particular, $\mu_M(m) = \mu_R(1)m$ for every $m \in M$, an identity that will be used repeatedly throughout the paper. Although some of the results remain valid without the compatibility condition, we assume it throughout the paper to simplify the exposition.

This article is devoted to the study of μ-prime ideals and μ-prime submodules. We establish several characterizations of these notions, investigate their behavior with respect to localization, homomorphic and inverse images, direct products, and quotient modules, prove a version of the prime avoidance lemma, and introduce the notions of μ-multiplicative systems and the μ-radical.

2. μ-Prime Submodules

Definition 2.1. Let $P \lneq M$. We say that P is *μ-prime* if, whenever $\mu(am) \in P$ for some $a \in R$ and $m \in M$, either $\mu(m) \in P$ or $\mu(a) \in (P : M)$.

We say that a proper ideal $\mathcal{P}$ of R is *μ-prime* whenever $\mathcal{P}$ is a μ-prime submodule of R.

We write $\mu\text{-}\operatorname{Spec}(M)$ for the collection of all μ-prime R-submodules of M.

By the above definition, each prime submodule of an R-module M is necessarily μ-prime. In particular, when $\mu = \mathrm{id}_M$, the concept of μ-prime submodules reduces to the usual notion of prime submodules. The next examples demonstrate that, for nonidentity multiplication endomorphisms, the family of μ-prime submodules may properly contain the family of prime submodules.

Example 2.1. (a) Let $R = \mathbb{Z}$ and $I = 12\mathbb{Z}$. Define the multiplication endomorphism $\mu : R \to R$ by $\mu(a) = 4a$. Then, $I \in \mu\text{-}\operatorname{Spec}(R)$, but $I \notin \operatorname{Spec}(R)$. On the other hand, for the multiplication endomorphism defined by $\mu(a) = 2a$, $I \notin \mu\text{-}\operatorname{Spec}(R)$

(b) Let $R = \mathbb{Z}_{p^3}$ and $I = p^2\mathbb{Z}_{p^3}$. If $\mu : R \to R$ is given by $\mu(a) = pa$, then $I \in \mu\text{-}\operatorname{Spec}(R)$, but $I \notin \operatorname{Spec}(R)$.

(c) Suppose that R is an integral domain, $M = R[x]$, and $P = rxR[x]$, where $r \in R$ is a nonunit. If μ denotes the multiplication endomorphism defined by $\mu(f) = rf$, then $P \in \mu\text{-}\operatorname{Spec}(M)$, but $P \notin \operatorname{Spec}(M)$.

Proposition 2.1. *The following assertions hold.*

(a) *If $\mu = 0_R$ or $\mu = 0_M$, then every proper submodule of M is μ-prime.*

(b) *If μ is an automorphism of both R and M, then* $\operatorname{Spec}(M) = \mu\text{-}\operatorname{Spec}(M)$.

Proof. (1) is clear.

(2) Assume that P is μ-prime and μ is an automorphism. Let $am \in P$. Then $\mu(am) \in P$. Since $P \in \mu\text{-Spec}(M)$, either $\mu(m) \in P$ or $\mu(a) \in (P : M)$. Applying μ^{-1} gives $m \in P$ or $a \in (P : M)$. Hence $P \in \text{Spec}(M)$. The other side is clear. □

We know that if $P \in \text{Spec}(M)$, then $(P : M) \in \text{Spec}(R)$. The next proposition gives an analogous result for μ-prime submodules.

Proposition 2.2. *Suppose that $P \in \mu\text{-Spec}(M)$ and $K \leq M$ with $P \nsubseteq K$. Then, $(P : K) \in \mu\text{-Spec}(R)$. In particular, $(P : M) \in \mu\text{-Spec}(R)$.*

Proof. Let $\mu(ab) \in (P : K)$ and assume that $\mu(b) \notin (P : K)$. Then $\mu(bm) = \mu(b)m \notin P$ for some $m \in K$. Since $\mu(ab) \in (P : K)$, we have $\mu(ab)m \in P$. Moreover, $\mu(ab)m = \mu(a(bm)) \in P$. Since $P \in \mu\text{-Spec}(M)$ and $\mu(bm) \notin P$, we have $\mu(a) \in (P : M)$. Hence $\mu(a) \in (P : K)$, because $(P : M) \subseteq (P : K)$. Therefore $(P : K) \in \mu\text{-Spec}(R)$.

The last assertion follows by taking $K = M$. □

Proposition 2.2 does not admit a converse in general. As an example, consider $R = \mathbb{Z}$, $M = \mathbb{Z}_{16} \times \mathbb{Z}$, and $P = \{(0, 0)\}$. Let $\mu_R : \mathbb{Z} \to \mathbb{Z}$ and $\mu_M : M \to M$ be the multiplication endomorphisms induced by 2, that is, $\mu_R(a) = 2a$ and $\mu_M(m) = 2m$. Then, $(P : M)$ is a μ-prime (indeed, prime) ideal of R, whereas P is not a μ-prime submodule of M.

Proposition 2.3. *Assume that M is a multiplication R-module and let $P \lneq M$. Then, $P \in \mu\text{-Spec}(M)$ if and only if $(P : M) \in \mu\text{-Spec}(R)$.*

Proof. The forward implication is a consequence of Proposition 2.2.

For the converse, suppose that $(P : M) \in \mu\text{-Spec}(R)$. Let $\mu(rm) \in P$ and $\mu(r) \notin (P : M)$. Take any $a \in (Rm : M)$. Since $aM \subseteq Rm$, for each $x \in M$, one can find $s \in R$ such that $ax = sm$. Hence, $\mu((ra)x) = \mu(r)(ax) = \mu(r)(sm) = s\mu(rm) \in P$, which yields $\mu(ra)M \subseteq P$. Consequently, $\mu(ra) \in (P : M)$. As $(P : M)$ is μ-prime and $\mu(r) \notin (P : M)$, it follows that $\mu(a) \in (P : M)$.

Now M is a multiplication module, so $Rm = (Rm : M)M$. Thus m admits an expression $m = \sum_{i=1}^{n} a_i m_i$, where $a_i \in (Rm : M)$ and $m_i \in M$. Since $\mu(a_i) \in (P : M)$ for every i, we have $\mu(a_i)m_i \in P$. Therefore $\mu(m) = \sum_{i=1}^{n} \mu(a_i)m_i \in P$, showing that P is μ-prime. □

Proposition 2.4. *Let $P \leq M$. The following conditions are equivalent.*

(1) $P \in \mu\text{-Spec}(M)$.
(2) *If $\mu(aK) \subseteq P$ for $a \in R$ and $K \leq M$, then $\mu(K) \subseteq P$ or $\mu(a) \in (P : M)$.*
(3) *If $\mu(Im) \subseteq P$ for an ideal I of R and $m \in M$, then $\mu(m) \in P$ or $\mu(I) \subseteq (P : M)$.*
(4) *If $\mu(IK) \subseteq P$ for an ideal I of R and $K \leq M$, then $\mu(K) \subseteq P$ or $\mu(I) \subseteq (P : M)$.*

Proof. (4) $\Rightarrow$ (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2) $\Rightarrow$ (1) are evident.

(1) $\Rightarrow$ (4). Assume that $P \in \mu\text{-Spec}(M)$, $\mu(IK) \subseteq P$, and $\mu(K) \nsubseteq P$, where I is an ideal of R and $K \leq M$.

Suppose that $\mu(I) \nsubseteq (P : M)$. Thus, for some $i \in I$ and $k \in K$, $\mu(i) \notin (P : M)$ and $\mu(k) \notin P$. Since P is μ-prime and $\mu(i) \notin (P : M)$ and $\mu(k) \notin P$, it follows that $\mu(ik) \notin P$. However, $\mu(ik) = \mu(i)k \in \mu(I)K \subseteq P$, a contradiction. Therefore, $\mu(I) \subseteq (P : M)$. □

The following characterization follows by a routine induction on the number of factors together with the definition of μ-prime ideals, and is therefore omitted.

Corollary 2.1. *For a proper ideal P of R, the following conditions are equivalent.*

(a) $P \in \mu\text{-Spec}(R)$.
(b) *For any $a_1, \ldots, a_k \in R$, if $\mu(a_1 \cdots a_k) \in P$, then $\mu(a_i) \in P$ for some i.*
(c) *For any ideals $I_1, \ldots, I_k$ of R, if $\mu(I_1 \cdots I_k) \subseteq P$, then $\mu(I_j) \subseteq P$ for some j.*

Proposition 2.5. *Let $P \in \mu\text{-Spec}(R)$, and suppose that $K_1, \ldots, K_l \leq M$ satisfy $K_1 \cap K_2 \cap \cdots \cap K_l \subseteq P$. Then, there exists $i \in \{1, \ldots, l\}$ such that either $\mu(K_i) \subseteq P$ or $\mu((K_i : M)) \subseteq (P : M)$.*

Proof. Suppose, to the contrary, that $\mu(K_i) \nsubseteq P$ and $\mu((K_i : M)) \nsubseteq (P : M)$ for every $i = 1, \ldots, l$.

Choose $x \in K_1$ such that $\mu(x) \notin P$, and for each $i = 2, \ldots, l$, choose $r_i \in (K_i : M)$ with $\mu(r_i) \notin (P : M)$.

Set $a = r_2 \cdots r_l$. Then, $ax \in K_1 \cap \cdots \cap K_l \subseteq P$, so $\mu(ax) \in P$. Since $\mu(ax) = \mu(a)x$, we obtain $\mu(a)x \in P$. As $\mu(x) \notin P$ and P is μ-prime, it follows that $\mu(a) \in (P : M)$.

By Proposition 2.2, $(P : M) \in \mu\text{-Spec}(R)$. Hence, by Corollary 2.1, $\mu(r_i) \in (P : M)$ for some $i \in \{2, \ldots, l\}$, a contradiction.

Then, for some $i \in \{1, \ldots, l\}$, either $\mu(K_i) \subseteq P$ or $\mu((K_i : M)) \subseteq (P : M)$. □

Proposition 2.6. *Suppose that $P \leq M$. The following statements are equivalent.*

(1) $P \in \mu\text{-Spec}(M)$.
(2) *If $m \in M$ with $m \notin \mu^{-1}(P)$, then $\mu^{-1}((P : K)) = \mu^{-1}((P : M))$.*
(3) *If $m \in M$ $m \notin \mu^{-1}(P)$, then $\mu^{-1}((P : m)) = \mu^{-1}((P : M))$.*

Proof. (1) $\Rightarrow$ (2)**.** Assume that $P \in \mu\text{-Spec}(M)$ and let $K \leq M$ with $K \nsubseteq \mu^{-1}(P)$. Choose $k \in K$ such that $\mu(k) \notin P$.

Let $a \in \mu^{-1}((P : K))$. Then, $\mu(a) \in (P : K)$, so $\mu(a)K \subseteq P$. In particular, $\mu(a)k \in P$. Since $\mu(ak) = \mu(a)k$, we have $\mu(ak) \in P$. As $\mu(k) \notin P$ and P is μ-prime, it follows that $\mu(a) \in (P : M)$. Hence, $a \in \mu^{-1}((P : M))$. Therefore, $\mu^{-1}((P : K)) \subseteq \mu^{-1}((P : M))$.

The reverse inclusion is obvious.

(2) $\Rightarrow$ (3)**.** Take $K = Rm$.

(3) $\Rightarrow$ (1)**.** Assume that $\mu(am) \in P$ and $\mu(m) \notin P$. Then, $m \notin \mu^{-1}(P)$. Moreover, $\mu(a)m = \mu(am) \in P$, so $\mu(a) \in (P : m)$. Hence, $a \in \mu^{-1}((P : m))$. By (3),

$\mu^{-1}((P : m)) = \mu^{-1}((P : M))$. Therefore, $a \in \mu^{-1}((P : M))$, and so $\mu(a) \in (P : M)$. Thus, $P \in \mu\text{-Spec}(M)$. □

The prime avoidance lemma for modules has been proved by Lu [7] as follows: Let $S, P_1, \ldots, P_k$ be submodules of M such that $S \subseteq \bigcup_{i=1}^{k} P_i$ and $(P_k : M) \nsubseteq (P_j : M)$ for all $j \neq k$. If at most two of the P_i are not prime, then $S \subseteq P_j$ for some $j \in \{1, \ldots, k\}$.

We next establish an μ-analogue of the prime avoidance lemma. To this end, the following lemma will be needed.

Lemma 2.1. *Assume that P and K are submodules of M. Then, the following conclusions hold.*

(1) *$K \subseteq \mu^{-1}(P)$ if and only if $\mu(K) \subseteq P$.*
(2) *$\mu^{-1}((P : M)) = (\mu^{-1}(P) : M)$.*
(3) *If $\mu^{-1}(P) \neq M$, then $P \in \mu\text{-Spec}(M)$ if and only if $\mu^{-1}(P) \in \text{Spec}(M)$.*

Proof. (1) Assume that $K \subseteq \mu^{-1}(P)$. Let $k \in K$. Then, $k \in \mu^{-1}(P)$, and hence, by the definition of the inverse image, $\mu(k) \in P$. Therefore, $\mu(K) \subseteq P$.

Conversely, suppose that $\mu(K) \subseteq P$. For every $k \in K$, we have $\mu(k) \in \mu(K) \subseteq P$. Hence, $k \in \mu^{-1}(P)$. Therefore, $K \subseteq \mu^{-1}(P)$, and the first assertion follows.

(2) Let $a \in \mu^{-1}((P : M))$. Hence, $\mu(a) \in (P : M)$. Then, for every $m \in M$, $\mu(a)m \in P$. By the given identity $\mu(am) = \mu(a)m$, it follows that $\mu(am) \in P$. Thus, $am \in \mu^{-1}(P)$ for every $m \in M$, and consequently $a \in (\mu^{-1}(P) : M)$. Therefore, $\mu^{-1}((P : M)) \subseteq (\mu^{-1}(P) : M)$.

Conversely, assume that $a \in (\mu^{-1}(P) : M)$. Thus, $am \in \mu^{-1}(P)$ for every $m \in M$. Then, $\mu(am) \in P$. Using the identity $\mu(am) = \mu(a)m$, we obtain $\mu(a)m \in P$ for every $m \in M$. Thus, $\mu(a) \in (P : M)$, so $a \in \mu^{-1}((P : M))$. Hence, $\mu^{-1}((P : M)) = (\mu^{-1}(P) : M)$.

(3) Assume that $\mu^{-1}(P) \neq M$. We prove that $P \in \mu\text{-Spec}(M)$ if and only if $\mu^{-1}(P) \in \text{Spec}(M)$. First, suppose that $P \in \mu\text{-Spec}(M)$. Assume that $a \in R$ and $m \in M$ with $am \in \mu^{-1}(P)$. Then, $\mu(am) \in P$. Since $P \in \mu\text{-Spec}(M)$, either $\mu(m) \in P$ or $\mu(a) \in (P : M)$.

When $\mu(m) \in P$, we have $m \in \mu^{-1}(P)$. If $\mu(m) \notin P$, then $\mu(a) \in (P : M)$. Hence, by part (2), $a \in \mu^{-1}((P : M)) = (\mu^{-1}(P) : M)$. Therefore, $\mu^{-1}(P)$ is a prime submodule of M, and hence $\mu^{-1}(P) \in \text{Spec}(M)$.

Conversely, suppose that $\mu^{-1}(P) \in \text{Spec}(M)$. Assume that $a \in R$ and $m \in M$ satisfy $\mu(am) \in P$. Thus, $am \in \mu^{-1}(P)$. As $\mu^{-1}(P)$ is prime, it follows that $a \in (\mu^{-1}(P) : M)$ or $m \in \mu^{-1}(P)$.

By part (2), $(\mu^{-1}(P) : M) = \mu^{-1}((P : M))$. Hence, in the first case, $\mu(a) \in (P : M)$. In the second case, $m \in \mu^{-1}(P)$, and therefore $\mu(m) \in P$. Consequently, whenever $\mu(am) \in P$, we have either $\mu(a) \in (P : M)$ or $\mu(m) \in P$. Thus, P is a μ-prime submodule of M. □

Theorem 2.1. *Let $P_1, P_2, \ldots, P_k$ be submodules of M such that at most two of them are not μ-prime. Assume also that $\mu^{-1}((P_k : M)) \nsubseteq \mu^{-1}((P_j : M))$ for all $j \neq k$. If S is a submodule of M with $S \subseteq \bigcup_{i=1}^{k} P_i$, then $\mu(S) \subseteq P_j$ for some $j \in \{1, 2, \ldots, k\}$.*

Proof. If $\mu^{-1}(P_i) = M$ for some $i \in \{1, \ldots, k\}$, then $S \subseteq \mu^{-1}(P_i)$, and hence $\mu(S) \subseteq P_i$ by Lemma 2.1 (1). Thus, we may assume that $\mu^{-1}(P_i) \neq M$ for all i.

By Lemma 2.1 (3), each $\mu^{-1}(P_i)$, except at most two of them, is a prime submodule of M.

For each $j \neq k$, Lemma 2.1 (2) yields $(\mu^{-1}(P_k) : M) \nsubseteq (\mu^{-1}(P_j) : M)$, since $\mu^{-1}((P_k : M)) \nsubseteq \mu^{-1}((P_j : M))$.

For every $x \in P_i$, we have $\mu(x) = \mu(1)x \in P_i$. Hence, $\mu(P_i) \subseteq P_i$, and consequently $P_i \subseteq \mu^{-1}(P_i)$ for all i. Therefore, $S \subseteq \bigcup_{i=1}^{k} \mu^{-1}(P_i)$. Applying the prime avoidance lemma for modules [7], we obtain $S \subseteq \mu^{-1}(P_j)$ for some j.

Therefore, by Lemma 2.1 (1), $\mu(S) \subseteq P_j$. □

Proposition 2.7. *Assume that $\phi : M_1 \to M_2$ is an R-module homomorphism. The following assertions hold.*

(a) *If $P \in \mu\text{-Spec}(M_1)$, ϕ is surjective, and $\text{Ker}(\phi) \subseteq P$, then $\phi(P) \in \mu\text{-Spec}(M_2)$.*
(b) *If $P \in \mu\text{-Spec}(M_2)$, then $\phi^{-1}(P) \in \mu\text{-Spec}(M_1)$.*

Proof. First note that $\phi\mu = \mu\phi$. Indeed, for every $m \in M_1$, we have $\phi(\mu(m)) = \phi(\mu(1m)) = \phi(\mu(1)m) = \mu(1)\phi(m) = \mu(1\phi(m)) = \mu(\phi(m))$.

(a) Let $P \in \mu\text{-Spec}(M_1)$, and suppose that $\mu(am) \in \phi(P)$ for some $a \in R$ and $m \in M_2$. As ϕ is surjective, $\phi(m_1) = m$ for some $m_1 \in M_1$. Hence, $\phi(\mu(am_1)) = \mu(am) \in \phi(P)$. Thus, $\phi(\mu(am_1)) = \phi(p)$ for some $p \in P$, so $\mu(am_1) - p \in \text{Ker}(\phi) \subseteq P$. Therefore, $\mu(am_1) \in P$.

Since $P \in \mu\text{-Spec}(M_1)$, either $\mu(m_1) \in P$ or $\mu(a) \in (P : M_1)$. In the first case, $\mu(m) = \phi(\mu(m_1)) \in \phi(P)$. In the second case, for every $m \in M_2$, choose $m_1 \in M_1$ with $\phi(m_1) = m$. Then $\mu(a)m = \phi(\mu(a)m_1) \in \phi(P)$, and hence $\mu(a) \in (\phi(P) : M_2)$. Thus, $\phi(P) \in \mu\text{-Spec}(M_2)$.

(b) Let $P \in \mu\text{-Spec}(M_2)$ and suppose that $\mu(am) \in \phi^{-1}(P)$ for some $a \in R$ and $m \in M_1$. Hence, $\mu(a\phi(m)) = \phi(\mu(am)) \in P$. Since $P \in \mu\text{-Spec}(M_2)$, either $\mu(\phi(m)) \in P$ or $\mu(a) \in (P : M_2)$.

If $\mu(\phi(m)) \in P$, then $\phi(\mu(m)) \in P$, and hence $\mu(m) \in \phi^{-1}(P)$. If $\mu(a) \in (P : M_2)$, then $\phi(\mu(a)m) = \mu(a)\phi(m) \in P$ for every $m \in M_1$. Therefore, $\mu(a)m \in \phi^{-1}(P)$ for every $m \in M_1$, and so $\mu(a) \in (\phi^{-1}(P) : M_1)$. Thus, $\phi^{-1}(P) \in \mu\text{-Spec}(M_1)$. □

Applying Proposition 2.7 to the canonical epimorphism $\pi : M \to M/K$, defined by $\pi(m) = m + K$, yields the following corollary.

Corollary 2.2. *Let $P, K \leq M$ satisfy $K \subseteq P$. Then, $P \in \mu\text{-Spec}(M)$ if and only if $P/K \in \mu\text{-Spec}(M/K)$.*

Theorem 2.2. *Let R be a unique factorization domain (UFD) and let aR be a nonzero proper ideal of R. Then, $aR \in \mu\text{-Spec}(R)$ if and only if $a = d$ or $a = dp$, where $d \mid \mu(1)$ and p is an irreducible element of R.*

Proof. Using the equality $\mu(a) = \mu(1)a$ for every $a \in R$, we distinguish two cases.

Suppose first that $\mu(1) \in U(R)$. Then μ is an automorphism of R. Hence, by Proposition 2.1 (2), $\mathrm{Spec}(R) = \mu\text{-}\mathrm{Spec}(R)$. Since every nonzero principal prime ideal of a UFD is generated by an irreducible element, the conclusion follows by taking $d = 1$.

Now assume that $\mu(1) \notin U(R)$. Identifying μ with multiplication by the fixed element $\mu(1)$, we prove the two implications.

($\Rightarrow$) Let $aR \in \mu\text{-}\mathrm{Spec}(R)$. Let $a = p^k q^\ell t$ and $\mu(1) = p^{k'} q^{\ell'} s$, where p and q are distinct irreducible elements, $p, q \nmid t$, and $p, q \nmid s$. If $k > k'$ and $\ell > \ell'$, then $\mu(1)p^{k-k'}q^{\ell-\ell'}t = sa \in aR$.

If $\mu(1)p^{k-k'} \in aR$, then $a \mid \mu(1)p^{k-k'}$, which is impossible since the exponent of q in $\mu(1)p^{k-k'}$ is only $\ell' < \ell$. Similarly, if $\mu(1)q^{\ell-\ell'}t \in aR$, then $a \mid \mu(1)q^{\ell-\ell'}t$, which is impossible since the exponent of p in $\mu(1)q^{\ell-\ell'}t$ is only $k' < k$. Hence $\mu(1)p^{k-k'} \notin aR$ and $\mu(1)q^{\ell-\ell'}t \notin aR$, contradicting the μ-primeness of aR.

Therefore, at most one irreducible divisor of a may occur with exponent greater than its exponent in $\mu(1)$.

If every irreducible factor of a occurs with exponent at most its exponent in $\mu(1)$, then $a \mid \mu(1)$, and we may take $d = a$. Otherwise, there is a unique irreducible element p whose exponent in a is greater than its exponent in $\mu(1)$. In this case, write $a = dp^k$, where $d \mid \mu(1)$, $k \geq 1$, and the exponent of p in d is the same as its exponent in $\mu(1)$.

If $k \geq 2$, then $\mu(pp^{k-1}) = \mu(1)p^k \in aR$. Since the exponent of p in d is the same as its exponent in $\mu(1)$, the exponent of p in $\mu(p) = \mu(1)p$ is smaller than its exponent in $a = dp^k$. Hence $\mu(p) \notin aR$. Similarly, the exponent of p in $\mu(p^{k-1}) = \mu(1)p^{k-1}$ is smaller than its exponent in $a = dp^k$, and hence $\mu(p^{k-1}) \notin aR$. This contradicts the μ-primality of aR. Thus $k = 1$, and hence $a = dp$.

($\Leftarrow$) Let $a = d$ or $a = dp$, where $d \mid \mu(1)$ and p is irreducible. Suppose that $\mu(xy) \in aR$ for some $x, y \in R$.

If $a = d$, then $a \mid \mu(1)$, and hence $\mu(x) = \mu(1)x \in aR$. So $aR \in \mu\text{-}\mathrm{Spec}(R)$.

Now suppose that $a = dp$ and assume that $\mu(x) \notin aR$. Write $\mu(1) = dc$, where $c \in R$. Moreover, $\mu(xy) = \mu(1)xy = dcxy \in dpR$, and hence $p \mid cxy$.

If $p \mid c$, then $\mu(x) = dcx \in dpR$, a contradiction. Hence $p \nmid c$. Therefore, $p \mid xy$. On the other hand, $\mu(x) = \mu(1)x = dcx \notin dpR$ implies that $p \nmid x$. Since p is irreducible in the UFD R, it is prime, and hence $p \mid y$. Therefore, $\mu(y) = \mu(1)y = dcy \in dpR = aR$. Hence aR is a μ-prime ideal. □

Corollary 2.3. *Let $\mu : \mathbb{Z} \to \mathbb{Z}$ be a multiplication endomorphism. Then, $m\mathbb{Z} \in \mu\text{-}\mathrm{Spec}(\mathbb{Z})$ if and only if $m = d$ with $1 \neq d \mid \mu(1)$, or $m = dp$, where $d \mid \mu(1)$ and p is a prime number.*

Corollary 2.4. *Let $\mu : \mathbb{Z} \to \mathbb{Z}$ be a multiplication endomorphism, and let $1 \neq m, n \in \mathbb{N}$ with $m \mid n$. Then, $\frac{m\mathbb{Z}}{n\mathbb{Z}} \in \mu\text{-}\mathrm{Spec}(\mathbb{Z}_n)$ if and only if $m = d$ or $m = dp$, where $d \mid \mu(1)$ and p is a prime number.*

Proof. Since every ideal of $\mathbb{Z}_n$ is of the form $\frac{m\mathbb{Z}}{n\mathbb{Z}}$, where $m \mid n$, it suffices to determine when such an ideal is μ-prime. By Corollary 2.2, $\frac{m\mathbb{Z}}{n\mathbb{Z}} \in \mu\text{-Spec}(\mathbb{Z}_n)$ if and only if $m\mathbb{Z} \in \mu\text{-Spec}(\mathbb{Z})$. The result now follows from Corollary 2.3. □

Before proceeding, we note that μ induces homomorphisms $\bar{\mu} : S^{-1}R \to S^{-1}R$ and $\bar{\mu} : S^{-1}M \to S^{-1}M$ defined by $\bar{\mu}(\frac{r}{s}) = \frac{\mu(r)}{s}$ and $\bar{\mu}(\frac{m}{s}) = \frac{\mu(m)}{s}$, which are well-defined.

Proposition 2.8. *Consider a multiplicatively closed subset S of R and a submodule $P \leq M$. If $P \in \mu\text{-Spec}(M)$, then $S^{-1}P \in \bar{\mu}\text{-Spec}(S^{-1}M)$. The converse holds provided* $\mathrm{Zd}(M/P) \cap S = \emptyset$.

Proof. ($\Rightarrow$) Let $P \in \mu\text{-Spec}(M)$ and suppose that $\bar{\mu}\left(\frac{r}{s} \cdot \frac{m}{t}\right) \in S^{-1}P$ for some $\frac{r}{s} \in S^{-1}R$ and $\frac{m}{t} \in S^{-1}M$. Thus, $\frac{\mu(rm)}{st} \in S^{-1}P$, so there exists $u \in S$ such that $u\mu(rm) \in P$. Since $u\mu(rm) = \mu(urm) = \mu(r(um))$, and P is μ-prime, either $\mu(um) \in P$ or $\mu(r) \in (P : M)$.

If $\mu(um) \in P$, then $\bar{\mu}\left(\frac{m}{t}\right) = \frac{\mu(m)}{t} = \frac{\mu(um)}{ut} \in S^{-1}P$.

If $\mu(r) \in (P : M)$, then for every $\frac{m'}{t'} \in S^{-1}M$, $\bar{\mu}\left(\frac{r}{s}\right)\frac{m'}{t'} = \frac{\mu(r)m'}{st'} \in S^{-1}P$. Hence, $\bar{\mu}\left(\frac{r}{s}\right) \in (S^{-1}P : S^{-1}M)$. Therefore $S^{-1}P \in \bar{\mu}\text{-Spec}(S^{-1}M)$.

($\Leftarrow$) Assume that $S^{-1}P \in \bar{\mu}\text{-Spec}(S^{-1}M)$ and $\mathrm{Zd}(M/P) \cap S = \emptyset$. Let $\mu(rm) \in P$ for some $r \in R$ and $m \in M$. Thus, $\bar{\mu}\left(\frac{r}{1}\frac{m}{1}\right) = \frac{\mu(rm)}{1} \in S^{-1}P$. Hence either $\bar{\mu}\left(\frac{m}{1}\right) \in S^{-1}P$ or $\bar{\mu}\left(\frac{r}{1}\right) \in (S^{-1}P : S^{-1}M)$.

If $\bar{\mu}\left(\frac{m}{1}\right) \in S^{-1}P$, then $\frac{\mu(m)}{1} \in S^{-1}P$, so there exists $u \in S$ with $u\mu(m) = \mu(um) \in P$. Since $\mathrm{Zd}(M/P) \cap S = \emptyset$, we obtain $\mu(m) \in P$.

If $\bar{\mu}\left(\frac{r}{1}\right) \in (S^{-1}P : S^{-1}M)$, then for every $m' \in M$, $\frac{\mu(r)m'}{1} = \bar{\mu}\left(\frac{r}{1}\right)\frac{m'}{1} \in S^{-1}P$. Hence, for every $m' \in M$ there exists $u \in S$ such that $u\mu(r)m' \in P$. Since $\mathrm{Zd}(M/P) \cap S = \emptyset$, it follows that $\mu(r)m' \in P$ for every $m' \in M$. Therefore, $\mu(r) \in (P : M)$, and consequently $P \in \mu\text{-Spec}(M)$. □

The lemma below follows directly.

Lemma 2.2. *Let I be an ideal of R and $K \leq M$. Then, the following statements hold:*

(a) $\mu^{-1}(I) = R$ *if and only if* $\mu(1) \in I$;
(b) $\mu^{-1}((K : M)) = R$ *if and only if* $\mu^{-1}(K) = M$.

Proposition 2.9. *For each $i \in \Lambda$, let R_i be a ring, P_i an ideal of R_i, and $\mu_i : R_i \to R_i$ be a multiplication endomorphism. Let $R = \prod_{i\in\Lambda} R_i$, $P = \prod_{i\in\Lambda} P_i$, and define $\mu : R \to R$ by $\mu((r_i)) = (\mu_i(r_i))$. Then, $P \in \mu\text{-Spec}(R)$ if and only if there exists $j \in \Lambda$ such that $P_j \in \mu_j\text{-Spec}(R_j)$ and $\mu_i^{-1}(P_i) = R_i$ for all $i \neq j$.*

Proof. ($\Rightarrow$) Suppose that $P \in \mu\text{-Spec}(R)$ and let $j \in \Lambda$ be such that $\mu_j^{-1}(P_j) \neq R_j$. By Lemma 2.2, $\mu_j(1) \notin P_j$.

Let $a = (a_i)$, $b = (b_i) \in R$, where $a_j = 1$, $a_i = 0$ for $i \neq j$, and $b_j = 0$, $b_i = 1$ for $i \neq j$. Then $\mu(ab) \in P$. Since $\mu(a) \notin P$, the μ-primeness of P yields $\mu(b) \in P$. Hence $\mu_i(1) \in P_i$ for every $i \neq j$. Again by Lemma 2.2, $\mu_i^{-1}(P_i) = R_i$ for all $i \neq j$.

Let $x, y \in R_j$ satisfy $\mu_j(xy) \in P_j$. Define $c = (c_i), d = (d_i) \in R$ by $c_j = x$, $c_i = 0$ for $i \neq j$, and $d_j = y$, $d_i = 0$ for $i \neq j$. Then $\mu(cd) \in P$. Since $P \in \mu\text{-Spec}(R)$, either $\mu(c) \in P$ or $\mu(d) \in P$. Therefore $\mu_j(x) \in P_j$ or $\mu_j(y) \in P_j$, showing that P_j is a μ_j-prime ideal of R_j.

($\Leftarrow$) Suppose that for some $j \in \Lambda$, the ideal P_j is μ_j-prime and $\mu_i^{-1}(P_i) = R_i$ for every $i \neq j$.

Take $a = (a_i)$ and $b = (b_i)$ in R with $\mu(ab) \in P$. Then, $\mu_j(a_j b_j) \in P_j$. Since P_j is μ_j-prime, either $\mu_j(a_j) \in P_j$ or $\mu_j(b_j) \in P_j$. Moreover, $\mu_i^{-1}(P_i) = R_i$ implies $\mu_i(1) \in P_i$ by Lemma 2.2, and therefore $\mu_i(a_i) = \mu_i(1)a_i \in P_i$ and $\mu_i(b_i) = \mu_i(1)b_i \in P_i$ for every $i \neq j$. Hence, $\mu(a) \in P$ or $\mu(b) \in P$, and so $P \in \mu\text{-Spec}(R)$. □

Proposition 2.10. *For each $i \in \Lambda$, let M_i be an R_i-module, P_i a submodule of M_i, and $\mu_i : R_i \to R_i$ and $\mu_i : M_i \to M_i$ be multiplication endomorphisms. Let $R = \prod_{i\in\Lambda} R_i$, $M = \prod_{i\in\Lambda} M_i$, $P = \prod_{i\in\Lambda} P_i$, and define $\mu : R \to R$ and $\mu : M \to M$ componentwise by $\mu((x_i)) = (\mu_i(x_i))$. Then, $P \in \mu\text{-Spec}(M)$ if and only if $P_j \in \mu_j\text{-Spec}(M_j)$ for some $j \in \Lambda$ and $\mu_i^{-1}(P_i) = M_i$ for all $i \neq j$.*

Proof. ($\Rightarrow$) Suppose that $P \in \mu\text{-Spec}(M)$. Then, by Proposition 2.2, $(P : M) \in \mu\text{-Spec}(R)$. Since $(P : M) = \prod_{i\in\Lambda}(P_i : M_i)$, Proposition 2.9 implies that there exists $j \in \Lambda$ such that $(P_j : M_j) \in \mu_j\text{-Spec}(R_j)$ and $\mu_i^{-1}((P_i : M_i)) = R_i$ for every $i \neq j$. By Lemma 2.2(2), $\mu_i^{-1}(P_i) = M_i$ for all $i \neq j$.

Suppose that $a \in R_j$ and $m \in M_j$ are such that $\mu_j(am) \in P_j$. Assume that $\bar{a} = (a_i) \in R$ and $\bar{m} = (m_i) \in M$, where $a_j = a$, $m_j = m$, and $a_i = 0$, $m_i = 0$ for all $i \neq j$. Then, $\mu(\bar{a}\bar{m}) \in P$. Since $P \in \mu\text{-Spec}(M)$, either $\mu(\bar{m}) \in P$ or $\mu(\bar{a}) \in (P : M)$. Hence, $\mu_j(m) \in P_j$ or $\mu_j(a) \in (P_j : M_j)$, showing that $P_j \in \mu\text{-Spec}(M_j)$.

($\Leftarrow$) Assume that there exists $j \in \Lambda$ such that $P_j \in \mu_j\text{-Spec}(M_j)$ and $\mu_i^{-1}(P_i) = M_i$ for every $i \neq j$.

Let $\bar{a} = (a_i) \in R$ and $\bar{m} = (m_i) \in M$ satisfy $\mu(\bar{a}\bar{m}) \in P$. Then, $\mu_i(a_i m_i) \in P_i$ for every $i \in \Lambda$. In particular, $\mu_j(a_j m_j) \in P_j$. Since P_j is μ_j-prime, either $\mu_j(m_j) \in P_j$ or $\mu_j(a_j) \in (P_j : M_j)$. Moreover, $\mu_i^{-1}(P_i) = M_i$ implies $\mu_i(m_i) \in P_i$ for every $i \neq j$, and $\mu_i^{-1}((P_i : M_i)) = R_i$ by Lemma 2.2(2). Hence, $\mu_i(a_i) \in (P_i : M_i)$ for every $i \neq j$. Therefore, $\mu(\bar{m}) \in P$ or $\mu(\bar{a}) \in (P : M)$, and so $P \in \mu\text{-Spec}(M)$. □

The endomorphism μ naturally induces an endomorphism $\widetilde{\mu} : R(+)M \to R(+)M$ defined by $\widetilde{\mu}(r, m) = (\mu(r), \mu(m))$. A direct argument yields the following proposition.

Proposition 2.11. *Let P be a proper ideal of R. Then, $P \in \mu\text{-Spec}(R)$ if and only if $P(+)M \in \widetilde{\mu}\text{-Spec}(R(+)M)$.*

An R-module M is said to be μ-prime whenever its zero submodule (0) is μ-prime.

Proposition 2.12. *An R-module M is μ-prime if and only if* $\mathrm{Zd}(\mu(M)) = \mathrm{Ann}(\mu(M))$.

Proof. First, observe that $\mathrm{Ann}(\mu(M)) = \mu^{-1}(\mathrm{Ann}(M))$.

($\Rightarrow$) Assume that $(0) \in \mu\text{-}\mathrm{Spec}(M)$. If $\mu(M) = 0$, then there is nothing to prove.

Let $r \in \mathrm{Zd}(\mu(M))$. Then, for some $m \in M$ with $\mu(m) \neq 0$, we have $r\mu(m) = 0$. Since $r\mu(m) = \mu(rm)$, the μ-primeness of (0) implies that $\mu(r) \in \mathrm{Ann}(M)$. Hence, $r \in \mu^{-1}(\mathrm{Ann}(M)) = \mathrm{Ann}(\mu(M))$. Therefore, $\mathrm{Zd}(\mu(M)) \subseteq \mathrm{Ann}(\mu(M))$. The reverse inclusion is obvious. Thus, $\mathrm{Zd}(\mu(M)) = \mathrm{Ann}(\mu(M))$.

($\Leftarrow$) Assume that $\mathrm{Zd}(\mu(M)) = \mathrm{Ann}(\mu(M))$. Let $\mu(rm) = 0$ and $\mu(m) \neq 0$. Then, $r\mu(m) = 0$. Hence, $r \in \mathrm{Zd}(\mu(M)) = \mathrm{Ann}(\mu(M)) = \mu^{-1}(\mathrm{Ann}(M))$. Then, $\mu(r) \in \mathrm{Ann}(M)$, that is, $\mu(r)M = 0$. Therefore, (0) is a μ-prime submodule of M. □

Definition 2.2. Suppose that M is an R-module and consider a subset $S \subseteq M \setminus \{0\}$. We say that S is a *μ-multiplicative system* of M if, for every ideal I of R and all submodules K and L of M, $(K + \mu(I)L) \cap S = \emptyset$ implies that $(K + \mu(L)) \cap S = \emptyset$ or $(K + \mu(I)M) \cap S = \emptyset$.

Proposition 2.13. *Suppose that $P \leq M$. Then, $P \in \mu\text{-}\mathrm{Spec}(M)$ if and only if $M \setminus P$ is a μ-multiplicative system.*

Proof. ($\Rightarrow$) Let $P \in \mu\text{-}\mathrm{Spec}(M)$, set $S = M \setminus P$, and suppose that $(K+\mu(I)L)\cap S = \emptyset$ for some ideal I of R and submodules $K, L \leq M$. Then, $K + \mu(I)L \subseteq P$. Since $\mu(IL) = \mu(I)L \subseteq P$ and P is μ-prime, Proposition 2.4 yields that either $\mu(L) \subseteq P$ or $\mu(I)M \subseteq P$. Consequently, either $(K + \mu(L)) \cap S = \emptyset$ or $(K + \mu(I)M) \cap S = \emptyset$.

($\Leftarrow$) Assume that $M \setminus P$ is a μ-multiplicative system. Let $\mu(IL) \subseteq P$ for some ideal I of R and submodule $L \leq M$. Taking $K = 0$, we obtain $(0+\mu(I)L)\cap(M\setminus P) = \emptyset$. Hence, either $\mu(L) \subseteq P$ or $\mu(I)M \subseteq P$. Therefore, by Proposition 2.4, $P \in \mu\text{-}\mathrm{Spec}(M)$. □

Proposition 2.14. *Assume that M is an R-module and $S \subseteq M$ is a μ-multiplicative system. If P is a submodule of M which is maximal among submodules disjoint from S, then $P \in \mu\text{-}\mathrm{Spec}(M)$.*

Proof. Take a submodule $P \leq M$ satisfying $P \cap S = \emptyset$ and maximal with this property. Suppose that $\mu(IK) \subseteq P$ and $\mu(K) \nsubseteq P$, where I is an ideal of R and K is a submodule of M. In view of Proposition 2.4, it is enough to prove that $\mu(I)M \subseteq P$.

Since $P \cap S = \emptyset$ and $\mu(IK) = \mu(I)K \subseteq P$, we have $(P + \mu(I)K) \cap S = \emptyset$. Because S is a μ-multiplicative system, either $(P + \mu(K)) \cap S = \emptyset$ or $(P + \mu(I)M) \cap S = \emptyset$.

If $(P+\mu(K))\cap S = \emptyset$, then $\mu(K) \subseteq P$, a contradiction. Then, $(P+\mu(I)M)\cap S = \emptyset$, and so, $\mu(I)M \subseteq P$. □

Definition 2.3. Let $K \leq M$ be contained in at least one μ-prime submodule. Define $\sqrt[\mu]{K}$ by

$$\sqrt[\mu]{K} = \{\, m \in M \mid \text{no } \mu\text{-multiplicative system containing } m \text{ is disjoint from } K\}.$$

If no μ-prime submodule of M contains K, we put $\sqrt[\mu]{K} = M$.

Theorem 2.3. *Let $K \leq M$ with $\sqrt[\mu]{K} \neq M$. Then, $\sqrt[\mu]{K}$ is the intersection of all μ-prime submodules of M containing K.*

Proof. Assume that $\sqrt[\mu]{K} \neq M$ and take $m \in \sqrt[\mu]{K}$. Let $P \in \mu\text{-Spec}(M)$ with $K \subseteq P$. By Proposition 2.13, $S = M \setminus P$ is a μ-multiplicative system. If $m \notin P$, then $K \cap S \neq \emptyset$ by Definition 2.3, which contradicts $K \subseteq P$. Hence, $m \in P$.

For the converse, take $m \notin \sqrt[\mu]{K}$. By Definition 2.3, there exists a μ-multiplicative system S containing m such that $K \cap S = \emptyset$. Take the collection of submodules of M containing K and disjoint from S. Zorn's Lemma yields a maximal member P of this family. Thus, $K \subseteq P$, $P \cap S = \emptyset$, and P is maximal with respect to these properties. By Proposition 2.14, $P \in \mu\text{-Spec}(M)$. Since $m \in S$ and $P \cap S = \emptyset$, we obtain $m \notin P$. □

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