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DECOMPOSABLE FILTERS OF LATTICES

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Abstract. In this paper, we introduce and investigate the latticial counterpart of the filter-theoretical concepts of prime, second and irreducible. In a manner analogous to modules over a commutative ring, the main purpose of this paper is to investigate L-prime filters, L-second filters, meet-irreducible filters and special decomposable filters of lattices.

1. Introduction

The notion of an order plays an important role not only throughout mathematics but also in adjacent such as logic and computer science. The beauty of lattice theory derives in part from the extreme simplicity of its basic concepts: (partial) ordering, least upper and greatest lower bounds. In this respect, it closely resembles group theory. Thus lattices and groups provide two of the most basic tools of universal algebra, and in particular the structure of algebraic systems is usually most clearly revealed through the analysis of appropriate lattices.

The idea of investigating a mathematical structure via its representations in simpler structures is commonly used and often successful. Modules arise when the representing subjects have an additive structure: representations as endomorphisms of abelian groups and vector spaces are the main cases. A representation of a group G over a field K is the same thing as a module over the corresponding group algebra $K[G]$. A module over a ring R is "really just" an abelian group M together with a ring morphism from to the endomorphism ring of M. It is an important feature of this strategy that any single representation will tell us only a certain amount about the original structure. So, for example, one looks at the set of all irreducible characters (simple modules) of a finite group, and even then for some purposes one has to look at more general

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modules. Thus arises the project of classifying all representations or, more realistically, all representations of a certain significant type. A commonly adopted strategy is to prove a decomposition theorem which says that every representation of the sort we are considering may be built up from certain simpler ones, and then to develop a description and structure theory for these simpler building blocks. An optimal structure theory for the blocks is one which provides us with a complete list and with representations of the members of the list, which are explicit enough to allow us to answer many questions about the blocks with relatively little effort. Here we make the general point that for the vast majority of rings, the description of arbitrary module is infeasible. In particular one is interested in the description of certain significant properties of filters of a lattice rather than in arbitrary filters: the second filters seem to form such a class of filters which arise in practice and where there is hope of some kind of description. Therefore the study of properties of lattices are very important.

The theory secondary representation is a sort of dual of the theory of primary decomposition in a module over a non-trivial commutative ring was introduced in [11]. The study of second modules (so secondary modules) is much newer having been instigated in [12] and has been improved in various areas by a number of authors (see, [1], [3], [6], [8]). In fact, the set of attached prime ideals of a module contains a lot of information about the module itself. Prime modules have been studied by various authors over the past 35 years (see, for example, [1, 4, 5, 7]). The structure theory of lattices have been studied by several authors (see, [9, 10]). Decomposability is not just the algebraic properties for some filters of lattices. There exist in other algebraic areas, such as rings, modules and lattice-ordered group. Decomposable filter of a lattice is the common tool to understand these properties. One point of this paper is to investigte decomposability of filters of lattices.

Let L be a distributive lattice with 1. In this paper, we are interested in investigating L-second filters and L-prime filters to use other notions of prime, second and associate which exist in the literature as laid forth in [4, 11]. Here is a brief outline of the article. Among many results in this paper, Section 2 lists some definitions, and prove the set of all identity join of an Artinian filter F of L is the union of all the associated primes of F and prime avoidance theorem. Also, we shall establish explicit descriptions of L-second filters of L and then investigate the relationship among L-second filters, L-prime filters and and meet-irreducible filters. It is shown that if $F \neq 1$ is an Artinian filter, then F is a meet-irreducible decomposable. Some special L-representable filters of L are investigated. Section 3 is devoted to full description of all L-prime filters, L-second filters, and meet-irreducible filters of the lattice $(L(Z), \subseteq)$ ($L(Z)$ is the collection of ideals of Z, the ring of integers) with respect to the following definitions: $mZ \vee nZ = (m,n)Z$ and $mZ \wedge nZ = [m,n]Z$, for all ideals mZ and nZ of Z, where (m,n) and $[m,n]$ are greatest common divisor and least common multiple of m,n, respectively.

Let us recall some notions and notations. By a lattice we mean a poset $(L, \leq)$ in which every couple elements x, y has a g.l.b. (called the meet of x and y, and written

$x \wedge y$) and a l.u.b. (called the join of x and y, and written $x \vee y$). A lattice L is complete when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that L is a lattice with 0 and 1). A lattice L is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A non-empty subset F of a lattice L is called a filter, if for $a \in F$, $b \in L$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if L is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of L). A proper filter F of L is called prime if $x \vee y \in F$, then $x \in F$ or $y \in F$. A proper filter F of L is said to be maximal if G is a filter in L with $F \subsetneqq G$, then $G = L$. Let L be a lattice with 0 and 1 (so if F is a filter, then $0 \in F$ if and only if $F = L$). If $a \in L$, then a complement of a in L is an element $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. The lattice L is complemented if every element of L has a complement in L. We say that a filter F of L is an Artinian if any non-empty set of subfilters of F has a minimal member with respect to set inclusion. This definition is equivalent to the descending chain condition on subfilters of F. First we need the following lemma proved in [2, 9].

Lemma 1.1. *Let L be a lattice.*

(i) *A non-empty subset F of L is a filter of L if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in L$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$.*

(ii) *If $F_1, \dots, F_n$ are filters of L and $a \in L$, then $\vee_{i=1}^n F_i = \{\vee_{i=1}^n a_i : a_i \in F_i\}$ and $a \vee F_i = \{a \vee a_i : a_i \in F_i\}$ are filters of L and $\vee_{i=1}^n F_i = \bigcap_{i=1}^n F_i$.*

(iii) *If D is an arbitrary non-empty subset of L, then the set $T(D)$ consisting of all elements of L of the form $(a_1 \wedge a_2 \wedge \cdots \wedge a_n) \vee x$ (with $a_i \in D$ for all $1 \leq i \leq n$ and $x \in L$) is a filter of L containing D (so if $D = \{a\}$, then $T(\{a\}) = T(a) = \{a \vee t : t \in L\}$).*

(iv) *If L is distributive, F, G are filters of L, and $x \in L$, then $(G :_L F) = \{x \in L : x \vee F \subseteq G\}$ and $(F :_L T(x)) = (F :_L x) = \{a \in L : a \vee x \in F\}$ are filters of L.*

(v) *If $\{F_i\}_{i \in \Delta}$ is a chain of filters of L, then $\cup_{i \in \Delta} F_i$ is a filter of L.*

(vi) *If L is distributive, $G, F_1, \dots, F_n$ are filters of L, then $G \vee (\wedge_{i=1}^n F_i) = \wedge_{i=1}^n (G \vee F_i)$.*

(vii) *If L is distributive and $F_1, \dots, F_n$ are filters of L, then $\wedge_{i=1}^n F_i = \{\wedge_{i=1}^n a_i : a_i \in F_i\}$ is a filter of L.*

(viii) *If L is a complemented lattice, then every prime filter of L is a maximal filter.*

2. Meet-irreducible Filters and L-prime Filters

In this section, we collect some basic properties concerning L-second filters, L-prime filters and meet-irreducible filters and then investigate the relationship among these filters. Throughout this paper, we shall assume unless otherwise stated, that L is a distributive lattice with 1. Our starting point is the following definition.

Definition 2.1. (i) An element x of L is called identity join of filter $F \neq 1$, if there exists $y \neq 1$ of F such that $x \vee y = 1$.

(ii) A filter F of L is called L-prime in case $F \neq 1$ and $(1 :_L F) = (1 :_L G)$ for every subfilter $G \neq 1$ of F.

(iii) A filter F of L is said to be meet-irreducible if $F \neq 1$ and the meet of any two proper subfilters of F is always a proper subfilter.

(iv) A filter F of L will be called a L-second filter provided $F \neq 1$ and $(1 :_L F) = (G :_L F)$ for every proper subfilter G of F.

(v) A prime filter P of L is called an associated prime filter for $F \neq 1$, if there exists a subfilter F' of F such that $(1 :_L F') = P$ (so $F' \neq 1$).

(vi) A filter F is called uniserial if its subfilters are linearly ordered by inclusion.

The set of all identity joins of a filter F is denoted $\mathrm{Id}(F)$ and the set of all associated prime filters of F is denoted $\mathrm{Ass}(F)$. The notion of prime ideal is central to commutative ring theory. There is a theorem that says if M is a non-zero module over a Noetherian ring R, then the set of zero-divisors for M is the union of all the associated primes of M. Motivated by this result and the prime avoidance theorem the following theorem is a lattice counterpart of these theorems.

Theorem 2.1. (i) *If $F \neq 1$ is an Artinian filter of L, then the set of identity joins for F is the union of all the associated primes of F.*

(ii) *Let $F_1, \ldots, F_n$, where $n \geq 2$, be filters of L such that at most two of $F_1, \ldots, F_n$ are not prime. If F is a filter of L such that $F \subseteq \cup_{i=1}^{n} F_i$, then $F \subseteq F_j$ for some j, with $1 \leq j \leq n$.*

Proof. (i) Let $x \in \cup_{P \in \mathrm{Ass}(F)} P$. Then there exist a subfilter $F' \neq 1$ of F and $P \in \mathrm{Ass}(F)$ such that $x \in (1 :_L F') = P$; hence $x \in \mathrm{Id}(F)$. For the other containment, let $x \in \mathrm{Id}(F)$. Then there exists $f \neq 1$ such that $x \vee f = 1$; so $H = \{a \in F : a \neq 1, x \vee a = 1\} \neq \emptyset$. Consider the poset $(H, \leq)$. Let $T = \{f_i\}_{i \in \Lambda}$ be a chain of H. Then $G = \{f_i \vee L : i \in \Lambda\}$ is a non-empty set of subfilters of F which is an Artinian filter. So there is an element $1 \neq f \in H$ such that $\min(G) = f \vee L$; so $f \vee L \subseteq f_i \vee L$ for every $i \in \Lambda$. Now for each $i \in \Lambda$, there exists $b_i \in L$ such that $f = f_i \vee b_i$; hence $f_i \leq f$ for every $i \in \Lambda$. Therefore by Zorn's Lemma, H has a maximal element, say f. Since $1 \neq f = f \vee f \in f \vee L$, $f \vee L \neq 1$. As $x \in (1 :_L f \vee L)$, it suffices to show that $(1 :_L f \vee L) = P \neq L$ is prime. Let $a \vee b \in P$ and $a \notin P$. (so $a \vee f \vee L \neq 1$). It follows that $a \vee f \neq 1$ and $x \vee (a \vee f) = a \vee (x \vee f) = 1$, so $a \vee f \in H$; hence $a \vee f = f$ by maximality of f. Thus $b \vee f = b \vee a \vee f = 1$ which implies that $b \vee f \vee L = 1$, and hence $b \in P$.

(ii) The proof is by induction on n. Consider first the case in which $n = 2$. Suppose that $F \subseteq F_1 \cup F_2$ such that $F \nsubseteq F_1$; we show that $F \subseteq F_2$. Let $a \in F$ be such that $a \notin F_1$. Let $x \in F \cap F_1$. Then F is a filter gives $a \wedge x \in F \subseteq F_1 \cup F_2$; then $a \wedge x \in F_2$ so $x \in F_2$. Therefore $F \cap F_1 \subseteq F_2$. Thus $F = F \cap (F_1 \cup F_2) = (F \cap F_1) \cup (F \cap F_2) \subseteq F_2$. Assume, inductively, that

$n = k+1$, where $k \geq 2$, and that result has been proves in the case where $n = k$. Since at most two of the F_i are not prime, we can assume that F_{k+1} is prime. We claim that there is a j with $1 \leq j \leq k+1$ such that $F \subseteq \cup_{i \neq j} F_i$ $(1 \leq i \leq k+1)$. Assume to the contrary, for each $j = 1, \dots, k+1$, it is the case that $F \nsubseteq \cup_{i \neq j} F_i$ $(1 \leq i \leq k+1)$. Thus for each j with $1 \leq j \leq k+1$, there exists $a_j \in F \setminus (F_1 \cup \cdots \cup F_{j-1} \cup F_{j+1} \cup \cdots \cup F_{k+1})$; $a_j \in F_j$ for all j, with $1 \leq j \leq k+1$. Since F_{k+1} is prime, we have $\vee_{i=1}^{k} a_i \notin F_{k+1}$. Therefore $\vee_{i=1}^{k} a_i \in \cup_{i=1}^{k} F_i \setminus F_{k+1}$ and $a_{k+1} \in F_{k+1}$. Set $c = a_{k+1} \wedge (\vee_{i=1}^{k} a_i) = \vee_{i=1}^{k}(a_i \wedge a_{k+1})$. Then $c \notin F_{k+1}$ (otherwise,$\vee_{i=1}^{k} a_i \in F_{k+1}$, a contradiction). By a similar argument we can not have $c \in F_j$ for all j with $1 \leq j \leq k$. But $c \in F$ since $a_j \in F$ for $j = 1, \dots, k+1$, and so we have a contradiction to the hypothesis that $F \subseteq \cup_{i=1}^{k+1} F_i$. So there is a j with $1 \leq j \leq k+1$ such that $F \subseteq \cup_{i \neq j} F_i$ $(1 \leq i \leq k+1)$. Now by inductive hypothesis to deduce that $F \subseteq F_i$ for some i with $1 \leq i \leq k+1$. □

We now to investigate the relationship among L-second filters, L-prime filters and meet-irreducible filters.

Proposition 2.1. *Let $F \neq 1$ be a filter of L. Then the following hold.*

(i) *F is L-second if and only if for each a in L, either $a \vee F = 1$ or $a \vee F = F$.*

(ii) *F is L-second if and only if $|F| = 2$.*

(iii) *F is L- second if and only if the only subfilters of F are 1 and F itself.*

(iv) *If F is L-second, then it is meet-irreducible.*

(v) *If F is L-second, then it is L-prime.*

(vi) *If F is L-second, then $(1 :_L F)$ is a prime filter of L.*

(vii) *If F and G are L-second filters of L with $(1 :_L F) = (1 :_L G) = P$, then $G = F$.*

Proof. (i) Assume that F is a L-second filter of L and let $a \in L$ be such that $a \vee F \neq F$. Then $a \in (a \vee F :_L F) = (1 :_L F)$ gives $a \vee F = 1$. Conversely, suppose that F' is a proper subfilter of F; we show that $(F' :_L F) = (1 :_L F)$. Since the inclusion $(1 :_L F) \subseteq (F' :_L F)$ is clear, we will prove the reverse inclusion. Let $x \in (F' :_L F)$. By assumption, either $x \vee F = 1$ or $x \vee F = F$; hence $x \vee F = 1$ since F' is a proper subfilter, and so we have equality.

(ii) Assume to the contrary, $|F| \geq 3$. Then there exist $a, b \in F$ such that $a \neq b$, $a \neq 1$ and $b \neq 1$ which implies that either $a \wedge b \neq a$ or $a \wedge b \neq b$. We may assume that $a \wedge b \neq a$. It suffices to show that $a \vee F \neq 1$ and $a \vee F \neq F$. If $a \vee F = 1$, then $a = a \vee a \in a \vee F = 1$ which is a contradiction. If $a \vee F = F$, then $a \wedge b \in F$ gives $a \wedge b = a \vee f$ for some $f \in F$, and so $a \wedge b \leq a \leq a \vee f = a \wedge b$; hence $a = a \wedge b$ that is a contradiction. Thus $|F| = 2$. Conversely, suppose, $F = \{1, a\}$. Then for every $b \in L$ we have $b \vee a = 1$ or $b \vee a = a$; therefore $b \vee F = 1$ or $b \vee F = F$, as required.

(iii) Let the only subfilters of F are 1 and F itself and $a \in L$. Since $a \vee F$ is a subfilter of F, we get $a \vee F = 1$ or $a \vee F = F$. Therefore F is a L-second filter. The other implication follows from (ii).

(iv) Apply (ii) and (iii).
(v) Let $a \in L$ and $x \in F$ such that $a \vee x = 1$ and $a \vee F \neq 1$. Then $a \vee F = F$ since F is a L-second filter. It follows that $x = a \vee y$ for some $y \in F$ which implies that $x = a \vee y = a \vee (a \vee y) = a \vee x = 1$. Thus F is L-prime.
(vi) Since $F \neq 1$, $(1 :_L F) \neq L$. Let $x \vee y \in (1 :_L F)$ with $x \notin (1 :_L F)$, so $x \vee F = F$. Then $(x \vee y) \vee F = (x \vee F) \vee (y \vee F) = F \vee (y \vee F) = 1$ gives $y \vee F = 1$. Thus $(1 :_L F)$ is prime.
(vii) By (ii) and (iii), set $F = \{1, a\}$ and $G = \{1, b\}$. An inspection will show that $a \vee F = F$ and $b \vee G = G$ which implies that $a, b \notin P$; hence $a \vee G \neq 1$ and $b \vee F \neq 1$. Now by (iii), F is a L-second gives $b \vee F = F$, and so $b \leq a \vee b = a$. Similarly, G is a L-second gives $a \leq a \vee b = b$. Thus $F = G$. □

Let F be a L-second filter of L. If this the case, then $(1 :_L F) = P$ is prime by Proposition 2.1 (vi), and we say that F is P-L-second. A L-second representation of a filter F of L is obtained by writing F as the meet of a finite number of L-second filters of L, say $F = \wedge_{i=1}^n F_i$. The representation is said to be minimal if the prime filters $(1 :_L F_i) = P_i$ are distinct and none of the subfilters F_i of F, is redundant. Any L-second representation of F can be refind to a minimal one (see Proposition 2.3 (vii). The set of prime filters attached to the filter F over the lattice L is denoted by $\mathrm{Att}_L(F)$. If $F = \wedge_{i=1}^n F_i$ is a minimal L-second representation of F, where F_i is P_i-L-second, then $\mathrm{Att}_L(F) = \{P_1, \ldots, P_n\}$ and thus $\mathrm{Att}_L(F)$ is a finite set whenever F has a L-second representation. If such representation exists, we shall say F is L-representable.

Example 2.1. Let $D = \{1, \ldots, n\}$. Then the set $L(D) = \{X : X \subseteq D\}$ forms a complemented distributive lattice under set inclusion with greatest element D and least element $\emptyset$ (note that if $x, y \in L(D)$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$). The set of all prime filters (resp. maximal filters) of $L(D)$ is denoted by $\mathrm{Spec}(L(D))$ (resp. $\mathrm{Max}(L(D))$). Then the following hold.

(i) If a is a minimal element of a filter F of a lattice L, then we show that $F = a \vee F$. Clearly, $a \vee F \subseteq F$. For the reverse inclusion, let $x \in F$. Since $a \wedge x \in F$, $a \wedge x = a \leq x$ by minimality of a. As $a = a \vee a \in a \vee F$ and it is a filter, we get $x \in a \vee F$, and so we have equality.
(ii) If F is a filter of $L(D)$, then there exists a subset $\mathbf{a}$ of D such that $F = \mathbf{a} \vee L(D)$ (since F is a finite set, the results follows from (i)).
(iii) We show that every prime filter F of $L(D)$ is of the form $\{m\} \vee L(D)$, where $m \in D$. By Lemma 1.1 (viii), it suffices to show that they are maximal. First we show that $\{m\} \vee L(D)$ $(m \in D)$ is a maximal filter. Since $\emptyset \notin \{m\} \vee L(D)$, $\{m\} \vee L(D) \neq L(D)$. Let G be a filter of $L(D)$ such that $\{m\} \vee L(D) \subsetneqq G \subseteq L(D)$; we show that $G = L(D)$. Then there is a subset $\mathbf{a}$ of D such that $\mathbf{a} \in G$ and $m \notin \mathbf{a}$. Since $\{m\} \in G$, G is a filter gives $\{m\} \wedge \mathbf{a} = \{m\} \cap \mathbf{a} = \emptyset \in G$, and so $G = L(D)$. Let F be a maximal filter of $L(D)$. Then $F = \mathbf{b} \vee L(D)$ for some subset $\mathbf{b}$ of D by (ii). It is enough to show that $|\mathbf{b}| = 1$. Assume

to the contrary, $|\mathbf{b}| \geq 2$. Then there exist $i_1, i_2 \in \mathbf{b}$ with $i_1 \neq i_2$. Therefore we have $\mathbf{b} \vee L(D) \subsetneqq \{i_1\} \vee L(D) \subsetneqq L(D)$ that is a contradiction. Thus $\mathrm{Max}(L(D)) = \mathrm{Spec}(L(D)) = \{\{m\} \vee L : m \in D\}$.

(iv) If $F \neq 1$ is a filter of $L(D)$, then we prove that F is L-representable and there exists a subset $\mathbf{b}$ of D such that $\mathrm{Att}_L(F) = \{\{n\} \vee L(D) : n \notin \mathbf{b}\}$. Clearly, for each $m \in D$, the set $\{D, D \setminus \{m\}\}$ is a L-second filter of $L(D)$. Conversely, assume that G is a L-second filter of $L(D)$. Then $G = \{D, X\}$ by Proposition 2.1. Since $X \neq D$, there are elements $m_1, \dots, m_k \in D$ such that $X = D \setminus \{m_1, \dots, m_k\}$. If $k \neq 1$, then $X \subsetneqq D \setminus \{m_1\} \neq D$; hence $D \setminus \{m_1\} \in G$ since G is a filter that is a contradiction. Thus a filter G of $L(D)$ is L-second if and only if $G = \{D, D \setminus \{m\}\}$ for some $m \in D$. Let F be a filter of $L(D)$ with $F \neq 1$. By (ii), $F = \mathbf{a} \vee L(D)$ for some subset $\mathbf{a}$ of D. Now we show that H is a L-second subfilter of F if and only if $H = \{D, D \setminus \{n\}\}$ for some $n \notin \mathbf{a}$. If H is L-second, then $H = \{D, D \setminus \{s\}\}$. As $F = \{X \subseteq D : \mathbf{a} \subseteq X\}$, we get $\mathbf{a} \subseteq D \setminus \{s\}$; thus $s \notin \mathbf{a}$. The other implication is clear. Set $\Lambda = \{i \in D : i \notin \mathbf{a}\}$ and $F_i = \{D, D \setminus \{i\}\}$, where $i \in \Lambda$. Then it is enough to show that $F = \wedge_{i \in \Lambda} F_i$. It is clear that $\wedge_{i \in \Lambda} F_i \subseteq F$. For the other containment, let $X \in F$. If $X = D$, we are done. Otherwise, there exist $i_1, \dots, i_k \in D$ such that $X = D \setminus \{i_1, \dots, i_k\}$. Since $\mathbf{a} \subseteq X$, $i_1, \dots, i_k \notin \mathbf{a}$. As $\wedge_{j=1}^{k}(D \setminus \{i_j\}) = \cap_{j=1}^{k}(D \setminus \{i_j\}) = D \setminus \{i_1, \dots, i_k\} = X$ and $\wedge_{j=1}^{k}(D \setminus \{i_j\}) \in \wedge_{i \in \Lambda}(\{D, D \setminus \{i\}\})$, we get $X \in \wedge_{i \in \Lambda}(\{D, D \setminus \{i\}\})$, and so we have equality. Now for each $i \in \Lambda$, we show that $(1 :_L F_i) = \{i\} \vee L(D)$ (see (iii)). Let $X \in \{i\} \vee L(D)$ (so $i \in X$). Then $X \vee F_i = \{D, X \cup D \setminus \{i\}\} = \{D\}$; so $\{i\} \vee L(D) \subseteq (1 :_L F_i)$. The reverse inclusion is clear, and so we have equality. If $i, j \in \Lambda$ with $i \neq j$, then $\{i\} \vee L(D) \neq \{j\} \vee L(D)$ since $\{i\} \vee L(D)$ is the set of subsets of D contains i. Thus F is L-representable with $\mathrm{Att}_L(F) = \{\{i\} \vee L(D) : i \in \Lambda\}$.

Let $F \neq 1$ be a filter of L. A meet-irreducible decomposition F is an expression for F as a meet of finitely many meet-irreducible filters of L. We say that F is a meet-irreducible decomposable filter of L precisely when it has a meet-irreducible decomposition.

Theorem 2.2. *Let F be an Artinian filter of L with $F \neq 1$. Then the following hold.*

(i) *F contains only a finite number of L-second subfilters.*

(ii) *F is a meet-irreducible decomposable.*

(iii) *If L is a complemented lattice, then F is a L-representable.*

Proof. (i) Suppose the result is false. Let F be an Artinian filter of L with $F \neq 1$ such that F does not contain a finite number of L-second subfilters. Let $H \neq 1$ be a subfilter of F minimal with respect to the property that H does not contain a finite number of L-second subfilters. (So H is not a L-second filter). So there exists $a \in L$ such that $a \vee H \neq 1$ and $a \vee H \subsetneqq H$

by Proposition 2.1. Set $K = \{h \in H : a \vee h = 1\}$. If $x, y \in K$ and $z \in L$, then $(x \wedge y) \vee a = (x \vee a) \wedge (y \vee a) = 1$ and $(x \vee z) \vee a = 1$ gives K is a subfilter of H by Lemma 1.1 (i) such that $K \vee a = 1$, and so $K \subsetneqq H$. If $K \neq 1$, then K contains only a finite number of L-second subfilters $K_1, \dots, K_m$ and $H \vee a$ contains only a finite number of L-second subfilters $H_1, \dots, H_n$. Let X be a L-second subfilter of H. Then $a \vee X = 1$ or $a \vee X = X$. If $a \vee X = 1$, then $X \subseteq K$ and hence $X = K_i$ for some i. Similarly, if $a \vee X = X$, then $X = H_j$ for some j. Therefore every L-second subfilter of H belongs to the list $K_1, \dots, K_m, H_1, \dots, H_n$ of subfilters of H. Thus H has at most $n + m$ of L-second subfilters which is a contradiction. If $K = 1$, then $X = H_j$ for some j and again H has at most a finite number of L-second subfilters, a contradiction.

(ii) Assume to the contrary, F is not meet-irreducible decomposable. Then the set of subfilters say $G \neq 1$ of F which are not a finite meet of meet-irreducible filters has a minimal element H. Clearly H is not meet-irreducible. Thus H is the meet of two strictly smaller subfilters H_1 , H_2. Then H_1 , H_2 are a finite meet of meet-irreducible filters by minimality of H. So H is a finite meet of meet-irreducible filters, that is a contradiction.

(iii) By (ii), it is enough show that if G is a meet-irreducible filter, then G is a L-second filter. Assume to the contrary, G is not L-second. Then by Proposition 2.1, $a \vee G \neq G$ and $a \vee G \neq 1$ for some $a \in L$. Set $G_1 = \{x \in G : a \vee x = 1\}$ and $G_2 = a \vee G$. Clearly, $G \neq G_1$ and $G \neq G_2$. Let $y \in G$. By assumption, $a \wedge a' = 0$ and $a \vee a' = 1$ for some $a' \in L$. Then we have $y = (y \vee a') \wedge (a \vee y)$, where $y \vee a' \in G_1$ and $a \vee y \in G_2$; hence $G = G_1 \wedge G_2$, and therefore G is not meet-irreducible, as required. □

The next proposition gives a more explicit description of L-prime filters.

Proposition 2.2. *Let $F \neq 1$ be a filter of L. Then the following hold.*

(i) *F is L-prime if and only if for any a in L and x in F, $a \vee x = 1$ implies that $x = 1$ or $a \vee F = 1$.*

(ii) *F is L-prime if only if $(1 :_L F)$ is a prime filter of L.*

(iii) *If $\{1\}$ is a prime filter, then F is L-prime.*

(iv) *If $P \neq 1$ is a prime filter with $(1 :_L P) \neq 1$, then there exists a L-prime filter F such that $(1 :_L F) = P$.*

(v) *If there exist $x, y \in F$ such that $x \neq 1$ and $y \neq 1$ with $x \vee y = 1$, then F is not L-prime.*

(vi) *If L is complemented and $|F| \geq 3$, then F is not a L-prime filter.*

Proof. (i) Assume that F is a L-prime filter of L and let $a \vee x = 1$ with $x \neq 1$, where $a \in L$ and $x \in F$. Then $T(x)$ is a subfilter of F with $T(x) \neq 1$; so $(1 :_L F) = (1 :_L T(x))$. As $a \vee T(x) = 1$, we have $a \vee F = 1$. Conversely, suppose that F' is a subfilter of F such that $F' \neq 1$; we show that $(1 :_L F) = (1 :_L F')$.

Clearly, $(1 :_L F) \subseteq (1 :_L F')$. For the other containment, let $z \in (1 :_L F')$. There exists $1 \neq w \in F'$ such that $z \vee w = 1$ which implies that $z \vee F = 1$; hence $z \in (1 :_L F)$, as required.

(ii) Since $F \neq 1$, $(1 :_L F) \neq L$. Let $x \vee y \in (1 :_L F)$ be such that $x \notin (1 :_L F)$, so $1 \neq x \vee a \in F$ for some $a \in F$ which implies that $y \vee (x \vee a) = 1$. Then F is L-prime gives $y \vee F = 1$. Conversely, suppose that $(1 :_L F) = P$ is a prime filter and $F' \neq 1$ is a proper subfilter of F; we show that $(1 :_L F') = (1 :_L F)$. Clearly, $(1 :_L F) \subseteq (1 :_L F')$. Let $1 \neq f' \in F'$. Then $f' = f' \vee f' \in f' \vee F' \subseteq f' \vee F$ gives $f' \notin P$. Let $x \in (1 :_L F')$. Then $x \vee f' = 1 \in P$ gives $x \in P$, and so we have equality. Thus F is a L-prime filter.

(iii) By (ii), It suffices to show that $(1 :_L F) = \{1\}$. Let $x \in (1 :_L F)$. Then there exists $1 \neq a \in F$ with $x \vee a \in \{1\}$. Now $\{1\}$ is a prime filter gives $x \in \{1\}$, and so we have equality.

(iv) Set $F = (1 :_L P)$. By (ii), it suffices to show that $(1 :_L F) = P$. Since the inclusion $P \subseteq (1 :_L F)$ is clear, we will prove the reverse inclusion. Let $x \in (1 :_L F)$. By assumption, there is an element $1 \neq f \in F$ with $x \vee f = 1 \in P$ (so $f \vee F \neq 1$); hence $f \notin P$. Now P is prime gives $x \in P$, as required.

(v) Assume to the contrary, F is L-prime. Since $x \vee y = 1$, either $x = 1$ or $y \vee F = 1$. So $y \vee F = 1$. Now $y = y \vee y \in y \vee F$ gives $y = 1$ that is a contradiction. Thus F is not a L-prime.

(vi) Suppose the result is false. Then $G = (1 :_L F)$ is a prime filter. Let x and y be distinct elements of F such that $x \neq 1$ and $y \neq 1$. By assumption, there exist $x', y' \in L$ such that $x \wedge x' = 0 = y \wedge y'$ and $x \vee x' = 1 = y \vee y'$. It follows that either $x \in G$ and $x' \notin G$ or $x \notin G$ and $x' \in G$ since G is a prime filter. Since $1 \neq x = x \vee x \notin G$, we get $x' \in G$. Similarly, $y' \in G$. Thus $x' \wedge y' \in G$; hence $(x' \wedge y') \vee (x \wedge y) = 1$ since $x \wedge y \in F$. It then follows that $x = x \wedge 1 = x \wedge [(x' \wedge y') \vee (x \wedge y)] = x \wedge (x \wedge y) = x \wedge y$. Similarly, $y = x \wedge y = x$ which is a contradiction. □

Let F be a L-prime filter of L. If this is the case, then $P = (1 :_L F)$ is a prime filter by Proposition 2.2, and we say that F is a P-L-prime filter. Also it is easy to see that every subfilter $G \neq 1$ of a L-prime filter F is L-prime.

Assume that F is a filter of L and let $(F_i)_{i \in \Lambda}$ be a non-empty family of subfilters of F. If $G = \wedge_{i \in \Lambda} F_i$, then each element $g \in G$ can be expressed in the form $g = \wedge_{i=1}^{n} f_{\lambda_i}$, where $\{\lambda_1, \ldots, \lambda_n\}$ is a finite subset of Λ and $f_{\lambda_i} \in F_{\lambda_i}$ for $1 \leq i \leq n$. We can actually write this as $g = \wedge_{\lambda \in \Lambda} f_\lambda$, where it is understood that $f_\lambda \in F_\lambda$ for all $\lambda \in \Lambda$ but only finitely many of the f_λ are different from 1. So G is a subfilter of F. A L-prime subfilter G of a filter F of L is called a maximal L-prime subfilter if G is not contained in another L-prime subfilter of F.

Proposition 2.3. *The following hold.*

(i) *Let P be a prime filter of L. Then the meet of any non-empty family of P-L-prime subfilters of a filter F of L is also a P-L-prime subfilter of F.*

(ii) *If F_1 and F_2 are maximal P-L-prime subfilters of F, then $F_1 = F_2$.*
(iii) *Every L-prime subfilter of a filter F of L is contained in a uniqe maximal L-prime subfilter of F.*

Proof. (i) Suppose that $(F_i)_{i\in\Lambda}$ is a non-empty family of P-L-prime subfilters of F and let $G = \wedge_{i\in\Lambda} F_i$. Then $P \subseteq (1 :_L G) \subseteq (1 :_L F_i) = P$ for all $i \in \Lambda$, gives $(1 :_L G) = P$. Now the assumption follows from Proposition 2.2 (ii).
(ii) By (i), $F_1 \wedge F_2$ is a P-L-prime filter. Since $F_1 \subseteq F_1 \wedge F_2$, $F_2 \subseteq F_1 \wedge F_2$, and$F_1$, F_2 are maximal P-L-prime, we get $F_1 = F_1 \wedge F_2 = F_2$.
(iii) Assume that G is a L-prime subfilter of F and let Δ be the set of L-prime subfilters of F, say F', with $G \subseteq F'$. So $G \in \Delta$. Of course, the relation of inclusion, $\subseteq$, is a partial order on Δ. Let $(F_i)_{i\in\Lambda}$ be a chain of elements Δ. Set $H = \cup_{i\in\Lambda} F_i$ (so $1 \neq G \subseteq H$). Clearly, H is a subfilter of F. Let $a \in L$ and $x \in H$ such that $a \vee x = 1$. Then $x \in F_i$ for some i, and so F_i is a L-prime gives either $x = 1$ or $a \vee F_i = 1$. If $x = 1$, then we are done. So suppose that $x \neq 1$ and $a \vee F_i = 1$. It follows that $a \vee H = 1$. Thus H is a L-prime subfilter of F. Now Δ is easily seen to be inductive under $\subseteq$, so by Zorn's Lemma Δ has a maximal element, i.e., F has a maximal L-prime subfilter contains G which is unique by (ii). □

Theorem 2.3. *Let F be an Artinian filter with $F \neq 1$. Then the following hold.*
(i) *F contains only a finite number of maximal L-prime subfilter.*
(ii) *A filter F of L is L-prime if and only if* $\mathrm{Ass}(F) = \{P\}$ *for some prime filter P.*

Proof. (i) We know that every maximal L-prime subfilter of F cotains a L-second subfilter (since F is an Artinian filter) which is a L-prime subfilter by Proposition 2.1 (v). Now by Proposition 2.3 (iii), every L-second subfilter of F is contained in a uniqe maximal L-prime subfilter. Now the assertion follows from Theorem 2.2 (i).
(ii) Clearly, if F is L-prime, then $\mathrm{Ass}(F) = \{(1 :_L F)\}$. Conversely, let $\mathrm{Ass}(F) = \{P\}$. By Proposition 2.2 (ii), it suffices to show that $(1 :_L F) = P$. Since $P \in \mathrm{Ass}(F)$, there exists a subfilter $F' \neq 1$ of F such that $(1 :_L F) \subseteq (1 :_L F') = P$. For the reverse inclusion, let $x \in P$. If $x \vee F \neq 1$, then $x \vee F$ has a L-second subfilter G (since $x \vee F$ is an Artinian filter). By Proposition 2.1, $|G| = 2$, so we can write $G = \{1, x \vee f\}$, where $f \in F$. As G is a L-prime filter by Proposition 2.2, so $(1 :_L G) = P \in \mathrm{Ass}(F)$. Then $x \in P$ gives $x \vee G = 1$, so $x \vee f = 1$, which is a contradiction; hence $x \vee F = 1$, and so we have equality. □

Theorem 2.4. *If $F \neq 1$ is an uniserial filter, then F is a meet-irreducible and a L-prime.*

Proof. Clearly, F is a meet-irreducible filter. Now we show that F is a L-prime filter. Suppose F is not L-prime. Then there exists a proper subfilter $F' \neq 1$ of F such that $(1 :_L F) \subsetneqq (1 :_L F')$. Then there exsits an element $x \in (1 :_L F')$ such that $x \notin (1 :_L F)$

(so $x \vee F \neq 1$). By assumption, either $x \vee F \subseteq F'$ or $F' \subseteq x \vee F$. If $x \vee F \subseteq F'$, then $x \vee (x \vee F) \subseteq x \vee F' = 1$ gives $x \vee F = 1$ that is a controdiction. Now we may assume that $F' \subseteq x \vee F$. Let $f' \in F'$. Then $f' = x \vee f$ for some $f \in F$; hence $1 = x \vee f' = x \vee (x \vee f) = x \vee f = 1$ which is a contradiction since $F' \neq 1$. Thus F is a L-prime filter. □

The following Example shows that L-prime filters and meet-irreducible filters are different concepts.

Example 2.2. (a) let L be the lattice as described in Figure 1.

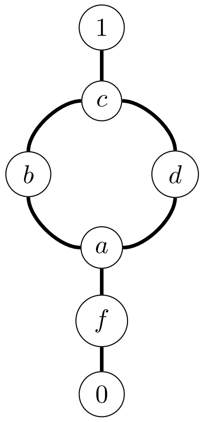

FIGURE 1. Lattice L

(i) Clearly, the filter $\{1\}$ is prime. Then the filter $F = \{1, c, d, b, a\}$ of L is L-prime by Proposition 2.2 (iii). An inspection will show that $F_1 = \{1, c\}$, $F_2 = \{1, c, d\}$, and $F_3 = \{1, c, b\}$ are all of proper subfilters of F with $F_i \neq 1$ $(1 \leq i \leq 3)$. Since $F = F_2 \wedge F_3$, we get F is not a meet-irreducible filter. Moreover, $F_2 \nsubseteq F_3$ and $F_3 \nsubseteq F_2$ gives F is not an uniserial filter. Thus a L-prime filter need not be meet-irreducible (resp. uniserial) filter.

(ii) Clearly, the filter $G = \{1, c, d, b, a, f\}$ of L is a L-prime filter and a meet-irreducible filter, but it is not an uniserial filter (note that F_1, F_2, F_3, F are all of proper subfilters of G not equal to 1). Thus the converse of Theorem 2.4 is not true. Moreover, since F is a subfilter of the meet-irreducible filter G, we obtain that a subfilter of a meet-irreducible filter, in general, is not meet irredeucible.

(b) let L' be the lattice as described in Figure 2. Consider the filter $F = \{1, a, b, c, d\}$ of L'. Since $a \vee b = 1$, F is not a L-prime filter by Proposition 2.2 (v). Clearly, F is a meet-irreducible filter, but it is not uniserial.

Example 2.3. (a) Let $(L, \leq)$ be a complete lattice which is a chain.

(i) If F is a proper filter of L, then there exists $b \in L$ with $b \notin F$; so $H = \{x \in L : x \notin F\} \neq \emptyset$. Assume that $\sup(H) = a$ and let $x \in L$. If

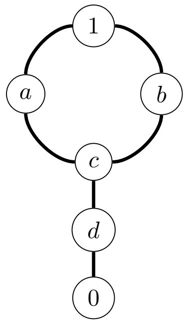

Figure 2. Lattice L'

$a \leq x$ with $a \neq x$, then $x \in F$ (otherwise, $x \notin F$ gives $x \in H$ that is a contradiction). If $x \leq a$ with $a \neq x$, then there exists $h \in H$ such that $x \leq h$ since L is a chain which implies that $x \notin F$ (otherwise, F is a filter gives $h \in F$, a contradiction). Thus F is of the form either $[a, 1]$ or $(a, 1]$.

(ii) Since L is a chain, every proper filter of L is prime (so $\{1\}$ is prime). Moreover, the only maximal filter of L is $(0, 1]$.

(iii) By Proposition 2.2 (iii), every filter $F \neq 1$ of L is L-prime.

(b) Assume that R is a local Dedekind domain with unique maximal ideal $P = Rp$ and let $E = E(R/P)$, the R-injective hull of R/P. For each positive integer n, set $A_n = (0 :_E P^n)$. Then by [8, Lemma 2.6], every non-zero proper submodule of E is equal to A_m for some m with a strictly increasing sequence of submodules $A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots$. The collection of submodules of E form a complete lattice which is a chain under set inclusion which we shall denote by $L(E)$ with respect to the following definitions: $A_n \vee A_m = A_n + A_m$ and $A_n \wedge A_m = A_n \cap A_m$ for all submodules A_n and A_m of E. Then by (a), the following hold.

(i) Every proper filter of $L(E)$ is of the form $[A_n, E]$ for some n.

(ii) Every proper filter of $L(E)$ is prime. Moreover, the only maximal filter of $L(E)$ is $[A_1, E]$.

(iii) Every filter $F \neq 1$ is $L(E)$-prime.

(iv) For every filter $F \neq 1$, F is not L-second (otherwise, $F = \{A_n, E\}$ for some n by Proposition 2.1, but $A_{n+1} \vee F \neq F$ and $A_{n+1} \vee F \neq 1$).

3. Further Results

The collection of ideals of Z, the ring of integers, form a lattice under set inclusion which we shall denote by $L(Z)$ with respect to the following definitions: $mZ \vee nZ = (m, n)Z$ and $mZ \wedge nZ = [m, n]Z$ for all ideals mZ and nZ of Z, where (m, n) and $[m, n]$ are greatest common divisor and least common multiple of m, n, respectively. Note that $L(Z)$ is a distributive complete lattice with least element the zero ideal and the greatest element Z. In this section we give a full description of all L-prime filters,

L-second filters, and meet-irreducible filters of $L(Z)$. First we need the following lemma proved in [9, Theorem 2.9].

Lemma 3.1. *For the lattice $L(Z)$ the following hold.*

(i) *If p is a prime number and k is a positive integer, then the set $F_{p^k} = \{mZ \in L(Z) : p^k \nmid m\}$ is a prime filter of $L(Z)$. Moreover, $L(Z) \setminus \{0\}$ is the only maximal filter of $L(Z)$.*

(ii) *Every prime filter of $L(Z)$ is of the form either F_{p^k} for some prime number p and positive integer k or $L(Z) \setminus \{0\}$.*

Theorem 3.1. *Let $\mathbf{P}$ be the set of all prime numbers. Then the following hold.*

(i) *The set of all $L(Z)$-prime filters of $L(Z)$ is*

$$(\cup_{p\in\mathbf{P}}\{\{Z, pZ, p^2Z, \dots\}\}) \cup (\cup_{p\in\mathbf{P}}\{\{Z, pZ, p^2Z, \dots, p^nZ\} : n \in N\})$$

(ii) *The set of all $L(Z)$-second filters of $L(Z)$ is $\cup_{p\in\mathbf{P}}\{\{Z, pZ\}\}$.*

(iii) *The set of all meet-irreducible filters of $L(Z)$ is*

$$(\cup_{p\in\mathbf{P}}\{\{Z, pZ, p^2Z, \dots\}\}) \cup (\cup_{p\in\mathbf{P}}\{\{Z, pZ, p^2Z, \dots, p^nZ\} : n \in N\})$$

Proof. (i) First we show that if p is a prime number, then $F = \{Z, pZ, p^2Z, \dots\}$ is a L-prime filter of $L(Z)$. Clearly, F is a filter of $L(Z)$. Let $mZ \in L(Z)$ and $p^lZ \in F$ for some positive integer l with $mZ \vee p^lZ = Z$ and $p^lZ \neq Z$; we show that $mZ \vee F = Z$. By assumption, $(m, p^l) = 1$ gives $(m, p^k) = 1$ for all positive integer k; hence $mZ \vee F = Z$. Thus F is L-prime by Proposition 2.2 (i). Next we prove that for every $L(Z)$-prime filter G of $L(Z)$, there exists a prime number p such that G is a subfilter of $F = \{Z, pZ, p^2Z, \dots\}$. To see that, let G be a $L(Z)$-prime filter of $L(Z)$, so $(1 :_{L(Z)} G)$ is a prime filter which implies that either $(1 :_{L(Z)} G) = F_{p^k}$ or $(1 :_{L(Z)} G) = L(Z) \setminus \{0\}$ for some prime number p by Lemma 3.1 (ii). Let $Z \neq mZ \in G$. If $(1 :_{L(Z)} G) = F_{p^k}$, then for each prime $q \neq p$, $qZ \in F_{p^k}$ implies that $qZ \vee mZ = Z$; so $(q, m) = 1$. Therefor $m = p^s$ for some s, and hence $mZ = p^sZ$ (note that in this case $(1 :_{L(Z)} G) = F_p$). If $(1 :_{L(Z)} G) = L(Z) \setminus \{0\}$, then $mZ \vee G = 1$ gives $G = 1$ which is a contradiction. If G is a proper subfilter of F, then there exists $p^tZ \in F$ with $t \geq 1$ and $p^tZ \notin G$. Let $m + 1$ be the least positive integer such that $p^{m+1}Z \notin G$; so $p^mZ \in G$. Then for each $t \leq m$, $p^mZ \subseteq p^tZ$; hence $p^tZ \in G$ since G is a filter. Thus $G = \{Z, pZ, \dots, p^mZ\}$. An inspection will show that for each $p \in \mathbf{P}$, every proper subfilter ($\neq 1$) of $F = \{Z, pZ, p^2Z, \dots\}$ is equal to $\{Z, pZ, \dots, p^mZ\}$ for some integer m with a strictly increasing sequence of subfilters $\{Z, pZ\} \subset \cdots \subset \{Z, pZ, \dots, p^mZ\} \subset \cdots$ which are L-prime Since every subfilter ($\neq 1$) of a L-prime filter F is L-prime, and the proof is complete.

(ii) It suffices to show that a filter $G \neq 1$ is $L(Z)$-second if and only if $G = \{Z, pZ\}$ for some prime number p. Assume that $G = \{Z, pZ\}$ and let $mZ \in L(Z)$. If $(m, p) = 1$, then $mZ \vee G = Z$. If $p \mid m$, then $mZ \vee Z = Z$ and $mZ \vee pZ = pZ$, and so $mZ \vee G = G$. Thus G is a $L(Z)$-second filter by Proposition 2.1. Now assume that F is a L-second filter of $L(Z)$; we show that there is a prime number q such that $F = \{Z, qZ\}$. By Proposition 2.1 (ii), we have $F = \{Z, mZ\}$ for some integer m. If m is prime, we are done. Otherwise, $m = d_1 d_2$ for some integers $1 < d_1 < m$ and $1 < d_2 < m$, so $mZ \subsetneqq d_1 Z \neq Z$; hence $d_1 Z \in F$ (since F is a filter) which is a contradiction.

(iii) By (i), it is enough to show that a filter of $L(Z)$ is meet-irreducible if and only if it is $L(Z)$-prime. By the structure of $L(Z)$-prime filters, we get every $L(Z)$-prime is meet-irreducible. Conversely, suppose that $F \neq 1$ is a meet-irreducible filter; we show that F is $L(Z)$-prime. Assume to the contrary, F is not L-prime. Then there exist distinct prime numbers p, q and $mZ, nZ \in F$ such that $p \mid m$, $q \mid n$. First we show that $F_p \wedge F_q = L(Z) \setminus \{0\}$. Clearly, $F_p \wedge F_q \subseteq L(Z) \setminus \{0\}$. For the reverse inclusion, assume that $kZ \in L(Z) \setminus \{0\}$. There exist integers $s \geq 0$, $t \geq 0$, and u such that $k = p^s q^t u$ with $(u, pq) = 1$. Then $q^t Z \in F_p$, $p^s u Z \in F_q$, and $p^s uZ \wedge q^t Z = [q^t, p^s u]Z = kZ \in F_p \wedge F_q$, and so we have equality. Now set $F_1 = F_p \vee F = F_p \cap F$ and $F_2 = F_q \vee F = F_q \cap F$. Since $q \mid n$, $nZ \subseteq qZ$; hence F is filter gives $qZ \in F$, but $qZ \notin F_2$. Thus $F \neq F_2$. Similarly, $F \neq F_1$. Therefore we have $F_1 \wedge F_2 = (F_p \wedge F_q) \vee F = (L(Z) \setminus \{0\}) \cap F = F$ that is a contradiction since F is meet-irreducible. Thus F is $L(Z)$-prime. □

Remark 3.1. (i) By Theorem 3.1, every L-second filter of $L(Z)$ is of the form $G = \{Z, pZ\}$ for some p, so we conclude that a $L(Z)$-prime (resp. meet-irreducible) filter need not be L-second.

(ii) By Lemma 3.1 (i), $F_2 = \{mz \in L(Z) : 2 \nmid m\}$ is a prime filter. Set $H = \{Z, 3Z\}$ and $G = \{Z, 5z\}$. Then H, G are subfilters of F_2 with $(1 :_{L(Z)} H) = F_3$, $(1 :_{L(Z)} G) = F_5$, and $F_3 \neq F_5$; hence F_2 is not a $L(Z)$-prime. Moreover, $F = \{Z, 2Z\}$ is a $L(Z)$-prime by Theorem 3.1, but $10Z \vee 6Z = 2Z \in F$ and $6Z, 10Z \notin F$ gives F is not a prime filter. Thus L-prime (resp. meet-irreducible) filters and prime filters are different concepts.

(iii) If $P = L(Z) \setminus \{0\}$, then $(1 :_L P) = 1$, so $(1 :_L P)$ is not L-prime. Thus it shows that the condition $(1 :_L P) \neq 1$ is necessary in Proposition 2.2 (iv).

Theorem 3.2. *If $m \geq 2$ is an integer number, then $G = \{dZ \in L(Z) : d \mid m\}$ is a meet-irreducible decomposable filter of $L(Z)$.*

Proof. There are distinct prime numbers $p_1, p_2, \ldots, p_n$ and positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$. For each $1 \leq i \leq n$, we set $G_i = \{Z, p_i Z, p_i^2 Z, \ldots, p_i^{\alpha_i} Z\}$. Then G_i is a meet-irreducible subfilter of G by Theorem 3.1 for all $1 \leq i \leq n$. Now we show that $G = \wedge_{i=1}^n G_i$. Since the inclusion $\wedge_{i=1}^n G_i \subseteq G$ is clear, we will prove the reverse inclusion. Let $dZ \in G$. There exist integers $\beta_1, \beta_2, \ldots, \beta_n$ such that $d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$ with $0 \leq \beta_i \leq \alpha_i$ $(1 \leq i \leq n)$. Then $p_i^{\beta_i} Z \in G_i$ $(1 \leq i \leq n)$ gives $\wedge_{i=1}^n p_i^{\beta_i} Z = [p_1^{\beta_1}, p_2^{\beta_2}, \ldots, p_n^{\beta_n}]Z = dZ \in \wedge_{i=1}^n G_i$, and so we have equality. Thus G is a meet-irreducible decomposable. □

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