Kragujevac Journal of Mathematics
Volume 42(4) (2018), Pages 579–590.

HYERS-ULAM-RASSIAS STABILITY FOR THE LINEAR ORDINARY DIFFERENTIAL EQUATION OF THIRD ORDER

R. MURALI[1] AND A. PONMANA SELVAN[1]

Abstract. The Hyers-Ulam stability of the Ordinary Differential Equations has been investigated and the investigation is ongoing. In this paper, by applying initial condition, we investigate the approximate solutions of the homogeneous and non-homogeneous linear differential equation in the sense of Hyers-Ulam-Rassias.

1. Introduction

In 1940, S. M. Ulam [25] gave a wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. It motivated the study of stability problems for various functional equations. Among the problem raised by S.M. Ulam, the following question is concerned about the stability of homomorphisms.

Theorem 1.1. *Let G_1 be a group and let G_2 be a group endowed with a metric ρ. Given $\epsilon > 0$, does there exists a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies*

$$\rho(f(xy), f(x)f(y)) < \delta,$$

for all $x, y \in G$, then we can find a homomorphism $h : G_1 \to G_2$ exists with

$$\rho(f(x), h(x)) < \epsilon,$$

for all $x \in G_1$?

Key words and phrases. Hyers-Ulam-Rassias stability, linear differential equation, initial condition, homogeneous and non-homogeneous.

2010 *Mathematics Subject Classification.* Primary: 34K20, 26D10. Secondary: 34A30, 34A40, 39B82, 44A10, 39A10, 34C20, 45D05.

Received: May 12, 2017.

Accepted: July 24, 2017.

If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [5] was the first Mathematician, who brilliantly answered the question of Ulam by considering G_1 and G_2 to be Banach spaces. The result of Hyers is stated in the following celebrated Theorem.

Theorem 1.2 (Hyers [5]). *Assume that G_1 and G_2 are Banach spaces. If a function $f : G_1 \to G_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon, \tag{1.1}$$

for some $\epsilon > 0$ and for all $x, y \in G_1$, then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x), \tag{1.2}$$

exists for each $x \in G_1$ and $A : G_1 \to G_2$ is the unique additive function such that

$$\|f(x) - A(x)\| \leq \epsilon, \tag{1.3}$$

for all $x \in G_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in G_1$, then A is linear.

Taking the above fact into account, the additive functional equation

$$f(x+y) = f(x) + f(y)$$

is said to have **Hyers-Ulam stability** on (G_1, G_2). In the above Theorem, an additive function A satisfying the inequality (1.3) is constructed directly from the given function f and it is the most powerful tool to study the stability of several functional equations. In course of time, the following Theorem is the Theorem formulated by Hyers was generalized by Aoki, Th. M. Rassias [3, 19, 20], for additive mappings.

There is no reason for the Cauchy difference $f(x+y) - f(x) - f(y)$ to be bounded as in the expression of (1.1). Towards this point, in the year 1978, Rassias [20] tried to weaken the condition for the Cauchy difference and succeeded in proving what is now known to be the Hyers-Ulam stability for the Additive Cauchy Equation. This terminology is justified because the Theorem of Th.M. Rassias has strongly influenced Mathematicians studying stability problems of functional equation. In fact, Th. M. Rassias proved the following Theorem.

Theorem 1.3 (Rassias [20]). *Let X and Y be Banach spaces. Let $\theta \in (0, \infty)$ and let $p \in [0, 1)$. If a function $f : X \to Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \left(\|x\|^p + \|y\|^p\right), \tag{1.4}$$

for all $x, y \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p, \tag{1.5}$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

The findings of Th. M. Rassias have exercised a delectable influence on the develop ment of what is addressed as the generalized Hyers-Ulam-Rassias stability of functional equations (see [14]).

A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation

$$\phi\left(f, x, x', x'', \ldots, x^{(n)}\right) = 0, \tag{1.6}$$

has the Hyers - Ulam stability if for a given $\epsilon > 0$ and a function x such that

$$\left|\phi\left(f, x, x', x'', \ldots, x^{(n)}\right)\right| \leq \epsilon,$$

there exists a solution x_a of the differential equation (1.6) such that

$$|x(t) - x_a(t)| \leq K(\epsilon)$$

and

$$\lim_{\epsilon \to 0} K(\epsilon) = 0.$$

If we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\varphi(t)$, where ϕ, φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers - Ulam stability or Hyers-Ulam-Rassias stability.

Obloza seems to be the first author who has investigated the Hyers - Ulam stability of linear differential equations [15, 16]. Thereafter, In 1998, C. Alsina and R. Ger [2] were the first authors who investigated the Hyers-Ulam stability of the following differential equation $x'(t) - x(t) = 0$. They proved in [2] the following Theorem.

Theorem 1.4. *Assume that a differentiable function $f : I \to R$ is a solution of the differential inequality*

$$\|x'(t) - x(t)\| \leq \epsilon,$$

where I is an open sub interval of R. Then there exists a solution $g : I \to R$ of the differential equation $x'(t) = x(t)$ such that for any $t \in I$, we have,

$$\|f(t) - g(t)\| \leq 3\epsilon.$$

This result of C. Alsina and R. Ger [2] has been generalized by Takahasi [24]. They proved in [24] that the Hyers-Ulam stability holds true for the Banach Space valued for differential equation $y'(t) = \lambda y(t)$.

Indeed, the Hyers-Ulam stability has been proved for the first order linear differential equations in more general settings in [7, 12, 13, 23]. While Jung [7] proved a similar result for the differential equation $\phi(t)x'(t) = x$.

S. M. Jung [6] investigated the Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients by using matrix method. Also, G. Wang, M. Zhou, Li Sun [26] studied the Hyers-Ulam stability of a class of first - order linear differential equations.

After that many authors were proved the Hyers-Ulam stability for some of the second order and nth order linear differential equation in [1, 4, 8–11, 21–23, 27].

In this paper, we study the Hyers-Ulam-Rassias stability of a homogeneous and non-homogeneous linear differential equation of third order. In the following theorems, we prove the Generalized Hyers-Ulam statbility of the homogeneous third order linear differential equation of the form

$$x'''(t) + \alpha(t)x(t) = 0, \tag{1.7}$$

and the non-homogeneous linear differential equation

$$x'''(t) + \alpha(t)x(t) = \psi(t), \tag{1.8}$$

by applying initial conditions

$$x(a) = x'(a) = x''(a) = 0, \tag{1.9}$$

where $x \in C^3(I)$, $\alpha(t) \in C^0(I)$, for all $t \in \mathbb{R}$, where as $I = [a, b] \subseteq \mathbb{R}$, $-\infty < a < b < \ \infty$.

2. Preliminaries

First, we give the definition of the Hyers-Ulam-Rassias stability of a homogeneous linear differential equation (1.7) and the non-homogeneous linear differential equation (1.8) using the initial conditions (1.9).

Definition 2.1. *We say that the differential equation* (1.7) *has the Hyers-Ulam stability property, if there exists a positive constant K, such that the following property holds. For every $\epsilon > 0$, $x \in C^3([a, b])$, if*

$$|x'''(t) + \alpha(t)x(t)| \leq \epsilon,$$

and $x(a) = x'(a) = x''(a) = 0$, then there exists some $y \in C^3([a, b])$ satisfying

$$y'''(t) + \alpha(t)y(t) = 0$$

and $y(a) = y'(a) = y''(a) = 0$, such that $|x(t) - y(t)| \leq K\epsilon$.

Definition 2.2. We say that the homogeneous differential equation (1.7) has the Hyers-Ulam-Rassias (or Generalized Hyers-Ulam) stability property, if there exists $\theta_\phi \in C(I)$, such that for each approximate solution $x \in C^3(I)$ such that

$$|x'''(t) + \alpha(t)x(t)| \leq \phi(t),$$

for all $t \in I$. Then there exists a solution $y_0(t) \in C^3(I)$ satisfies the differential equation (1.7) with

$$|x(t) - y_0(t)| \leq \theta_\phi(t), \quad \text{for all } t \in I. \tag{2.1}$$

Definition 2.3. We say that the non-homogeneous differential equation (1.8) has the Hyers-Ulam-Rassias (or Generalized Hyers-Ulam) stability property, if there exists $\theta_\phi \in C(I)$, such that for each approximate solution $x \in C^3(I)$ such that

$$|x'''(t) + \alpha(t)x(t) - \psi(t)| \leq \phi(t),$$

for all $t \in I$. Then there exists a solution $y_0(t) \in C^3(I)$ satisfies the differential equation (1.8) with

$$|x(t) - y_0(t)| \leq \theta_\phi(t) \quad \text{for all } t \in I. \tag{2.2}$$

In the course of this work we shall need the following Replacement lemma.

Lemma 2.1 (Repalcement Lemma)**.** *Suppose that the function $f : [a, b] \to \mathbb{R}$ is a continuous function. Then*

$$\int_a^t \int_a^s f(\tau) d\tau ds = \int_a^t (t - \tau) f(\tau) d\tau, \quad t \in [a, b].$$

Theorem 2.1. *If* $\max |\alpha(x)| \leq \dfrac{6}{(b-a)^3}$*, then the differential equation*

$$x'''(t) + \alpha(t)x(t) = 0$$

with the initial conditions

$$x(a) = x'(a) = x''(a) = 0,$$

where $x \in C^3([a, b])$, $\alpha \in C([a, b])$ has the Hyers-Ulam stability property.

Lemma 2.2 (Gronwall Inequality)**.** *Let $x, y : [0, \infty) \to [0, \infty)$ be integrable functions, let $c > 0$ be a constant and let $t_0 \geq 0$ be given. If x satisfies the inequality*

$$x(t) \leq c + \int_{t_0}^t x(\tau)y(\tau) d\tau, \tag{2.3}$$

for all $t > t_0$, then we have

$$x(t) \leq c \exp\left(\int_{t_0}^t y(\tau) d\tau\right). \tag{2.4}$$

3. Hyers-Ulam-Rassias Stability

In the following Theorems, we would like to prove the Hyers-Ulam-Rassias stability of the linear differential equation (1.7) and (1.8) with initial conditions (1.9).

Theorem 3.1. *Suppose that $|\alpha(t)| < M$ where $M = \frac{6}{(b-a)^3}$ for $t \in [a, b]$. Assume that $\phi : [a, b] \to [0, \infty)$ is an increasing function. The homogeneous differential equation* (1.7) *has the Generalized Hyers-Ulam stability property with initial condition* (1.9), *if for $\theta_\phi \in C(I)$ and for each approximate solution $x \in C^3(I)$ of* (1.7) *satisfying*

$$|x'''(t) + \alpha(t)x(t)| \leq \phi(t),$$

there exists a solution $y_0 \in C^3([a, b])$ of (1.7) *such that $|x(t) - y_0(t)| \leq \theta_\phi(t)$.*

Proof. Suppose that $\phi : [a,b] \to [0,\infty)$ is an increasing function and suppose if for $\theta_\phi \in C(I)$ and for each approximate solution $x \in C^3(I)$ of (1.7) satisfying

$$|x'''(t) + \alpha(t)x(t)| \leq \phi(t). \tag{3.1}$$

From (3.1) we have that

$$-\phi(t) \leq x'''(t) + \alpha(t)x(t) \leq \phi(t).$$

Integrating from a to t, and applying the condition (1.9) we have

$$-\int_a^t \phi(\tau)d\tau \leq \int_a^t x'''(\tau)d\tau + \int_a^t \alpha(\tau)x(\tau)d\tau \leq \int_a^t \phi(\tau)d\tau,$$

$$-\int_a^t \phi(\tau)d\tau \leq x''(t) + \int_a^t \alpha(\tau)x(\tau)d\tau \leq \int_a^t \phi(\tau)d\tau. \tag{3.2}$$

Again integrating the equation (3.2) and applying the condition (1.9), we get

$$-\int_a^s\int_a^t \phi(\tau)d\tau ds \leq \int_a^s x''(s)ds + \int_a^s\int_a^t \alpha(\tau)x(\tau)d\tau ds \leq \int_a^s\int_a^t \phi(\tau)d\tau ds,$$

$$-\int_a^s\int_a^t \phi(\tau)d\tau ds \leq x'(t) + \int_a^s\int_a^t \alpha(\tau)x(\tau)d\tau ds \leq \int_a^s\int_a^t \phi(\tau)d\tau ds. \tag{3.3}$$

On further integration and also applying the condition (1.9) we arrive that,

$$\begin{aligned}
-\int_a^r\int_a^s\int_a^t \phi(\tau)d\tau ds dr &\leq \int_a^r x'(r)dr + \int_a^r\int_a^s\int_a^t \alpha(\tau)x(\tau)d\tau ds dr \\
&\leq \int_a^r\int_a^s\int_a^t \phi(\tau)d\tau ds dr,
\end{aligned}$$

$$\begin{aligned}
-\int_a^r\int_a^s\int_a^t \phi(\tau)d\tau ds dr &\leq x(t) + \int_a^r\int_a^s\int_a^t \alpha(\tau)x(\tau)d\tau ds dr \\
&\leq \int_a^r\int_a^s\int_a^t \phi(\tau)d\tau ds dr.
\end{aligned} \tag{3.4}$$

Now applying the Replacement Lemma to (3.4), we obtain

$$-\int_a^s\int_a^t (t-s)\phi(\tau)d\tau ds \leq x(t) + \int_a^s\int_a^t (t-s)\alpha(\tau)x(\tau)d\tau ds \leq \int_a^s\int_a^t (t-s)\phi(\tau)d\tau ds,$$

again applying the Repalcement Lemma, we have

$$-\int_a^t (t-s-\tau)\phi(s-\tau)d\tau \le x(t) + \int_a^t (t-s-\tau)\alpha(s-\tau)x(s-\tau)d\tau$$
$$\le \int_a^t (t-s-\tau)\phi(s-\tau)d\tau.$$

Repalce $(s+\tau)$ by ξ, we get

$$-\int_a^t (t-\xi)\phi(\xi)d\xi \le x(t) + \int_a^t (t-\xi)\alpha(\xi)x(\xi)d\xi \le \int_a^t (t-\xi)\phi(\xi)d\xi$$

and

$$\left| x(t) + \int_a^t (t-\xi)\alpha(\xi)x(\xi)d\xi \right| \le \int_a^t (t-\xi)\phi(\xi)d\xi.$$

If we choose that $y_0(t)$ such that it solves the equation (1.7) with the initial conditions (1.9) such that

$$y_0(t) = \int_a^t (t-\xi)\alpha(\xi)x(\xi)d\xi,$$

thus we estimate,

$$\begin{aligned}
|x(t)-y_0(t)| &\le \left| x(t) + \int_a^t (t-\xi)\alpha(\xi)x(\xi)d\xi \right| \\
&\quad + \int_a^t \left| (t-\xi)\alpha(\xi)x(\xi)d\xi - \int_a^t (t-\xi)\alpha(\xi)y_0(\xi)d\xi \right| d\xi \\
&\le \int_a^t (t-\xi)\phi(\xi)d\xi + \int_a^t \left| (t-\xi)\alpha(\xi)\left\{x(\xi)-y_0(\xi)\right\} d\xi \right| \\
&\le \int_a^t (t-\xi)\phi(\xi)d\xi + |\alpha(\xi)| \int_a^t (t-\xi)\left|x(\xi)-y_0(\xi)\right| d\xi.
\end{aligned}$$

Applying Gronwall inequality, we have

$$\begin{aligned}
|x(t)-y_0(t)| &\le \int_a^t (t-\xi)\phi(\xi)d\xi \exp\left\{ M \int_a^t (t-\xi)d\xi \right\} \\
&\le \int_a^t (t-\xi)\phi(\xi)d\xi \exp\left\{ M\frac{(t-a)^2}{2} \right\} \\
&\le K \int_a^t (t-\xi)\phi(\xi)d\xi,
\end{aligned}$$

where $K = \exp\left\{M\frac{(t-a)^2}{2}\right\}$. Hence, we have $|x(t) - y_0(t)| \leq \theta_\phi(t)$, where $\theta_\phi(t) = K\int\limits_a^t (t-\xi)\phi(\xi)d\xi$, and the proof is completed. □

Now, we are going to prove the Hyers-Ulam-Rassias stability of non-homogeneous linear differentiable equation (1.8) with initial conditions (1.9)

Theorem 3.2. *Suppose that $|\alpha(t)| < M$ where $M = \frac{6}{(b-a)^3}$ for $t \in [a,b]$. Assume that $\phi : [a,b] \to [0,\infty)$ is an increasing function. The homogeneous differential equation* (1.8) *has the Generalized Hyers-Ulam stability property with initial condition* (1.9)*, if for $\theta_\phi \in C(I)$ and for each approximate solution $x \in C^3(I)$ of* (1.8) *satisfying*

$$|x'''(t) + \alpha(t)x(t) - \psi(t)| \leq \phi(t),$$

there exists a solution $y_0 \in C^3([a,b])$ of (1.7) *such that $|x(t) - y_0(t)| \leq \theta_\phi(t)$.*

Proof. Suppose that $\phi : [a,b] \to [0,\infty)$ is an increasing function and suppose if for $\theta_\phi \in C(I)$ and for each approximate solution $x \in C^3(I)$ of (1.8) satisfying

$$|x'''(t) + \alpha(t)x(t) - \psi(t)| \leq \phi(t). \tag{3.5}$$

From (3.5) we have that $-\phi(t) \leq x'''(t) + \alpha(t)x(t) - \psi(t) \leq \phi(t)$. Integrating the above from a to t, and applying the condition (1.9) we have

$$-\int\limits_a^t \phi(\tau)d\tau \leq \int\limits_a^t x'''(\tau)d\tau + \int\limits_a^t \alpha(\tau)x(\tau)d\tau - \int\limits_a^t \psi(\tau)d\tau \leq \int\limits_a^t \phi(\tau)d\tau,$$

$$-\int\limits_a^t \phi(\tau)d\tau \leq x''(t) + \int\limits_a^t \alpha(\tau)x(\tau)d\tau - \int\limits_a^t \psi(\tau)d\tau \leq \int\limits_a^t \phi(\tau)d\tau. \tag{3.6}$$

Again integrating the equation (3.6) and applying the condition (1.9), we get

$$\begin{aligned}
-\int\limits_a^s\int\limits_a^t \phi(\tau)d\tau ds &\leq \int\limits_a^s x''(s)ds + \int\limits_a^s\int\limits_a^t \alpha(\tau)x(\tau)d\tau ds - \int\limits_a^s\int\limits_a^t \psi(\tau)d\tau ds \\
&\leq \int\limits_a^s\int\limits_a^t \phi(\tau)d\tau ds,
\end{aligned}$$

$$\begin{aligned}
-\int\limits_a^s\int\limits_a^t \phi(\tau)d\tau ds &\leq x'(t) + \int\limits_a^s\int\limits_a^t \alpha(\tau)x(\tau)d\tau ds - \int\limits_a^s\int\limits_a^t \psi(\tau)d\tau ds \\
&\leq \int\limits_a^s\int\limits_a^t \phi(\tau)d\tau ds.
\end{aligned} \tag{3.7}$$

On further integration and also applying the condition (1.9) we arrive that,

$$-\int_a^r\int_a^s\int_a^t \phi(\tau)d\tau ds dr \le \int_a^r x'(r)dr + \int_a^r\int_a^s\int_a^t \alpha(\tau)x(\tau)d\tau ds dr$$
$$-\int_a^r\int_a^s\int_a^t \psi(\tau)d\tau ds dr \le \int_a^r\int_a^s\int_a^t \phi(\tau)d\tau ds dr,$$

$$-\int_a^r\int_a^s\int_a^t \phi(\tau)d\tau ds dr \le x(t) + \int_a^r\int_a^s\int_a^t \alpha(\tau)x(\tau)d\tau ds dr$$
$$-\int_a^r\int_a^s\int_a^t \psi(\tau)d\tau ds dr \le \int_a^r\int_a^s\int_a^t \phi(\tau)d\tau ds dr. \tag{3.8}$$

Now applying the Replacement Lemma to (3.8), we obtain

$$-\int_a^s\int_a^t (t-s)\phi(\tau)d\tau ds \le x(t) + \int_a^s\int_a^t (t-s)\alpha(\tau)x(\tau)d\tau ds,$$
$$-\int_a^s\int_a^t (t-s)\psi(\tau)d\tau ds \le \int_a^s\int_a^t (t-s)\phi(\tau)d\tau ds,$$

again applying the Repalcement Lemma, we have

$$-\int_a^t (t-s-\tau)\phi(s-\tau)d\tau \le x(t) + \int_a^t (t-s-\tau)\alpha(s-\tau)x(s-\tau)d\tau$$
$$-\int_a^t (t-s-\tau)\psi(s-\tau)d\tau$$
$$\le \int_a^t (t-s-\tau)\phi(s-\tau)d\tau.$$

Repalce $(s+\tau)$ by ξ, we get

$$-\int_a^t (t-\xi)\phi(\xi)d\xi \le x(t) + \int_a^t (t-\xi)\alpha(\xi)x(\xi)d\xi - \int_a^t (t-\xi)\psi(\xi)d\xi$$
$$\le \int_a^t (t-\xi)\phi(\xi)d\xi$$

and

$$\left| x(t) + \int_a^t (t-\xi)\left\{\alpha(\xi)x(\xi)d\xi + \psi(\xi)\right\}d\xi \right| \le \int_a^t (t-\xi)\phi(\xi)d\xi.$$

If we choose that $y_0(t)$ such that it solves the equation (1.7) with the initial conditions (1.9) such that

$$y_0(t) = \int_a^t (t-\xi)\left\{\alpha(\xi)x(\xi)d\xi + \psi(\xi)\right\}d\xi,$$

thus we estimate

$$\begin{aligned}
|x(t) - y_0(t)| &\le \left| x(t) + \int_a^t (t-\xi)\left\{\alpha(\xi)x(\xi)d\xi + \psi(\xi)\right\}d\xi \right| \\
&\quad + \int_a^t \Big|(t-\xi)\left\{\alpha(\xi)x(\xi)d\xi + \psi(\xi)\right\}d\xi \\
&\quad - \int_a^t (t-\xi)\left\{\alpha(\xi)y_0(\xi)d\xi + \psi(\xi)\right\}\Big| d\xi \\
&\le \int_a^t (t-\xi)\phi(\xi)d\xi + \int_a^t \left|(t-\xi)\alpha(\xi)\left\{x(\xi) - y_0(\xi)\right\}d\xi\right| \\
&\le \int_a^t (t-\xi)\phi(\xi)d\xi + |\alpha(\xi)| \int_a^t (t-\xi)\left|x(\xi) - y_0(\xi)\right| d\xi.
\end{aligned}$$

Applying Gronwall inequality, we have

$$\begin{aligned}
|x(t) - y_0(t)| &\le \int_a^t (t-\xi)\phi(\xi)d\xi \exp\left\{M \int_a^t (t-\xi)d\xi\right\} \\
&\le \int_a^t (t-\xi)\phi(\xi)d\xi \exp\left\{M \frac{(t-a)^2}{2}\right\} \\
&\le K \int_a^t (t-\xi)\phi(\xi)d\xi,
\end{aligned}$$

where $K = \exp\left\{M\frac{(t-a)^2}{2}\right\}$. Hence, we have $|x(t) - y_0(t)| \le \theta_\phi(t)$, where

$$\theta_\phi(t) = K \int_a^t (t-\xi)\phi(\xi)d\xi,$$

and hence the proof. □

Acknowledgements. The authours are deeply grateful to the editor for useful comments. Also, we would like to thank the referees for their valuable suggestions.

References

[1] M. R. Abdollahpoura, R. Aghayaria and M. Th. Rassias, *Hyers-Ulam stability of associated Laguerre differential equations in a subclass of analytic functions*, J. Math. Anal. Appl. **437** (2016), 605–612.

[2] C. Alsina and R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. **2** (1998), 373–380.

[3] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[4] P. Gavruta, S. M. Jung and Y. Li, *Hyers-Ulam stability for second order linear differential equations with boundary conditions*, Electron. J. Differential Equations **2011**(80) (2011), 1–5.

[5] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[6] S. M. Jung, *Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients*, J. Math. Anal. Appl. **320** (2006), 549–561.

[7] S. M. Jung, *Hyers-Ulam stability of linear differential equation of first order*, Appl. Math. Lett. **17** (2004), 1135–1140.

[8] S. M. Jung, *Approximate solution of a linear differential equation of third order*, Bull. Malays. Math. Sci. Soc. **35**(4) (2012), 1063–1073.

[9] S. M. Jung, *Hyers-Ulam stability of linear differential equations of first order* (*II*), Appl. Math. Lett. **19** (2006), 854–858.

[10] S. M. Jung, *Hyers-Ulam stability of linear differential equations of first order* (*III*), J. Math. Anal. Appl. **311** (2005), 139–146.

[11] Y. Li and Y. Shen, *Hyers-Ulam stability of linear differential equations of second order*, Appl. Math. Lett. **23** (2010), 306–309.

[12] T. Miura, S. Miyajima and S. E. Takahasi, *A characterization of Hyers-Ulam stability of first order linear differential operators*, J. Math. Anal. Appl. **286** (2003), 136–146.

[13] T. Miura, S. M. Jung and S. E. Takahasi, *Hyers-Ulam-Rassias stability of the Banach space valued linear differential equation $y' = \lambda y$*, J. Korean Math. Soc. **41** (2004), 995–1005.

[14] J. F. Nash and M. Th. Rassias, *Open Problems in Mathematics*, Springer, New York, 2016.

[15] M. Obloza, *Connection between Hyers and Lyapunov stability of the ordinary differential equations*, Rocznik Nauk-Dydakt. Prace. Mat. **14** (1997), 141–146.

[16] M. Obloza, *Hyers stability of the linear differential equation*, Rocznik Nauk-Dydakt. Prace. Math. **13** (1993), 259–270.

[17] M. N. Qarawani, *Hyers-Ulam stability of a generalized second order nonlinear differential equation*, Appl. Math. **3** (2012), 1857–1861.

[18] M. N. Qarawani, *Hyers-Ulam stability of linear and nonlinear differential equation of second order*, Int. J. Appl. Math. **1**(4) (2012), 422–432.

[19] Th. M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000) 23–130.

[20] Th. M. Rassias, *On the stability of the linear mappings in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[21] K. Ravi, R. Murali and A. Ponmanaselvan, *Hyers-Ulam stability of n-th order linear differential equation with initial and boundary condition*, Asian J. Math. Comp. Res. **11**(3) (2016), 201–207.

[22] K. Ravi, R. Murali and A. Ponmanaselvan, *Ulam stability of a General n-th order linear differential equation with constant coefficients*, Asian J. Math. Comp. Res. **11**(1) (2016), 61–68.

[23] I. A. Rus, *Ulam Stabilities of ordinary differential equations in a Banach space*, Carpathian J. Math. **26**(1) (2010), 103–107.

[24] S. E. Takahasi, T. Miura and S. Miyajima, *On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$*, Bull. Korean Math. Soc. **39** (2002), 309–315.

[25] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.

[26] G. Wang, M. Zhou and L. Sun, *Hyers-Ulam stability of linear differential equations of first order*, Appl. Math. Lett. **21** (2008), 1024–1028.

[27] J. Xue, *Hyers-Ulam stability of linear differential equations of second order with constant coefficient*, Ital. J. Pure Appl. Math. **32** (2014), 419–424.

[1]PG and Research Department of Mathematics,
Sacred Heart College (Autonomous),
Tirupattur - 635 601, Vellore Dt., Tamil Nadu, India
E-mail address: shcrmurali@yahoo.co.in

[1]PG and Research Department of Mathematics,
Sacred Heart College (Autonomous),
Tirupattur - 635 601, Vellore Dt., Tamil Nadu, India
E-mail address: selvaharry@yahoo.com