

GENERALIZED MEASURE OF NONCOMPACTNESS AND FIXED POINT THEORY IN GENERALIZED LOCALLY CONVEX SPACES AND APPLICATIONS

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ABSTRACT. In this paper, we give some fixed point results in the so-called generalized locally convex space. These results, formulated in terms of families of generalized measures of noncompactness, extend many fixed point theorems in Banach spaces. As an application, we establish an existence result for a system of integral equations in the space of continuous functions.

1. INTRODUCTION AND PRELIMINARIES

The measure of noncompactness is a very useful tool in nonlinear functional analysis, especially when proving the existence of solutions for functional operator equations. Recently, there are many papers providing some fixed point results in locally convex spaces using measures of noncompactness (see for example [2, 3, 8–10]). In [1], Bahidi, Boudaoui, and Krichen introduced the concept of generalized locally convex space and proved new versions of fixed point theorems in this space. Further, they used their findings to prove the existence of solutions of a first order differential equations with impulses. Motivated by these findings, we are interested in this paper to continue the study of [1] by introducing the concept of a generalized family of measures of noncompactness. Then, we establish some extensions of classical fixed point theorems as Darbo, and Sadovskii fixed point theorem, in generalized locally convex spaces. The obtained results are applied to study the existence of solutions of a system of

Key words and phrases. Fixed point, generalized measure of noncompactness, generalized locally convex space, generalized seminorms, continuous functions, differential equation.

2020 *Mathematics Subject Classification.* Primary: 46A03. Secondary: 46A04, 47H10, 47H08.

DOI

Received: July 22, 2025.

Accepted: December 04, 2025.

nonlinear integral equations having the form:

$$(1.1) \quad \begin{cases} u = \xi_1(u, v), \\ v = \xi_2(u, v), \end{cases}$$

where $\xi_j : \Gamma \times \Gamma \rightarrow \Gamma$, $j = 1, 2$ are continuous operators defined on a generalized locally convex space Γ .

First, let us give some notations, and definitions. By \mathbb{R}_+ we mean the set of positive real numbers, $\mathcal{M}_{n \times n}(\mathbb{R}_+)$ we mean the set of $n \times n$ matrices with positive elements, and \mathbb{R}^n the set of $n \times 1$ real matrices. We denote by \preceq , the coordinate-wise ordering on

\mathbb{R}^n , that is, for $\eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$, $\kappa = \begin{pmatrix} \kappa_1 \\ \vdots \\ \kappa_n \end{pmatrix}$, $\eta \preceq \kappa$ if and only if $\eta_i \leq \kappa_i$ for all $i = 1, \dots, n$.

For $\eta, \kappa \in \mathbb{R}^n$, we denote by

$$\widetilde{\max}\{\eta, \kappa\} = \begin{pmatrix} \max\{\eta_1, \kappa_1\} \\ \max\{\eta_2, \kappa_2\} \\ \vdots \\ \max\{\eta_n, \kappa_n\} \end{pmatrix}.$$

Let $\eta \in \mathbb{R}^n$, the symbol $\eta \succ \mathbf{0}$ mean that $\eta_i > 0$ for each $i = 1, \dots, n$, and $\mathbf{0}$ is the zero $n \times 1$ matrix. By \mathbb{R}_+^n , I is the identity in $\mathcal{M}_{n \times n}(\mathbb{R}_+)$ and θ is the zero $n \times n$ matrix. Let us introduce the definition of a vector-valued seminorm as follows.

Definition 1.1. Let Γ be a topological vector space. A mapping $\mu : \Gamma \rightarrow \mathbb{R}_+^n$ is called a generalized seminorm (GSN, in short) if it satisfies the following:

- (i) $\mu(u) \succeq \mathbf{0}$, for each $u \in \Gamma$;
- (ii) $\mu(\alpha u) = |\alpha| \mu(u)$, $u \in \Gamma$, $\alpha \in \mathbb{R}$;
- (iii) $\mu(u + v) \preceq \mu(u) + \mu(v)$,

where

$$\mu(u) := \begin{pmatrix} \mu^1(u) \\ \mu^2(u) \\ \vdots \\ \mu^n(u) \end{pmatrix}.$$

Note that μ is a GSN on Γ if and only if μ^s are seminorms on Γ , for $s = 1, \dots, n$.

For the generalized family of seminorm $\Lambda = \{\mu_i, i \in I_\Gamma\}$, the pair $(\Gamma, \mu_i)_{i \in I_\Gamma}$ is said to be generalized locally convex space (*GLCS*, in short).

Remark 1.1. We say that the pair $(\Gamma, \mu_i)_{i \in I_\Gamma}$ is *GLCS* if each point of the space possesses a fundamental system of neighborhoods formed of convex sets with respect to the topology associated with the generalized family of seminorm Λ .

Definition 1.2. Let Γ and Δ be two Hausdorff *GLCSs*, $\Lambda_\Gamma = \{\mu_i, i \in I_\Gamma\}$ and $\Lambda_\Delta = \{\nu_j, j \in I_\Delta\}$ their families of GSNs. Let \mathfrak{S} be a subset of Γ and let A be a mapping from \mathfrak{S} into Δ . Then, the following hold.

- (i) A is continuous at a point u_0 of \mathfrak{S} if, for all $j \in I_\Delta$ and $\varepsilon \in \mathbb{R}^n$, $\varepsilon \succ \mathbf{0}$, there exist a finite set $\{i_1, \dots, i_k\} \subset I_\Gamma$ and $\delta \in \mathbb{R}^n$, $\delta \succ \mathbf{0}$ such that for $u \in \mathfrak{S}$

$$\widetilde{\max}_{1 \leq m \leq k} \mu_{im}(u_0 - u) = \widetilde{\max} \{ \mu_{i_1}(u_0 - u), \dots, \mu_{i_k}(u_0 - u) \} \text{ preceq} \delta,$$

then $\nu_j(Au_0 - Au) \preceq \varepsilon$.

- (ii) We say that A is continuous if it is continuous at any point of \mathfrak{S} .

Definition 1.3. Let Γ be a *GLCS* endowed with the family of GSNs $\Lambda_\Gamma = \{ \mu_i, i \in I_\Gamma \}$. Let \mathfrak{S} be a subset of Γ . We say that \mathfrak{S} is bounded if there exists $\{i_1, \dots, i_k\} \subset I_\Gamma$ such that for all $j = 1, \dots, k$, there exist $C_j \succ \mathbf{0}$ and for all $u \in \mathfrak{S}$, we have $\mu_{i_j}(u) \preceq C_j$.

For the following, $(\Gamma, \mu_i)_{i \in I_\Gamma}$, is a *GLCS* endowed with a family $\Lambda := \{ \mu_i, i \in I_\Gamma \}$ of GSNs. We denote by $\mathcal{B}(\Gamma)$ the family of all nonempty, bounded subsets of Γ .

Definition 1.4. Let $\Lambda = \{ \mu_i, i \in I_\Gamma \}$ be a family of GSNs on Γ . Let ρ_Λ be a family of functions $\rho_{\mu_i} : \mathcal{B}(\Gamma) \rightarrow \mathbb{R}_+^n$, $\mu_i \in \Lambda$ such that

$$\rho_{\mu_i}(\mathfrak{S}) = \begin{pmatrix} \rho_{\mu_i}^1(\mathfrak{S}) \\ \rho_{\mu_i}^2(\mathfrak{S}) \\ \vdots \\ \rho_{\mu_i}^n(\mathfrak{S}) \end{pmatrix}, \quad \mathfrak{S} \subset \mathcal{B}(\Gamma).$$

ρ_Λ is called a family of generalized measures of noncompactness (ρ_Λ -GMNC, in short) in Γ if for each $\mu_i \in \Lambda$, $i \in I_\Gamma$ it satisfies the following conditions.

- (i) $\rho_{\mu_i}(\overline{\text{conv}}(\mathfrak{S})) \preceq \rho_{\mu_i}(\mathfrak{S})$, for each $\mathfrak{S} \in \mathcal{B}(\Gamma)$, where $\overline{\text{conv}}(\mathfrak{S})$ is the closure of the convex hull of \mathfrak{S} in Γ .
- (ii) If $\mathfrak{S}_1 \subset \mathfrak{S}_2$, then $\rho_{\mu_i}(\mathfrak{S}_1) \preceq \rho_{\mu_i}(\mathfrak{S}_2)$, where $\mathfrak{S}_1, \mathfrak{S}_2 \in \mathcal{B}(\Gamma)$.
- (iii) $\rho_{\mu_i}(x \cup \mathfrak{S}) = \rho_{\mu_i}(\mathfrak{S})$, for each $\mathfrak{S} \in \mathcal{B}(\Gamma)$ and $x \in \mathfrak{S}$.
- (iv) $\rho_{\mu_i}(\mathfrak{S}) = \mathbf{0}$ if and only if \mathfrak{S} is relatively compact.
- (v) If $(\mathfrak{S}_k)_k$ is a sequence of nonempty closed subsets of $\mathcal{B}(\Gamma)$ such that $\mathfrak{S}_{k+1} \subset \mathfrak{S}_k$, $k = 1, 2, \dots, n$, and $\lim_{k \rightarrow +\infty} \rho_{\mu_i}(\mathfrak{S}_k) = \mathbf{0}$ for each $\mu_i \in \Lambda$, $i \in I_\Gamma$, then $\mathfrak{S}_{+\infty} = \bigcap_{k=1}^{+\infty} \mathfrak{S}_k$ is nonempty relatively compact subset of Γ .

The family ρ_Λ -GMNC is called:

- (vi) semiadditive, if for each $\mu_i \in \Lambda$, $i \in I_\Gamma$, $\rho_{\mu_i}(\mathfrak{S}_1 \cup \mathfrak{S}_2) = \widetilde{\max} \{ \rho_{\mu_i}(\mathfrak{S}_1), \rho_{\mu_i}(\mathfrak{S}_2) \}$, for every $\mathfrak{S}_1, \mathfrak{S}_2 \in \mathcal{B}(\Gamma)$,
- (vii) subadditive, if for each $\mu_i \in \Lambda$, $\rho_{\mu_i}(\mathfrak{S}_1 + \mathfrak{S}_2) \preceq \rho_{\mu_i}(\mathfrak{S}_1) + \rho_{\mu_i}(\mathfrak{S}_2)$, where $\mathfrak{S}_1, \mathfrak{S}_2 \in \mathcal{B}(\Gamma)$.

Remark 1.2. For each $\mu_i \in \Lambda$, $i \in I_\Gamma$, $\rho_\Lambda = \{ \rho_{\mu_i}, \mu_i \in \Lambda \}$ is a generalized family of measures of noncompactness in X if, and only if, for each $s = 1, 2, \dots, n$, $\{ \rho_{\mu_i}^s \}_{\mu_i \in \Lambda}$ is a family of measures of noncompactness in Γ .

Example 1.1. Let $\mathfrak{C} := C(\mathbb{R}_+)$ the set of continuous functions defined on \mathbb{R}_+ with the family of seminorms $|u|_T = \sup\{u(t), t \in [0, T]\}$ for all $T > 0$, and $u \in \mathfrak{C}$. We denote by $\{ \sigma_T, T > 0 \}$ the generalized family of seminorms defined from $\mathfrak{C} \rightarrow \mathbb{R}_+ \times \mathbb{R}_+$

by $\sigma_T(u) = \begin{pmatrix} |u|_T \\ |u|_T \end{pmatrix}, u \in \mathfrak{C}$. Then, $\Gamma = (\mathfrak{C}, \sigma_T)$ is a generalized locally convex space. In order to define the generalized family of measure of noncompactness, let us fix a bounded subset C of Γ and $T > 0$. For $u \in C$ and $\varepsilon > 0$, we define

$$\omega^T(u, \varepsilon) = \sup\{|u(t) - u(s)|, t, s \in [0, T], |t - s| \leq \varepsilon\}.$$

Moreover, we set

$$\bar{\omega}^T(C, \varepsilon) = \sup\{\omega_T(u, \varepsilon), u \in C\}, \quad \omega_0^T(C) = \lim_{\varepsilon \rightarrow 0} \bar{\omega}^T(C, \varepsilon).$$

It was proved in [4] that $\{\omega_0^T(C), T > 0\}$ is a family of measures of noncompactness in \mathfrak{C} , then we can define the generalized family of measures of noncompactness $\Phi_T(C) = \{\rho_T, T > 0\}$ as follow:

$$\rho_T(C) = \begin{pmatrix} \omega_0^T(C) \\ \omega_0^T(C) \end{pmatrix}.$$

Definition 1.5. Let $n \in \mathbb{N} \setminus \{0\}$. Let \mathcal{L} be a square matrix of real numbers. We say that \mathcal{L} is convergent to zero if $\lim_{k \rightarrow +\infty} \mathcal{L}^k = \theta$.

Example 1.2. Let u, v and $w \in \mathbb{R}_+$. In the following examples, \mathcal{L} converges to zero.

- (i) $\mathcal{L} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ and $\max\{u, v\} < 1$.
- (ii) $\mathcal{L} = \begin{pmatrix} u & -w \\ 0 & v \end{pmatrix}$ and $u + v < 1, w < 1$.
- (iii) $\mathcal{L} = \begin{pmatrix} u & -u \\ v & -v \end{pmatrix}$ and $|u - v| < 1, u > 1, v > 0$.
- (iv) $\mathcal{L} = \begin{pmatrix} u & u \\ v & v \end{pmatrix}$ or $\mathcal{L} = \begin{pmatrix} u & v \\ u & v \end{pmatrix}$, with $u + v < 1$.

Definition 1.6. Let Γ be a *GLCS*, and $\Phi_{\Lambda_\Gamma} := \{\rho_{\mu_i}, \mu_i \in \Lambda_\Gamma\}_{i \in I_\Gamma}$ a generalized family of measures of noncompactness. Let $A : \Gamma \rightarrow \Gamma$ be an operator.

- (i) A is called a generalized contraction operator with respect to ρ_Λ (for short, G - ρ_Λ -contraction) if for $\mu_i \in \Lambda, i \in I_\Gamma$ there exists a matrix $\mathcal{L}_{\mu_i} \in M_{n \times n}(\mathbb{R}_+)$ converging to θ , such that for every $\mathfrak{S} \subset \Gamma, A(\mathfrak{S}) \subset \mathfrak{B}(\Gamma)$ and for each $\mu_i \in \Lambda, i \in I_\Gamma$

$$\rho_{\mu_i}(A(\mathfrak{S})) \preceq \mathcal{L}_{\mu_i} \rho_{\mu_i}(\mathfrak{S}).$$

- (ii) A is called generalized condensing operator with respect to ρ_Λ (for short, G - ρ_Λ -condensing) if for any bounded subset \mathfrak{S} of $\Gamma, A(\mathfrak{S}) \in \mathfrak{B}(\Gamma)$ and for each $\mu_i \in \Lambda, i \in I_\Gamma$ such that $\rho_{\mu_i}(\mathfrak{S}) \succ \mathbf{0}$, we have

$$\rho_{\mu_i}(A(\mathfrak{S})) \prec \rho_{\mu_i}(\mathfrak{S}).$$

Definition 1.7. A function A of \mathbb{R}^n into \mathbb{R}^n is nondecreasing, if for all $u, v \in \mathbb{R}^n$ such that $u \preceq v$, then $A(u) \preceq A(v)$.

Now, we recall a version of Schauder-Tychonoff fixed point theorem in *GLCS* [1].

Theorem 1.1. *Let Γ be a Hausdorff GLCS, \mathfrak{S} be a nonempty convex subset of Γ , and let C be a compact subset of Γ with $C \subset \mathfrak{S}$. If $A : \mathfrak{S} \rightarrow \Gamma$ is a continuous operator with $A(\mathfrak{S}) \subset C \subset \mathfrak{S}$, then A has at least one fixed point in \mathfrak{S} .*

2. FIXED POINT THEOREMS VIA GMNC IN GLCS

In this section, we work to give some fixed point results in GLCS using GMNC. We start by proving some versions of Darbo and Sadovskii’s fixed point theorems.

Theorem 2.1. *Let Γ be a Hausdorff GLCS. Let $\rho_\Lambda := \{\rho_{\mu_i}, \mu_i \in \Lambda\}_{i \in I_\Gamma}$ be a family of ρ_Λ -GMNC. Let \mathfrak{S} be a nonempty bounded, closed, and convex subset of Γ and $A : \mathfrak{S} \rightarrow \mathfrak{S}$. If A is a G - ρ_Λ -contraction continuous mapping, then A has a fixed point in \mathfrak{S} .*

Proof. We define the sequence $(\mathfrak{S}_k)_k$ by

$$(2.1) \quad \begin{cases} \mathfrak{S}_1 = \mathfrak{S}, \\ \mathfrak{S}_{k+1} = \overline{\text{conv}}(A(\mathfrak{S}_k)), \quad k \in \mathbb{N}. \end{cases}$$

It is clear that the sequence $(\mathfrak{S}_k)_k$ consists of decreasing nonempty closed convex subsets of \mathfrak{S} . Let $\mu_i \in \Lambda, i \in I_\Gamma$. Then,

$$\rho_{\mu_i}(\overline{\text{conv}}(A(\mathfrak{S}_1))) \preceq \rho_{\mu_i}(A(\mathfrak{S}_1)).$$

Since A is G - ρ_Λ -contraction, so there exists a converging matrix \mathcal{L}_{μ_i} such that

$$\rho_{\mu_i}(\mathfrak{S}_2) = \rho_{\mu_i}(\overline{\text{conv}}(A(\mathfrak{S}_1))) \preceq \mathcal{L}_{\mu_i} \rho_{\mu_i}(\mathfrak{S}_1).$$

By induction, we get

$$\rho_{\mu_i}(\mathfrak{S}_{k+1}) \preceq \mathcal{L}_{\mu_i}^k \rho_{\mu_i}(\mathfrak{S}_1),$$

then $\lim_{k \rightarrow +\infty} \rho_{\mu_i}(\mathfrak{S}_k) = \mathbf{0}$, and consequently $\mathfrak{S}_\infty = \bigcap_{k=1}^\infty \mathfrak{S}_k$ is nonempty convex compact subset of \mathfrak{S} such that $A(\mathfrak{S}_\infty) \subseteq \mathfrak{S}_\infty$. Hence, by Theorem 1.1, there exists a point $u \in \mathfrak{S}_\infty$ such that $u = Au$. □

Theorem 2.2. *Let Γ be a Hausdorff GLCS, and $\rho_\Lambda = \{\mu_i, i \in I_\Gamma\}$ be a family of GMNC. Let \mathfrak{S} be a bounded, closed, convex subset of Γ . Assume that $A : \mathfrak{S} \rightarrow \mathfrak{S}$ is a continuous operator and G - ρ_Λ -condensing. Then, A has at least a fixed point in \mathfrak{S} .*

Proof. Let u_0 be a point in \mathfrak{S} . We define the set

$$\mathcal{S} = \{D \subset \Gamma, u_0 \in D \subset \mathfrak{S}, D \text{ is bounded convex and } A(D) \subset D\}.$$

It is clear that $\mathfrak{S} \in \mathcal{S}$, so \mathcal{S} is nonempty. Denote by $C = \bigcap_{D \in \mathcal{S}} D$. Obviously, $u_0 \in C$ and C is a bounded convex subset of Γ , and $A(C) \subset C$. Notice that

$$(2.2) \quad \overline{\text{conv}}\{A(C) \cup \{u_0\}\} \subset C.$$

It follows that $A(\overline{\text{conv}}\{A(C) \cup \{u_0\}\}) \subset A(C) \subset \overline{\text{conv}}\{A(C) \cup \{u_0\}\}$, and so,

$$\overline{\text{conv}}\{A(C) \cup \{u_0\}\} \in \mathcal{S}.$$

Hence,

$$(2.3) \quad C \subset \overline{\text{conv}}\{A(C) \cup \{u_0\}\}.$$

From (2.2) and (2.3), we get $C = \overline{\text{conv}}\{A(C) \cup \{u_0\}\}$. Using the properties (i) and (ii) of ρ_Λ -GMNC. We have for $\mu_i \in \Lambda$, $i \in I_\Gamma$

$$\begin{aligned} \rho_{\mu_i}(C) &= \rho_{\mu_i}(\overline{\text{conv}}\{A(C) \cup \{u_0\}\}) = \rho_{\mu_i}(\overline{\text{conv}}A(C)) \\ &\preceq \rho_{\mu_i}(A(C)). \end{aligned}$$

If $\varphi_{\mu_i}(C) \succ \mathbf{0}$, then by the ρ_Λ -condensibility of A , we obtain

$$\rho_{\mu_i}(C) \preceq \rho_{\mu_i}(A(C)) \prec \rho_{\mu_i}(C),$$

which means that

$$\rho_{\mu_i}^s(C) < \rho_{\mu_i}^s(C), \quad \text{for all } s = 1, \dots, n,$$

which is a contradiction. Then, $\rho_{\mu_i}(C) = \mathbf{0}$. Hence, C is relatively compact. Thus, from Theorem 1.1, there is a fixed point for the mapping A in \mathfrak{S} . \square

In what follows, we prove some versions of fixed point theorems proved by Harjani and Sadarangani in [6, 7]. Firstly, let us list some definitions needed later.

Definition 2.1. Let A and B be two matrices of the same dimension $m \times n$, the Hadamard product $A \odot B$ is a matrix of the same dimension as the operands, with elements given by

$$(A \odot B)_{ij} = (A)_{ij}(B)_{ij}.$$

For matrices of different dimensions ($m \times n$ and $p \times q$, where $m \neq p$ or $n \neq q$), the Hadamard product is undefined.

We denote by Q the class of functions $\gamma : [0, +\infty)^n \rightarrow [0, 1] \times \dots \times [0, 1]$

$$\gamma(t) = \begin{pmatrix} \gamma^1(t) \\ \vdots \\ \gamma^n(t) \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \in [0, +\infty],$$

which satisfies the condition $\gamma(t_k) \rightarrow \mathbf{1}$ implies $t_k \rightarrow \mathbf{0}$, where $\mathbf{1} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. Also, we

consider the two functions $\varphi, \psi : [0, +\infty)^n \rightarrow [0, +\infty)^n$ defined by $\varphi(t) = \begin{pmatrix} \varphi^1(t) \\ \vdots \\ \varphi^n(t) \end{pmatrix}$,

$$\psi(t) = \begin{pmatrix} \psi^1(t) \\ \vdots \\ \psi^n(t) \end{pmatrix}, \quad t \in [0, +\infty)^n, \text{ which satisfy}$$

- (\mathcal{H}_1) φ is continuous, nondecreasing, and $\varphi(t) = \mathbf{0}$ if and only if $t = \mathbf{0}$;
- (\mathcal{H}_2) ψ is a continuous function such that $\varphi(t) \succ \psi(t)$ for all $t \succ \mathbf{0}$.

Remark 2.1. (i) $\gamma \in Q$ if, and only if, for all $s = 1, \dots, n$, $\gamma^s(t_k) \rightarrow 1$, implies $t_k \rightarrow \mathbf{0}$.

(ii) Note that $\psi(\mathbf{0}) = \mathbf{0}$. Indeed, since $\psi(t) \prec \varphi(t)$ and φ, ψ are continuous we have $\mathbf{0} \preceq \psi(\mathbf{0}) = \lim_{t \rightarrow \mathbf{0}} \psi(t) \preceq \lim_{t \rightarrow \mathbf{0}} \varphi(t) = \varphi(\mathbf{0}) = \mathbf{0}$.

Theorem 2.3. *Let Γ be a Hausdorff GLCS. Let $\rho_\Lambda := \{\rho_{\mu_i}, \mu_i \in \Lambda\}_{i \in I_\Gamma}$ and $\rho_\Lambda = \{\mu_i, i \in I_\Gamma\}$ be a family of GMNC. Let \mathfrak{S} be a bounded, closed, convex subset of Γ and $A : \mathfrak{S} \rightarrow \mathfrak{S}$ be a continuous mapping such that for each $\mu_i \in \Lambda, i \in I_\Gamma$, there exists a function $\gamma \in Q$, such that*

$$(2.4) \quad \rho_{\mu_i}(A(C)) \preceq \gamma(\rho_{\mu_i}(C)) \odot \rho_{\mu_i}(C),$$

for C be a nonempty subset of \mathfrak{S} . Then A has at least one fixed point.

Proof. Clearly the sequence $(\mathfrak{S}_k)_k$ defined by (2.1) consists of a nonempty closed, bounded, convex and decreasing subset of \mathfrak{S} . If there exist an integer $N \geq 0$ such that for each $\mu_i \in \Lambda$ and $i \in I_\Gamma, \rho_{\mu_i}(\mathfrak{S}_N) = \mathbf{0}$. Then, \mathfrak{S}_N is relatively compact. Since $A(\mathfrak{S}_N) \subseteq \mathfrak{S}_N$, it follows by Schauder-Tychonoff fixed point theorem 1.1, A has a fixed point in \mathfrak{S}_N .

If for all $k \in \mathbb{N}, \rho_{\mu_i}(\mathfrak{S}_k) \neq \mathbf{0}$. Clearly, for $s = 1, \dots, n, (\rho_{\mu_i}^s(\mathfrak{S}_k))_k$ is a positive decreasing sequence. So, there is $r_s \geq 0, s = 1, \dots, n$ such that $\lim_{k \rightarrow +\infty} \rho_{\mu_i}^s(\mathfrak{S}_k) = r_s$. Suppose that for all $s = 1, \dots, n, r_s \neq 0$. By (2.4), we obtain that for $\mu_i \in \Lambda$ and $i \in I_\Gamma$

$$\begin{aligned} \rho_{\mu_i}^s(\mathfrak{S}_{k+1}) &= \rho_{\mu_i}^s(\overline{\text{conv}}(A(\mathfrak{S}_k))) \\ &\leq \rho_{\mu_i}^s(A(\mathfrak{S}_k)) \leq \gamma^s(\rho_{\mu_i}^s(\mathfrak{S}_k))\rho_{\mu_i}^s(\mathfrak{S}_k), \quad s = 1, \dots, n. \end{aligned}$$

Hence,

$$\frac{\rho_{\mu_i}^s(\mathfrak{S}_{k+1})}{\rho_{\mu_i}^s(\mathfrak{S}_k)} \leq \gamma^s(\rho_{\mu_i}^s(\mathfrak{S}_k)) \leq 1, \quad \text{for all } s = 1, \dots, n,$$

which yields $\lim_{k \rightarrow +\infty} \gamma^s(\rho_{\mu_i}^s(\mathfrak{S}_k)) = 1$. Since $\gamma \in Q$, Remark 2.1 (i) implies that $\lim_{k \rightarrow +\infty} \rho_{\mu_i}(\mathfrak{S}_k) = \mathbf{0}$. Then, $\mathfrak{S}_{+\infty} = \bigcap_{k=1}^{+\infty} \mathfrak{S}_k$ is a nonempty convex closed relatively compact subset of Γ , and $A(\mathfrak{S}_{+\infty}) \subseteq \mathfrak{S}_{+\infty}$. So by Schauder-Tychonoff fixed point Theorem 1.1, A has a fixed point in $\mathfrak{S}_{+\infty}$. \square

Theorem 2.4. *Let $\Lambda = \{\mu_i, i \in I_\Gamma\}$ be family of GSNs on a Hausdorff GLCS Γ and $\rho_\Lambda = \{\mu_i, i \in I_\Gamma\}$ be a family of GMNC. Let \mathfrak{S} be a nonempty bounded closed and convex subset of Γ . Assume $\varphi : [0, +\infty)^n \rightarrow [0, +\infty)^n$ be a function verify condition (\mathcal{H}_1) , and $A : \mathfrak{S} \rightarrow \mathfrak{S}$ be a continuous operator such that for each $\mu_i \in \Lambda, i \in I_\Gamma$*

$$(2.5) \quad \rho_{\mu_i}(A(C)) \preceq \rho_{\mu_i}(C) - \varphi(\rho_{\mu_i}(C)),$$

for any subset C of \mathfrak{S} . Then, A has at least one fixed point in \mathfrak{S} .

Proof. As we show in the proof of Theorem 2.3, we can easily prove that the sequence $(\mathfrak{S}_k)_k$ in (2.1) is nonempty, convex, bounded, closed and nondecreasing subset of \mathfrak{S} . Using the properties of family of GMNC we obtain that $(\rho_{\mu_i}(\mathfrak{S}_k))_k$ is also a positive decreasing sequence. Thus, for all $s = 1, \dots, n$, there exist an $r_s \geq 0$ such that

$\lim_{k \rightarrow +\infty} \rho_{\mu_i}^s(\mathfrak{S}_k) = r_s$, hence there exists $r = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \succeq \mathbf{0}$ such that $\lim_{k \rightarrow +\infty} \rho_{\mu_i}(\mathfrak{S}_k) = r$. Suppose that $r \succ \mathbf{0}$, Then, by (2.5) and for each $\mu_i \in \Lambda, i \in I_\Gamma$, we have

$$(2.6) \quad \begin{aligned} \rho_{\mu_i}(\mathfrak{S}_{k+1}) &= \rho_{\mu_i}(\overline{\text{conv}}(A(\mathfrak{S}_k))) \\ &\preceq \rho_{\mu_i}(A(\mathfrak{S}_k)) \preceq \rho_{\mu_i}(\mathfrak{S}_k) - \varphi(\rho_{\mu_i}(\mathfrak{S}_k)) \end{aligned}$$

So, from (2.6), we obtain

$$r \preceq r - \varphi(r),$$

and consequently $\varphi(r) \preceq \mathbf{0}$. Using properties of function φ , we get $r = \mathbf{0}$ which is absurd. Then, $\lim_{k \rightarrow +\infty} \rho_{\mu_i}(\mathfrak{S}_k) = \mathbf{0}$. Using the same arguments as in Theorem 2.3, we prove that A has a fixed point in \mathfrak{S} . \square

Now, we prove that Theorem 2.4 is still valid for changing condition (2.5) by the following one, for each $\mu_i \in \Lambda, i \in I_\Gamma$, we have

$$(2.7) \quad \varphi(\rho_{\mu_i}(A(C))) \preceq \varphi(\rho_{\mu_i}(C)) - \psi(\rho_{\mu_i}(C)), \quad C \subset \mathfrak{S}.$$

where φ and ψ verified conditions (\mathcal{H}_1) and (\mathcal{H}_2) , respectively.

Theorem 2.5. *Let $\Lambda = \{\mu_i, i \in I_\Gamma\}$ be family of GSNs on a Hausdorff GLCS Γ , and $\rho_\Lambda = \{\mu_i, i \in I_\Gamma\}$ be a family of GMNC. Let \mathfrak{S} be a nonempty bounded, closed, convex subset of Γ and $A : \mathfrak{S} \rightarrow \mathfrak{S}$ be a continuous mapping verified conditions (2.7). Then, A has at least one fixed point in \mathfrak{S} .*

Proof. From (2.7), we have

$$\begin{aligned} \varphi(\rho_{\mu_i}(\mathfrak{S}_{k+1})) &= \varphi(\rho_{\mu_i}(\overline{\text{conv}}(A(\mathfrak{S}_k)))) \\ &\preceq \varphi(\rho_{\mu_i}(A(\mathfrak{S}_k))) \preceq \varphi(\rho_{\mu_i}(\mathfrak{S}_k)) - \psi(\rho_{\mu_i}(\mathfrak{S}_k)) \\ &\preceq \varphi(\rho_{\mu_i}(\mathfrak{S}_k)) \end{aligned}$$

Using the fact that φ is nondecreasing, we obtain $\rho_{\mu_i}(\mathfrak{S}_{k+1}) \preceq \rho_{\mu_i}(\mathfrak{S}_k)$. Hence, the sequence $(\rho_{\mu_i}(\mathfrak{S}_k))_k$ is a positive decreasing sequence. So, $\lim_{k \rightarrow +\infty} \rho_{\mu_i}(\mathfrak{S}_k) = \mathbf{0}$. By the same arguments as in Theorem 2.3, we prove that A has a fixed point in \mathfrak{S} . \square

In what follows, we will give a generalization of Theorem 2.4.

Theorem 2.6. *Let $\Lambda = \{\mu_i, i \in I_\Gamma\}$ be family of GSNs on a Hausdorff GLCS X , and $\rho_\Lambda = \{\mu_i, i \in I_\Gamma\}$ be a family of GMNC. Let \mathfrak{S} be a nonempty, bounded, closed, convex subset of Γ . Assume that $\varphi, \psi : [0, +\infty)^n \rightarrow [0, +\infty)^n$ two functions verified conditions (\mathcal{H}_1) and (\mathcal{H}_2) . $A : \mathfrak{S} \rightarrow \mathfrak{S}$ be a continuous mapping such that for all $p \in \Lambda$ and $i \in I_\Gamma$ we have*

$$(2.8) \quad \varphi(\rho_{\mu_i}(A(C))) \preceq \psi(\rho_{\mu_i}(C)),$$

for C be subset of \mathfrak{S} . Then, A has at least one fixed point in \mathfrak{S} .

Proof. Let $(\mathfrak{S}_k)_k$ the sequence define by (2.1). Using (2.8), we obtain

$$\begin{aligned}
 \varphi(\rho_{\mu_i}(\mathfrak{S}_{k+1})) &= \varphi(\rho_{\mu_i}(\overline{\text{conv}}(A(\mathfrak{S}_k)))) \\
 &\preceq \varphi(\rho_{\mu_i}(A(\mathfrak{S}_k))) \preceq \psi(\rho_{\mu_i}(\mathfrak{S}_k)) \\
 (2.9) \qquad \qquad \qquad &\prec \varphi(\rho_{\mu_i}(\mathfrak{S}_k)).
 \end{aligned}$$

Clearly, $(\varphi(\rho_{\mu_i}(\mathfrak{S}_k)))_k$ is a positive decreasing sequence. Using (\mathcal{H}_1) , we obtain that $\rho_{\mu_i}(\mathfrak{S}_k)_k$ is also a positive decreasing sequence of real numbers. Thus, for all $i \in I_\Gamma$, there exist an $r \succeq \mathbf{0}$ such that $\lim_{k \rightarrow +\infty} \rho_{\mu_i}(\mathfrak{S}_k) = r$. When $k \rightarrow +\infty$ in (2.9), we get $\varphi(r) = \mathbf{0}$ and so $r = \mathbf{0}$. This completes the proof. \square

3. COUPLED FIXED POINT VIA A GMNC IN GLCSs

Now, motivated and inspired by the work of Banaś and Gobel [4], we state the following results.

Lemma 3.1. *Let $\Lambda = \{\mu_i, i \in I_\Gamma\}$ be a family of GSN, and $\rho_\Lambda = \{\rho_{\mu_i}, \mu_i \in \Lambda\}_{i \in I_\Gamma}$ be a family of GMNC on X . Let C be a bounded subset of $\Gamma \times \Gamma$. Consider the operator $\tilde{\rho}_{\mu_i} : \mathcal{B}(\Gamma \times \Gamma) \rightarrow \mathbb{R}_+^n$ defined by*

$$\tilde{\rho}_{\mu_i}(C) = \widetilde{\text{max}}\{\rho_{\mu_i}(C_1), \rho_{\mu_i}(C_2)\}, \quad \text{for each } \mu_i \in \Lambda, i \in I_\Gamma,$$

where C_1 and C_2 are the natural projection of C . Then, $\tilde{\rho}_\Lambda := \{\tilde{\rho}_{\mu_i}, \mu_i \in \Lambda\}$ is a family of GMNC on $\Gamma \times \Gamma$.

Proof. (i) Let C be a nonempty bounded subset of $\Gamma \times \Gamma$ then for $\mu_i \in \Lambda, i \in I_\Gamma$ Let C be a nonempty bounded subset of $\Gamma \times \Gamma$. Then, for $\mu_i \in \Lambda, i \in I_\Gamma$

$$\tilde{\rho}_{\mu_i}(\overline{\text{conv}}(C)) = \widetilde{\text{max}}\{\rho_{\mu_i}(\overline{\text{conv}}(C_1)), \rho_{\mu_i}(\overline{\text{conv}}(C_2))\}.$$

Since $\{\rho_{\mu_i}\}_{i \in I_\Gamma}$ is a family of GMNC, and the function max is nondecreasing, then

$$\begin{aligned}
 \tilde{\rho}_{\mu_i}(\overline{\text{conv}}(C)) &:= \left(\begin{array}{c} \max\{\rho_{\mu_i}^1(\overline{\text{conv}}(C_1)), \rho_{\mu_i}^1(\overline{\text{conv}}(C_2))\} \\ \vdots \\ \max\{\rho_{\mu_i}^n(\overline{\text{conv}}(C_1)), \rho_{\mu_i}^n(\overline{\text{conv}}(C_2))\} \end{array} \right) \\
 &\preceq \left(\begin{array}{c} \max\{\rho_{\mu_i}^1(C_1), \rho_{\mu_i}^1(C_2)\} \\ \vdots \\ \max\{\rho_{\mu_i}^n(C_1), \rho_{\mu_i}^n(C_2)\} \end{array} \right) \\
 &= \widetilde{\text{max}}\{\rho_{\mu_i}(C_1), \rho_{\mu_i}(C_2)\} \\
 &= \tilde{\rho}_{\mu_i}(C).
 \end{aligned}$$

(ii) Let C_a, C_b be two bounded subsets of $\Gamma \times \Gamma$ such that $C_a \subset C_b$, then $C_{a_j} \subset C_{b_j}$, for $j = 1, 2$. Since $\{\rho_{\mu_i}\}_{i \in I_\Gamma}$ is a family of GMNC in Γ , then for all $\mu_i \in \Lambda, i \in I_\Gamma$, we obtain

$$\rho_{\mu_i}(C_{a_j}) \preceq \rho_{\mu_i}(C_{b_j}), \quad j = 1, 2,$$

$$\begin{aligned}
\tilde{\rho}_{\mu_i}(C_a) &= \widetilde{\max}\{\rho_{\mu_i}(C_{a_1}), \rho_{\mu_i}(C_{a_2})\} \\
&:= \begin{pmatrix} \max\{\rho_{\mu_i}^1(C_{a_1}), \rho_{\mu_i}^1(C_{a_2})\} \\ \vdots \\ \max\{\rho_{\mu_i}^n(C_{a_1}), \rho_{\mu_i}^n(C_{a_2})\} \end{pmatrix} \\
&\preceq \begin{pmatrix} \max\{\rho_{\mu_i}^1(C_{b_1}), \rho_{\mu_i}^1(C_{b_2})\} \\ \vdots \\ \max\{\rho_{\mu_i}^n(C_{b_1}), \rho_{\mu_i}^n(C_{b_2})\} \end{pmatrix} \\
&= \widetilde{\max}\{\rho_{\mu_i}(C_{b_1}), \rho_{\mu_i}(C_{b_2})\} \\
&= \tilde{\rho}_{\mu_i}(C_b).
\end{aligned}$$

(iii) Let $u = (u_1, u_2) \in \Gamma \times \Gamma$ and $C \in \mathcal{B}(\Gamma \times \Gamma)$. For $\mu_i \in \Lambda$, $i \in I_\Gamma$,

$$\begin{aligned}
\tilde{\rho}_{\mu_i}(u \cup C) &= \widetilde{\max}\{\rho_{\mu_i}(u_1 \cup C_1), \rho_{\mu_i}(u_2 \cup C_2)\} \\
&:= \begin{pmatrix} \max\{\rho_{\mu_i}^1(u_1 \cup C_1), \rho_{\mu_i}^1(u_2 \cup C_2)\} \\ \vdots \\ \max\{\rho_{\mu_i}^n(u_1 \cup C_1), \rho_{\mu_i}^n(u_2 \cup C_2)\} \end{pmatrix} \\
&= \begin{pmatrix} \max\{\rho_{\mu_i}^1(C_1), \rho_{\mu_i}^1(C_2)\} \\ \vdots \\ \max\{\rho_{\mu_i}^n(C_1), \rho_{\mu_i}^n(C_2)\} \end{pmatrix} \\
&= \widetilde{\max}\{\rho_{\mu_i}(C_1), \rho_{\mu_i}(C_2)\} \\
&= \tilde{\rho}_{\mu_i}(C).
\end{aligned}$$

(iv) Let $C \in \mathcal{B}(\Gamma \times \Gamma)$ such that for $\mu_i \in \Lambda$, $i \in I_\Gamma$; $\tilde{\rho}_{\mu_i}(C) = \mathbf{0}$,

$$\begin{aligned}
\tilde{\rho}_{\mu_i}(C) &= \widetilde{\max}\{\rho_{\mu_i}(C_1), \rho_{\mu_i}(C_2)\} \\
&= \begin{pmatrix} \max\{\rho_{\mu_i}^1(C_1), \rho_{\mu_i}^1(C_2)\} \\ \vdots \\ \max\{\rho_{\mu_i}^n(C_1), \rho_{\mu_i}^n(C_2)\} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.
\end{aligned}$$

It follows that

$$\max\{\rho_{\mu_i}^s(C_1), \rho_{\mu_i}^s(C_2)\} = 0, \quad \text{for all } s = 1, \dots, n,$$

which yields that for all $s = 1, \dots, n$ and $j = 1, 2$, $\rho_{\mu_i}^s(C_j) = 0$. Since $\{\rho_{\mu_i}^s\}_{\mu_i \in \Lambda}$ is a family of MNC, then C_j , $j = 1, 2$ are relatively compact and by Tychonoff product Theorem [5, p. 19], we obtain that $C_1 \times C_2$ is relatively compact and $\overline{C} \subseteq \overline{C_1} \times \overline{C_2}$ so C is also relatively compact.

(v) Let $(\mathfrak{S}_k)_k = (V_k, W_k)$ be a sequence of closed subset of $\mathfrak{B}(\Gamma \times \Gamma)$ such that $\mathfrak{S}_{k+1} \subset \mathfrak{S}_k$ and $\lim_{k \rightarrow +\infty} \tilde{\rho}_{\mu_i}(\mathfrak{S}_k) = \mathbf{0}$, for $\mu_i \in \Lambda, i \in I_\Gamma$. Hence,

$$(3.1) \quad V_{k+1} \subset V_k, \quad W_{k+1} \subset W_k,$$

and for $\mu_i \in \Lambda, i \in I_\Gamma$, we have

$$(3.2) \quad \lim_{k \rightarrow +\infty} \rho_{\mu_i}(V_k) = \lim_{k \rightarrow +\infty} \rho_{\mu_i}(W_k) = \mathbf{0}.$$

Known that $\{\rho_{\mu_i}\}_{\mu_i \in \Lambda}$ is a family of GMNC in X , and from (3.1), (3.2), we get

$$(3.3) \quad \lim_{k \rightarrow +\infty} \rho_{\mu_i}(V_{+\infty}) = \lim_{k \rightarrow +\infty} \rho_{\mu_i}(W_{+\infty}) = \mathbf{0}.$$

Moreover,

$$(3.4) \quad \tilde{\rho}_{\mu_i}(\mathfrak{S}_{+\infty}) = \tilde{\rho}_{\mu_i}(\bigcap_{k=1}^{+\infty} \mathfrak{S}_k) = \widetilde{\max}\{\rho_{\mu_i}(\bigcap_{k=1}^{+\infty} V_k), \rho_{\mu_i}(\bigcap_{k=1}^{+\infty} W_k)\}.$$

By (3.3), (3.4), we obtain $\lim_{k \rightarrow +\infty} \tilde{\rho}_{\mu_i}(\mathfrak{S}_{+\infty}) = \mathbf{0}$, and by Definition 1.4 (v), $\mathfrak{S}_{+\infty}$ is relatively compact subset of $\Gamma \times \Gamma$. Consequently, $\tilde{\rho}$ is a family of GMNC in $\Gamma \times \Gamma$. □

Lemma 3.2. *Let $\Lambda = \{\mu_i, i \in I_\Gamma\}$ be a family of GSN and $\rho_\Lambda = \{\rho_{\mu_i}, \mu_i \in \Lambda\}_{i \in I_\Gamma}$ be a family of GMNC on Γ . Let C be bounded subset of $\Gamma \times \Gamma$. Consider the operator $\tilde{\rho}_{\mu_i} : \mathfrak{B}(\Gamma \times \Gamma) \rightarrow \mathbb{R}_+^n$ defines by*

$$\tilde{\rho}_{\mu_i}(C) = \frac{\rho_{\mu_i}(C_1) + \rho_{\mu_i}(C_2)}{2}, \quad \text{for each } \mu_i \in \Lambda, i \in I_\Gamma,$$

where C_1 and C_2 are the natural projection of C . Then, $\tilde{\rho}_\Lambda := \{\tilde{\rho}_{\mu_i}, \mu_i \in \Lambda\}$ is a family of GMNC on $\Gamma \times \Gamma$.

Proof. We can check the proof easily. □

Example 3.1. For C a nonempty bounded subset of \mathfrak{C} , $T > 0$, we have shown in Example 1.1 that $\rho_T(C) = \begin{pmatrix} \omega_0(C) \\ \omega_0(C) \end{pmatrix}$ defines a family of GMNC in \mathfrak{C} . Let C be bounded subset of $\mathfrak{C} \times \mathfrak{C}$. Let $T > 0$,

$$\begin{aligned} \text{(i)} \quad \tilde{\rho}_T^1(C) &= \widetilde{\max}\{\rho_T(C_1), \rho_T(C_2)\} = \begin{pmatrix} \max\{\omega_0(C_1), \omega_0(C_2)\} \\ \max\{\omega_0(C_1), \omega_0(C_2)\} \end{pmatrix}, \\ \text{(ii)} \quad \tilde{\rho}_T^2(C) &= \frac{\rho_T(C_1) + \rho_T(C_2)}{2} = \frac{1}{2} \begin{pmatrix} \omega_0(C_1) + \omega_0(C_2) \\ \omega_0(C_1) + \omega_0(C_2) \end{pmatrix}, \\ \text{(iii)} \quad \tilde{\rho}_T^3(C) &= \begin{pmatrix} \omega_0(C_1) + \omega_0(C_2) \\ \max\{\omega_0(C_1), \omega_0(C_2)\} \end{pmatrix}, \end{aligned}$$

with C_1 and C_2 are the natural projection of C . Then, for $s = 1, 2, 3$, $\tilde{\rho}_T^s = \{\tilde{\rho}_T^s, T > 0\}$ is a family of GMNC in $C(\mathbb{R}_+) \times C(\mathbb{R}_+)$.

In the same way as the above proofs, we can extend Lemma 3.1 and Lemma 3.2 for Γ^n .

Lemma 3.3. *Let $\Lambda = \{\mu_i, i \in I_\Gamma\}$ be a family of GSN, and $\rho_\Lambda = \{\rho_{\mu_i}, \mu_i \in \Lambda\}_{i \in I_\Gamma}$ be a family of GMNC on Γ , and let C be a bounded subset of Γ^n . For $\mu_i \in \Lambda, i \in I_\Gamma$, we define the operator $\tilde{\rho}_{\mu_i} : \mathcal{B}(\Gamma^n) \rightarrow \mathbb{R}_+^n$ by*

$$\tilde{\rho}_{\mu_i}(C) = \widetilde{\max}\{\rho_{\mu_i}(C_1), \rho_{\mu_i}(C_2), \dots, \rho_{\mu_i}(C_n)\},$$

or

$$\tilde{\rho}_{\mu_i}(C) = \frac{1}{n}(\rho_{\mu_i}(C_1) + \rho_{\mu_i}(C_2) + \dots + \rho_{\mu_i}(C_n)),$$

where $C_j, j = 1, \dots, n$ are the natural projection of C . Then, $\tilde{\rho}_\Lambda = \{\tilde{\rho}_{\mu_i}, \mu_i \in \Lambda\}$ define a family of GMNC in Γ^n .

In what follows we state some results to solve the system (1.1).

Theorem 3.1. *Let $\Lambda = \{\mu_i, i \in I_\Gamma\}$ be family of GSNs on a Hausdorff GLCS X , and $\rho_\Lambda = \{\rho_{\mu_i}, i \in I_\Gamma\}$ be a family of GMNC on Γ . Let \mathfrak{S} be a nonempty, bounded, closed, convex subset of Γ . Assume that*

- (i) $\xi_j : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathfrak{S}, j = 1, 2$ are continuous operators,
- (ii) $\psi : [0, +\infty)^n \rightarrow [0, +\infty)^n$ be a function verified condition (\mathcal{H}_2) and $\psi(t) \prec t$, for all $t \succ \mathbf{0}$, such that

$$(3.5) \quad \rho_{\mu_i}(\xi_j(C)) \preceq \psi(\widetilde{\max}\{\rho_{\mu_i}(C_1), \rho_{\mu_i}(C_2)\}), \quad j = 1, 2,$$

where C is a bounded subset of $\Gamma \times \Gamma$ and $C_j, j = 1, 2$ denotes the natural projection of C .

Then, the system (1.1) has at least one solution in $\mathfrak{S} \times \mathfrak{S}$.

Proof. Let C be a nonempty subset of $\Gamma \times \Gamma$, for $\mu_i \in \Lambda, i \in I_\Gamma$. By Lemma 3.1

$$\tilde{\rho}_{\mu_i}(C) = \widetilde{\max}\{\rho_{\mu_i}(C_1), \rho_{\mu_i}(C_2)\},$$

defines a family of GMNC in $\Gamma \times \Gamma$. Let us defines the operator $A : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathfrak{S} \times \mathfrak{S}$ as

$$A(x, y) = (\xi_1(x, y), \xi_2(x, y))$$

It is clear that A is continuous. From (3.5), and for $\mu_i \in \Lambda, i \in I_\Gamma$, we obtain that

$$\begin{aligned} \tilde{\rho}_{\mu_i}(A(C)) &\preceq \tilde{\rho}_{\mu_i}(\xi_1(C) \times \xi_2(C)) \\ &:= \widetilde{\max}\{\rho_{\mu_i}(\xi_1(C)), \rho_{\mu_i}(\xi_2(C))\} \\ &\preceq \widetilde{\max}\{\psi(\widetilde{\max}\{\rho_{\mu_i}(C_1)\}, \rho_{\mu_i}(C_2)), \psi(\widetilde{\max}\{\rho_{\mu_i}(C_1), \rho_{\mu_i}(C_2)\})\} \\ &\preceq \psi(\widetilde{\max}\{\rho_{\mu_i}(C_1), \rho_{\mu_i}(C_2)\}), \\ &= \psi(\tilde{\rho}_{\mu_i}(C)). \end{aligned}$$

We choose $\rho(t) = t, t > 0$. Then, by Theorem 2.6, there exists a point $(x', y') \in \mathfrak{S} \times \mathfrak{S}$ such that $(x', y') = A(x', y')$, which is a solution of the system (1.1). □

Corollary 3.1. *Let $\Lambda = \{\mu_i, i \in I_\Gamma\}$ be family of GSNs on a Hausdorff GLCS X , and $\rho_\Lambda = \{\rho_{\mu_i}, i \in I_\Gamma\}$ be a family of GMNC on X . Let \mathfrak{S} be a nonempty, bounded, closed, convex subset of Γ . Assume that*

- (i) $\xi_j : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathfrak{S}, j = 1, 2$ are continuous operators,
- (ii) for all $\mu_i \in \Lambda, i \in I_\Gamma$ there exists a convergent matrix \mathcal{L}_{μ_i} , such that

$$(3.6) \quad \rho_{\mu_i}(\xi_j(C)) \preceq \mathcal{L}_{\mu_i} \left(\frac{\rho_{\mu_i}(C_1) + \rho_{\mu_i}(C_2)}{2} \right), \quad j = 1, 2,$$

where C is a bounded subset of $\Gamma \times \Gamma$ and $C_j, j = 1, 2$ denotes the natural projection of C . Then, the system (1.1) has at least one solution in $\mathfrak{S} \times \mathfrak{S}$.

Proof. Let C be a bounded subset of $\Gamma \times \Gamma$, then by Lemma 3.2, we have that for $\mu_i \in \Lambda, i \in I_\Gamma$

$$\tilde{\rho}_{\mu_i}(C) = \frac{\rho_{\mu_i}(C_1) + \rho_{\mu_i}(C_2)}{2},$$

is a family of GMNC in $\Gamma \times \Gamma$. Let us defines the operator $A : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathfrak{S} \times \mathfrak{S}$ as

$$A(u, v) = (\xi_1(u, v), \xi_2(u, v)).$$

It is clear that A is continuous. From (3.6), and for $\mu_i \in \Lambda, i \in I_\Gamma$, we get

$$\begin{aligned} \tilde{\rho}_{\mu_i}(A(C)) &\preceq \tilde{\rho}_{\mu_i}(A_1(C), \xi_2(C)) \\ &:= \frac{1}{2}(\rho_{\mu_i}(\xi_1(C)) + \rho_{\mu_i}(\xi_2(C))) \\ &\preceq \mathcal{L}_{\mu_i} \left(\frac{\rho_{\mu_i}(C_1) + \rho_{\mu_i}(C_2)}{2} \right) \\ &= \mathcal{L}_{\mu_i} \tilde{\rho}_{\mu_i}(C). \end{aligned}$$

Then, the result follows from Theorem 2.1 □

4. SOLVING A SYSTEM OF INTEGRAL EQUATIONS IN $\mathfrak{E} \times \mathfrak{E}$

In this section, we study the existence of solutions to the following system of integral equations:

$$(4.1) \quad \begin{cases} u(t) = f_1(t, u(t), v(t), \int_0^{\beta_1(t)} g_1(t, s, u(\alpha_1(s)), v(\alpha_1(s)))ds), \\ v(t) = f_2(t, u(t), v(t), \int_0^{\beta_2(t)} g_2(t, s, u(\alpha_2(s)), v(\alpha_2(s)))ds) \end{cases}$$

such that for $j = 1, 2$, we have the following hypothesis.

- (\mathcal{H}_1) $\alpha_j, \beta_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous functions.
- (\mathcal{H}_2) $f_j : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}$ are continuous. Moreover, there exists function $k_j \in L^1_{loc}(\mathbb{R}_+)$ such that

$$|f_j(t, u_1, v_1, w_1) - f_j(t, u_2, v_2, w_2)| \leq k_j(t)[|u_1 - u_2| + |v_1 - v_2|] + |w_1 - w_2|,$$

and

$$\eta_j = \sup\{|f_j(t, 0, 0, 0)|, t \in \mathbb{R}_+\} < +\infty.$$

(\mathcal{H}_3) $g_j : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous and

$$|g_j(t, s, u(s), v(s))| \leq \max\{|u(s)|, |v(s)|\}.$$

(\mathcal{H}_4) For $j=1,2$,

$$2\|k_j\|_{L^1_{loc}} + |\alpha|_T \beta_j^T < 1,$$

where $\beta_j^T := \sup\{\beta_j(t), t \in [0, T], T > 0\}$.

Theorem 4.1. *Let the hypothesis (\mathcal{H}_1)-(\mathcal{H}_4) hold. Then, the system of equations (4.1) has at least one solution in $\mathfrak{C} \times \mathfrak{C}$.*

Proof. (i) Let $\Gamma = \mathfrak{C}$. For $j = 1, 2$, let us consider the map $\xi_j : \Gamma \times \Gamma \rightarrow \Gamma$ defined by

$$\xi_j(u, v)(t) = f_j\left(t, u(t), v(t), \int_0^{\beta_j(t)} g_j(t, s, u(\alpha_j(s)), v(\alpha_j(s))) ds\right).$$

Clearly that for $j = 1, 2$, ξ_j is continuous on \mathbb{R}_+ for all $u, v \in \Gamma$. Also, for $u, v \in \Gamma$, and by assumption (\mathcal{H}_2) and (\mathcal{H}_3), we have

$$\begin{aligned} |\xi_j(u, v)(t)| &= \left| f_j\left(t, u(t), v(t), \int_0^{\beta_j(t)} g_j(t, s, u(\alpha(s)), v(\alpha(s))) ds\right) - f_j(t, 0, 0, 0) \right| \\ &\quad + |f_j(t, 0, 0, 0)| \\ (4.2) \quad &\leq k_j(t)[|u(t)| + |v(t)|] + \int_0^{\beta_j(t)} \max\{|u(\alpha(s))|, |v(\alpha(s))|\} ds + |f_j(t, 0, 0, 0)|. \end{aligned}$$

Thus,

$$|\xi_j(u, v)|_T \leq \|k_j\|_{L^1_{loc}}[|u|_T + |v|_T] + \beta_j^T |\alpha|_T \max\{|u|_T, |v|_T\} + \eta_j.$$

Hence, ξ_j maps $B_r \times B_r$ into B_r for $r = \max\left\{\frac{\eta_j}{1 - (2\|k_j\|_{L^1_{loc}} + |\alpha|_T \beta_j^T)}, j = 1, 2\right\}$.

(ii) Let C_1, C_2 be nonempty subset of B_r . Let $(u, v) \in C_1 \times C_2$, $\varepsilon \geq 0$ and $t_1, t_2 \in \mathbb{R}_+$ such that $|t_1 - t_2| \leq \varepsilon$. We assume that $t_1 \geq t_2$. For all $j = 1, 2$, we have

$$\begin{aligned} \left| \xi_j(u, v)(t_1) - \xi_j(u, v)(t_2) \right| &\leq \left| f_j\left(t, u(t), v(t), \int_0^{\beta_j(t_1)} g_j(t_1, s, x(\alpha_j(s)), y(\alpha_j(s))) ds\right) \right. \\ &\quad \left. - f_j\left(t, u(t), v(t), \int_0^{\beta_j(t_2)} g_j(t_2, s, u(\alpha_j(s)), v(\alpha_j(s))) ds\right) \right| \\ &\leq \|k_j\|_{L^1_{loc}}[|u(t_1) - v(t_2)| + |v(t_1) - v(t_2)|] \\ &\quad + \int_0^{\beta_j(t_1)} |g_j(t_1, s, u(\alpha_j(s)), v(\alpha_j(s))) \\ &\quad - g_j(t_2, s, u(\alpha_j(s)), v(\alpha_j(s)))| ds \\ &\quad + \int_{\beta_j(t_2)}^{\beta_j(t_1)} |g_j(t_2, s, u(\alpha_j(s)), v(\alpha_j(s)))| ds. \end{aligned}$$

Hence,

$$\omega^T(\xi_j(u, v), \varepsilon) \leq \|k_j\|_{L^1_{loc}} [\omega^T(u, \varepsilon) + \omega^T(v, \varepsilon)] + \beta^T \omega^T(g_j, \varepsilon) + r\omega^T(\beta_j, \varepsilon),$$

with $\beta^T = \max \{ \beta_j^T, j = 1, 2 \}$,

$$\omega^T(g_j, \varepsilon) = \sup \left\{ \left| g_j(t_1, s, u(\alpha_j(s)), v(\alpha_j(s))) - g_j(t_2, s, u(\alpha_j(s)), v(\alpha_j(s))) \right| \right\},$$

$t_1, t_2 \in [0, T]$, $|t_1 - t_2| \leq \varepsilon$, $u, v \in [-r, r]$, $s \in [0, \beta^T]$, $\omega^T(\beta_j, \varepsilon) = \sup \{ |\beta_j(t_1) - \beta_j(t_2)|, t_1, t_2 \in [0, T], |t_1 - t_2| \leq \varepsilon \}$.

By the uniform continuity of g_j on $[0, T] \times [0, \beta^T] \times [-r, r] \times [-r, r]$, we obtain $\lim_{\varepsilon \rightarrow 0} \omega^T(g_j, \varepsilon) = 0$ and also by the uniform continuity of β_j on $[0, T]$, we derive $\lim_{\varepsilon \rightarrow 0} \omega^T(\beta_j, \varepsilon) = 0$. Consequently,

$$\omega_0^T(\xi_j(C_1, C_2)) \leq \|k_j\|_{L^1_{loc}} [\omega_0^T(C_1) + \omega_0^T(C_2)],$$

which yields that for $j = 1, 2$

$$\rho_T(\xi_j(C)) \preceq \mathcal{L}_T \left(\frac{\rho_T(C_1) + \rho_T(C_2)}{2} \right), \quad j = 1, 2,$$

with $\mathcal{L}_T = \begin{pmatrix} k & k \\ k & k \end{pmatrix}$ and $k = \max_{j=1,2} \|k_j\|_{L^1_{loc}}$.

From assumption (\mathcal{H}_4) , the matrix \mathcal{L}_T is convergent and by Corollary 3.1, the operator $A(u, v) = (\xi_1(u, v), \xi_2(u, v))$ has a fixed point in, which is a solution of system of integral equation (4.1). \square

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