Kragujevac Journal of Mathematics
Volume 51(7) (2027), Pages 1107–1122.

COALITION IN n-INORDINATE INVARIANT NON-INTERSECTION GRAPHS

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Abstract. A class of algebraic intersection graphs, called the n-inordinate invariant non-intersection graphs, was introduced and various properties of these graphs are investigated, in the literature. Domination is an important structural property in graphs and the notion of coalition has been recently introduced in graphs based on domination in graphs. In this article, we analyze the structure of the n-inordinate invariant non-intersection graphs by investigating coalition partitions in them.

1. Introduction

For terminology in group theory, we refer to [4]. For basic definitions and results in graph theory, see [23] and for further concepts in domination in graphs, see [3]. Also, the reader may refer to [5], for the fundamentals in combinatorics.

Algebraic graph theory is a promising area of research in which graphs, and intersection graphs are constructed based on algebraic structures and their properties are investigated (see [6, 8]).

In this regard, a class of algebraic derived graphs, called *invariant intersection graphs* were introduced in [15] based on the automorphism groups of graphs and their properties were studied in [18]. Based on the investigation in [15] and [18], a specific class of these algebraic derived graphs were identified as the *n-inordinate invariant intersection graphs* and their properties were studied in [21].

Key words and phrases. Invariant intersection graphs, invariant non-intersection graphs, n-inordinate invariant intersection graphs, n-inordinate invariant non-intersection graphs, domination, coalition.

2020 *Mathematics Subject Classification.* Primary: 05C25. Secondary: 05C62, 05C69, 05C70.

DOI

Received: September 09, 2025.

Accepted: April 16, 2026.

In [21], the n-inordinate invariant intersection graphs were defined independently of the automorphism groups of graphs, relying instead on the symmetric group S_n-the group of all permutations of a set S with n elements. Along with this, the complement of the n-inordinate invariant intersection graphs, called the *n-inordinate invariant non-intersection graphs* was also introduced, as follows. The *n-inordinate invariant non-intersection graphs*, denoted by $\overline{\Lambda}_{K_n}$, is the graph with $V(\overline{\Lambda}_{K_n}) = \{v_\pi : \pi \in S_n\}$ and any two distinct vertices $v_{\pi_1}, v_{\pi_2} \in V(\overline{\Lambda}_{K_n})$ corresponding to permutations $\pi_1,\ \pi_2 \in S_n$ are adjacent in $\overline{\Lambda}_{K_n}$, when $\text{fix}(\pi_1) \cap \text{fix}(\pi_2) = \emptyset$.

In [21], the structural properties of $\overline{\Lambda}_{K_n}$ were investigated, and it has been found that $\overline{\Lambda}_{K_n}$ contains $\rho(n)$ universal vertices, where $\rho(n)$ denotes the number of derangements of n elements. Following this, different properties of these graphs such as proper vertex colorings, dominator colorings, color connections and centrality based graph entropy were discussed in [7, 9–14, 16, 17, 19, 20].

An illustration of an n-inordinate invariant non-intersection graph is given in Figure 1. Note that the identity permutation is denoted by π_0 and the vertex corresponding to any permutation $\pi \in S_n$ is denoted by $v_\pi \in V(\overline{\Lambda}_{K_n})$, throughout the study.

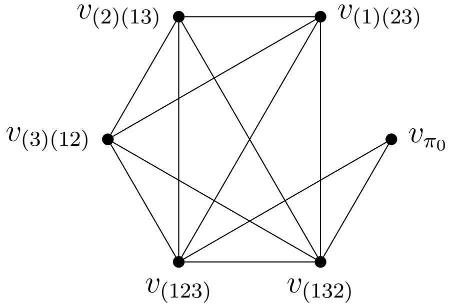

FIGURE 1. The 3-inordinate invariant non-intersection graph.

A set $D \subseteq V(G)$ is said to be a *dominating set* of a graph G, if for all $v \in V(G) - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$. Based on this, the notion of *coalition* in a graph G has been recently introduced as follows. For any two disjoint subsets $D, D' \subset V(G)$, that are not dominating sets of G, $D \cup D'$ must be a dominating set of G (see [1]).

Partitioning the vertex set of a graph with a specific property is a well-known problem in graph theory. As domination in graphs is an important structural property of graphs that has real-life applications in networks, the notion of partitioning the vertex set of a graph such that its partite sets form coalitions to possess the property of different types of domination in the graph are being introduced in the literature.

As coalition partitions of graphs give interesting structural insights about the graphs, we analyse the structure of $\overline{\Lambda}_{K_n}$ by discussing some variants of coalition partitions of these graphs, in this article. Also, as coalition in graphs is very recently introduced, they have not been investigated for any of the algebraic graphs defined thus far, except one, in the literature, making our study the first of its kind (see [14]).

2. Coalition in n-Inordinate Invariant Non-Intersection Graphs

In this section, we discuss coalitions in the n-inordinate invariant non-intersection graphs $\overline{\Lambda}_{K_n}$, with respect to the property of domination, for which we use the notations V_i and V_i^k to denote the non-disjoint subsets $\{v_\pi : i \in \mathrm{fix}(\pi)\}$ and $\{v_\pi \in V_i : |\,\mathrm{fix}(\pi)| = k\}$, for $1 \leq i \leq n$ and $1 \leq k \leq n$, of $V(\overline{\Lambda}_{K_n})$, respectively.

It has been understood from the literature that the study of coalitions in a graph with universal or isolated vertices depends on the structure of its subgraph induced by its non-universal, non-isolated vertices. Hence, this study focuses on discussing coalitions in the subgraphs $\overline{\Lambda^{**}}_{K_n}$ and $\overline{\Lambda^{*}}_{K_n}$ of $\overline{\Lambda}_{K_n}$, that are induced by the vertices in $\bigcup\limits_{i=1}^{n} \bigcup\limits_{k=1}^{n-2} V_i^k$ and $\bigcup\limits_{i=1}^{n} \bigcup\limits_{k=1}^{n} V_i^k$, respectively.

A *coalition partition* σ of a graph G is a partition of $V(G)$ such that every partite set of σ is either a universal vertex of G, or is not a dominating set of G which on forming coalition with another partite set in σ becomes a dominating set of G. The *coalition number* of G, denoted by $C(G)$, is the maximum order of a coalition partition of G (see [2]). First, we examine coalition in $\overline{\Lambda^{**}}_{K_n}$, and obtain its coalition number as follows.

Theorem 2.1. *For $n \geq 3$,*

$$C(\overline{\Lambda^{**}}_{K_n}) = \begin{cases} 3, & \textit{if } n = 3, \\ 11, & \textit{if } n = 4, \\ 55, & \textit{if } n = 5, \\ 2 + \left(\binom{n}{2} - \binom{\lceil \frac{n}{2} \rceil - 1}{2}\right)(3-n) & \\ + \sum\limits_{k=1}^{2} \binom{n}{k} \rho(n-k) + \sum\limits_{k=3}^{\lceil \frac{n}{2} \rceil - 1} \binom{\lceil \frac{n}{2} \rceil - 1}{k} \rho(n-k), & \textit{if } n \geq 6. \end{cases}$$

Proof. As $\overline{\Lambda^{**}}_{K_3} \cong K_3$, the result follows. Therefore, consider a partition $\sigma_c = \{D_j : 1 \leq j \leq x\}$ of $\overline{\Lambda^{**}}_{K_n}$, $n \geq 4$, where $x = \left(\binom{n}{2} - \binom{\lceil \frac{n}{2} \rceil - 1}{2}\right)(3-n) + \sum\limits_{k=1}^{2} \binom{n}{k} \rho(n-k)$, of $V(\overline{\Lambda^{**}}_{K_n})$ defined as follows:

(i) $D_1 = \bigcup\limits_{i=1}^{n} \bigcup\limits_{k=\lceil \frac{n}{2} \rceil}^{n-2} V_i^k - \{v_{(1)(2)\ldots(\lfloor \frac{n}{2} \rfloor+1)(\lfloor \frac{n}{2} \rfloor+1\,\lfloor \frac{n}{2} \rfloor+2\,\ldots\,n-1\,n)}\}$,

(ii) $D_2 = \bigcup\limits_{i=1}^{\lfloor \frac{n}{2} \rfloor+1} \bigcup\limits_{k=3}^{\lceil \frac{n}{2} \rceil-1} V_i^k \cup \{v_{(1)(2)\ldots(\lfloor \frac{n}{2} \rfloor+1)(\lfloor \frac{n}{2} \rfloor+1\,\lfloor \frac{n}{2} \rfloor+2\,\ldots\,n-1\,n)}\}$,

(iii) $D_{j'}$, $3 \leq j' \leq 2+t$, where $t = \sum\limits_{k=3}^{\lceil \frac{n}{2} \rceil-1} \binom{\lceil \frac{n}{2} \rceil - 1}{k} \rho(n-k)$, are singletons containing one of the vertices of $\bigcup\limits_{i=1}^{n} \bigcup\limits_{k=3}^{n-2} V_i^k - (D_1 \cup D_2)$,

(iv) $D_{j'}$, $t+3 \le j' \le 2+t+\binom{\lceil \frac{n}{2}\rceil -1}{2}\rho(n-2)$, are singletons where each $D_{j'}$ contains one of the $\binom{\lceil \frac{n}{2}\rceil -1}{2}\rho(n-2)$ vertices of $\bigcup\limits_{i=1}^{n} V_i^2 - \bigcup\limits_{i=1}^{\lfloor \frac{n}{2}\rfloor+1} V_i^2$,

(v) $D_{j'}$, $t+3+\binom{\lceil \frac{n}{2}\rceil -1}{2}\rho(n-2) \le j' \le t+2+\binom{n}{2}+\binom{\lceil \frac{n}{2}\rceil -1}{2}(\rho(n-2)-1)$, contain $(n-2)$ vertices of $\bigcup\limits_{i=1}^{n} V_i^1$ with $\bigcap\limits_{\pi\in D_{j'}} \mathrm{fix}(\pi)=\emptyset$, $\{i : i \notin \bigcup\limits_{\pi\in D_{j'}} \mathrm{fix}(\pi)\} \subset \bigcup\limits_{i=1}^{\lfloor \frac{n}{2}\rfloor+1} V_i^2$, and $\bigcup\limits_{\pi\in D_{j'}} \mathrm{fix}(\pi) \ne \bigcup\limits_{\pi\in D_{j''}} \mathrm{fix}(\pi)$, for any $t+3+\binom{\lceil \frac{n}{2}\rceil -1}{2}\rho(n-2) \le j' \ne j'' \le t+2+\binom{n}{2}+\binom{\lceil \frac{n}{2}\rceil -1}{2}(\rho(n-2)-1)$,

(vi) $D_{j'}$, $t+3+\binom{n}{2}+\binom{\lceil \frac{n}{2}\rceil -1}{2}(\rho(n-2)-1) \le j' \le t+2+\left(\binom{n}{2}-\binom{\lceil \frac{n}{2}\rceil -1}{2}\right)(3-n)+\sum\limits_{k=1}^{2}\binom{n}{k}\rho(n-k)$, are singletons containing all the remaining vertices of $\bigcup\limits_{i=1}^{n} V_i^1 \cup \bigcup\limits_{i=1}^{\lfloor \frac{n}{2}\rfloor+1} V_i^2$.

The partition σ_c is a coalition partition of $\overline{\Lambda^{**}}_{K_n}$, as no $D_j \in \sigma_c$ is a dominating set of $\overline{\Lambda^{**}}_{K_n}$, and any singleton in σ_c containing a vertex of $\bigcup\limits_{i=1}^{\lfloor \frac{n}{2}\rfloor+1} V_i^k$, $k=1,2$, forms coalition with one of the non-singleton partite sets of σ_c constructed from the vertices of $\bigcup\limits_{i=1}^{n} V_i^1$ and all the other partite sets of σ_c forms a coalition with D_1. Hence,

$$C(\overline{\Lambda^{**}}_{K_n}) \ge 2+\left(\binom{n}{2}-\binom{\lceil \frac{n}{2}\rceil -1}{2}\right)(3-n)+\sum_{k=1}^{2}\binom{n}{k}\rho(n-k)+\sum_{k=3}^{\lceil \frac{n}{2}\rceil -1}\binom{\lceil \frac{n}{2}\rceil -1}{k}\rho(n-k).$$

Now, consider another partition $\sigma'_c = \{D'_1, D'_2, \ldots, D'_{n!-\rho(n)-(n-1)!+2}\}$ such that $D'_1 = V_1 - \{v_\pi\}$, $D'_2 = \{v_\pi\}$, for any vertex $v_\pi \in V_1^1$, and all the other partite sets of σ'_c are singletons consisting a vertex from $V(\overline{\Lambda^{**}}_{K_n}) - V_1$. The partition σ'_c is a maximal coalition partition of $V(\overline{\Lambda^{**}}_{K_n})$, as the partite set $D'_1 \in \sigma'_c$ dominates all the vertices of $\overline{\Lambda^{**}}_{K_n}$, except v_π, and all the other singletons in σ'_c are also not dominating sets of $\overline{\Lambda^{**}}_{K_n}$; whereas, $D'_1 \cup D'_j$, $2 \le j \le n!-\rho(n)-(n-1)!+1$, is a dominating set of $\overline{\Lambda^{**}}_{K_n}$. Hence, $C(\overline{\Lambda^{**}}_{K_n}) \ge n!-\rho(n)-(n-1)!+2$.

As $n!-\rho(n)-(n-1)!+2 > 2+\left(\binom{n}{2}-\binom{\lceil \frac{n}{2}\rceil -1}{2}\right)(3-n)+\sum\limits_{k=1}^{2}\binom{n}{k}\rho(n-k)+\sum\limits_{k=3}^{\lceil \frac{n}{2}\rceil -1}\binom{\lceil \frac{n}{2}\rceil -1}{k}\rho(n-k)$, for $n=4,5$,

$$C(\overline{\Lambda^{**}}_{K_n}) \ge \begin{cases} n!-\rho(n)-(n-1)!+2, & \text{if } n=4,5,\\ 2+\left(\binom{n}{2}-\binom{\lceil \frac{n}{2}\rceil -1}{2}\right)(3-n) & \\ +\sum\limits_{k=1}^{2}\binom{n}{k}\rho(n-k)+\sum\limits_{k=3}^{\lceil \frac{n}{2}\rceil -1}\binom{\lceil \frac{n}{2}\rceil -1}{k}\rho(n-k), & \text{if } n\ge 6. \end{cases}$$

If $C(\overline{\Lambda^{**}}_{K_4}) \geq 12$, it implies that $D'_1 \in \sigma'_c$ is altered, but it can be verified that $C(\overline{\Lambda^{**}}_{K_4}) = 11$, owing to the small value of n.

When $n = 5$, a modified version of the partition σ_c of order 55; greater than the previous two values, is obtained as follows. The partition $\sigma''_c = \{D'_j : 1 \leq j \leq 55\}$ of $V(\overline{\Lambda^{**}}_{K_5})$ such that D'_j, $1 \leq j \leq 10$, contains three vertices of $\bigcup\limits_{i=1}^{5} V_i^1$ with $\bigcap\limits_{\pi \in D'_{j'}} \text{fix}(\pi) = \emptyset$, $\left\{ i : i \notin \bigcup\limits_{\pi \in D'_{j'}} \text{fix}(\pi) \right\} \subset \bigcup\limits_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1} V_i^2$ and $\bigcup\limits_{\pi \in D'_{j'}} \text{fix}(\pi) \neq \bigcup\limits_{\pi \in D'_{j''}} \text{fix}(\pi)$, and D'_j, $11 \leq j \leq 55$, are singletons containing the remaining vertices of $\overline{\Lambda^{**}}_{K_5}$.

The only vertex that any non-singleton in the partition σ''_c of $\overline{\Lambda^{**}}_{K_5}$ does not dominate is a vertex $v_\pi \in V_i^3$ that has $\text{fix}(\pi) = \bigcup\limits_{v_{\pi'} \in D'_j} \text{fix}(\pi')$. Hence, there exists a coalition partner for any of the singletons that contain vertices that correspond to one of the $\binom{5}{2}$ distinct 3 or 2-element fix. Therefore, a coalition partition of $\overline{\Lambda^{**}}_{K_5}$ of order 55 is obtained; thereby, settling the problem when $n = 5$.

For $\overline{\Lambda^{**}}_{K_n}$, $n \geq 6$, let $C(\overline{\Lambda^{**}}_{K_n}) \geq 3 + \left(\binom{n}{2} - \binom{\lceil \frac{n}{2} \rceil - 1}{2} \right)(3 - n) + \sum\limits_{k=1}^{2} \binom{n}{k} \rho(n-k) + \sum\limits_{k=3}^{\lceil \frac{n}{2} \rceil - 1} \binom{\lceil \frac{n}{2} \rceil - 1}{k} \rho(n-k)$, and σ_c^* be such a coalition partition of $\overline{\Lambda^{**}}_{K_n}$. This implies that at least one of the $2 + \binom{n}{2} - \binom{\lceil \frac{n}{2} \rceil - 1}{2}$ non-singletons of σ_c is altered to obtain σ_c^*.

Assume that one of the $\binom{n}{2} - \binom{\lceil \frac{n}{2} \rceil - 1}{2}$ non-singletons, say $D_{t+3+\binom{\lceil \frac{n}{2} \rceil - 1}{2}\rho(n-2)}$, of σ_c that contains the vertices of $\bigcup\limits_{i=1}^{n} V_i^1$ is altered in σ_c^*. The removal of at least one of the $n-2$ vertices of $D_{t+3+\binom{\lceil \frac{n}{2} \rceil - 1}{2}\rho(n-2)}$ reduces the number of singletons in σ_c^* by at least $\rho(n-1) + \rho(n-2)$ because, if a vertex $v_\pi \in V_i^1$ or $v_\pi \in V_i^2$, for some $1 \leq i \leq n$, must be a singleton in any coalition partition of $\overline{\Lambda^{**}}_{K_n}$, its coalition partner must consist of vertices that dominate all the vertices v_{π_1} such that $i \in \text{fix}(\pi_1)$. That is, its coalition partner must either contain $(n-2)$ vertices of $V_{i'}^1$ that correspond to permutations with mutually disjoint fixes such that $i' \notin \text{fix}(\pi)$, $1 \leq i \neq i' \leq n$, or it must contain $\left(\binom{n}{2} - 1 \right)$ vertices of V_i^2 that correspond to all distinct 2-element fixes, except $\text{fix}(\pi)$, or it must contain all $\binom{n}{2}$ vertices of $\bigcup\limits_{i=1}^{n} V_i^{n-2}$ and the vertices of $\bigcup\limits_{k=3}^{n-3} V_i^k$, for all $i \in \text{fix}(\pi)$.

As $\rho(n-1) + \rho(n-2) \geq n$, for any $n \geq 6$, and altering these non-singletons of σ_c to obtain another coalition partition σ_c^* gives at most $n-2$ additional sets in σ_c^*, we get $|\sigma_c^*| \leq |\sigma_c|$, leading to a contradiction.

Therefore, based on our assumption, it can be deduced that the possibility of obtaining such a coalition partition σ_c^* of $\overline{\Lambda^{**}}_{K_n}$ is only when $D_1 \in \sigma_c$ or $D_2 \in \sigma_c$ are altered.

The partite sets $D_1 \in \sigma_c$ or $D_2 \in \sigma_c$, are independent sets, and D_1 dominates all the vertices in $V(\overline{\Lambda^{**}}_{K_n}) - \{v_{(1)(2)\cdots(\lfloor\frac{n}{2}\rfloor+1)(\lfloor\frac{n}{2}\rfloor+1\,\lfloor\frac{n}{2}\rfloor+2\cdots n-1\,n)}\}$, as any vertex in $\bigcup\limits_{i=1}^{n} V_i^k$, for any $1 \leq k \leq n-2$, is adjacent only to the vertices of $\bigcup\limits_{k=1}^{n-k} V_{i'}^k$, where $1 \leq i \neq i' \leq n$. Hence, as no vertex of $\bigcup\limits_{i=1}^{n} V_i^3$ is adjacent to any vertex of $\bigcup\limits_{i=1}^{n} V_i^{n-2}$, a coalition partner for any subset of vertices from $\bigcup\limits_{k=3}^{n-3} V_i^k$ cannot contain only the vertices of $\bigcup\limits_{k=1}^{2} V_i^k$. This is because such a coalition partner, which contains only the vertices of $\bigcup\limits_{k=1}^{2} V_i^k$ that dominate all the vertices of V_i^{n-2}, becomes a dominating set of $\overline{\Lambda^{**}}_{K_n}$.

Case 1. If D_1 is altered as $D_1^* = D_1 - \{v_\pi\}$, for some $v_\pi \in D_1$, in σ_c^* then D_1^* does not dominate v_π, $v_{(1)(2)\ldots(\lfloor\frac{n}{2}\rfloor+1)(\lfloor\frac{n}{2}\rfloor+1\,\lfloor\frac{n}{2}\rfloor+2\ldots n-1\,n)}$, and the vertices of $V(\overline{\Lambda^{**}}_{K_n}) - D_1$ to which only v_π is adjacent to, if such a set of vertices exists. Therefore, this implies that the coalition partner of D_1^* must dominate v_π, $v_{(1)(2)\ldots(\lfloor\frac{n}{2}\rfloor+1)(\lfloor\frac{n}{2}\rfloor+1\,\lfloor\frac{n}{2}\rfloor+2\ldots n-1\,n)}$, and the vertices $V(\overline{\Lambda^{**}}_{K_n}) - D_1$ to which only v_π is adjacent to, if exists.

Hence, the vertex v_π cannot be a singleton in σ_c^*, as it does not dominate the vertex $v_{(1)(2)\ldots(\lfloor\frac{n}{2}\rfloor+1)(\lfloor\frac{n}{2}\rfloor+1\,\lfloor\frac{n}{2}\rfloor+2\ldots n-1\,n)}$. Also, as any vertex in $\bigcup\limits_{i=1}^{n} \bigcup\limits_{k=1}^{\lceil\frac{n}{2}\rceil-1} V_i^k$ that dominates only $v_{(1)(2)\ldots(\lfloor\frac{n}{2}\rfloor+1)(\lfloor\frac{n}{2}\rfloor+1\,\lfloor\frac{n}{2}\rfloor+2\ldots n-1\,n)}$ is a singleton in σ_c, the removal of v_π reduces the number of singletons in σ_c^* by at least $\rho(n-1) > 1$, leading to a contradiction.

Case 2. If $D_1^* = D_1 \cup \{v_\pi\}$ in σ_c^*, where v_π can be any vertex that does not dominate the vertex $v_{(1)(2)\ldots(\lfloor\frac{n}{2}\rfloor+1)(\lfloor\frac{n}{2}\rfloor+1\,\lfloor\frac{n}{2}\rfloor+2\ldots n-1\,n)}$. However, such vertices are either singletons, or in $D_2 \in \sigma_c$, or in one of the other non-singletons of σ_c, which cannot be altered. Hence, $v_\pi \in D_2$, or it must be a singleton in σ_c, in this case. If $\{v_\pi\} \in \sigma_c$, $D_1^* = D_1 \cup \{v_\pi\}$ makes $|\sigma_c^*| \leq |\sigma_c| - 1$, as $v_\pi \in \bigcup\limits_{i=1}^{\lfloor\frac{n}{2}\rfloor+1} V_i^1 \cup V_i^2$. Therefore, v_π must be a vertex in $D_2 \in \sigma_c$. However, transferring any number of vertices from D_2 to D_1 to obtain σ_c^* does not make any difference in the order of σ_c^*. Hence, in this case, $|\sigma_c^*| \leq |\sigma_c|$; a contradiction.

Case 3. If D_2 is altered as $D_2^* = D_2 \cup \{v_\pi\}$ in σ_c^*, for any v_π, it implies that v_π can only be a singleton in σ_c, by *Case* 1 and *Case* 2. As this makes $|\sigma_c^*| \leq |\sigma_c| - 1$, D_2 cannot be altered as $D_2^* = D_2 \cup \{v_\pi\}$, for any vertex in $V(\overline{\Lambda^{**}}_{K_n}) - D_2$.

Case 4. If $D_2^* = D_2 - \{v_\pi\}$, for some vertex $v_\pi \in D_2$, in σ_c^*, then v_π must either be added to any other partite of σ_c, or it must be a singleton, to get σ_c^*. We know that $\mathrm{fix}(\pi) \subseteq \{1, 2, \ldots, \lfloor\frac{n}{2}\rfloor + 1\}$, for any $v_\pi \in D_2$. If $\mathrm{fix}(\pi) \subset \{1, 2, \ldots, \lfloor\frac{n}{2}\rfloor + 1\}$, then it cannot be a singleton, as cannot form a coalition with any of the other partite sets of σ_c. Hence, it can only be added to some other partite set of σ_c, to obtain σ_c^*, which does not increase the order of σ_c^*. If $\mathrm{fix}(\pi) = \{1, 2, \ldots, \lfloor\frac{n}{2}\rfloor + 1\}$, the vertex v_π cannot be added to D_1, upon removing it from D_2, as it makes D_1 a dominating set of $\overline{\Lambda^{**}}_{K_n}$.

This completes the proof as we eliminate the possibility of altering any non-singletons of σ_c to obtain a larger coalition partition of $\overline{\Lambda^{**}}_{K_n}$. $\square$

Theorem 2.2. *For $n \geq 3$,*

(i) $C(\overline{\Lambda^{*}}_{K_n}) = n! - \rho(n) - (n-1)! + 2$,

(ii) $C(\overline{\Lambda}_{K_n}) = C(\overline{\Lambda^{*}}_{K_n}) + \rho(n) = n! - (n-1)! + 2$.

Proof. As v_{π_0} is an isolated vertex of $\overline{\Lambda^{*}}_{K_n}$, any partite set in any coalition partition of $\overline{\Lambda^{*}}_{K_n}$ that does not contain the vertex v_{π_0} can form a coalition only with the partite set that contains v_{π_0}, say D'. Based on this, the following cases arise.

Case 1. If $D' = \{v_{\pi_0}\}$, it demands all the other partite sets of this coalition partition of $\overline{\Lambda^{*}}_{K_n}$ to be the dominating sets of $\overline{\Lambda^{**}}_{K_n}$. As it is known that $V(\overline{\Lambda^{**}}_{K_n})$ can be partitioned into at most $\delta(\overline{\Lambda^{**}}_{K_n}) + 1$ dominating sets (see [3]) and $\delta(\overline{\Lambda^{**}}_{K_n}) = 2\rho(n-1) + \rho(n-2)$, it is deduced that the order of the coalition partition of $\overline{\Lambda^{*}}_{K_n}$ in this case can at most be $2\rho(n-1) + \rho(n-2) + 2$.

Case 2. Let D' be a non-singleton such that $v_{\pi_0} \in D'$. As any other partite set in this the coalition partition of $\overline{\Lambda^{*}}_{K_n}$ can form coalition only with D', a coalition partition of $\overline{\Lambda^{**}}_{K_n}$ such that D' is a coalition partner of all its other partite sets must be obtained. Based on this, the following cases arise.

Subcase 2a. Let D' contain either contains $n-2$ vertices of $\bigcup\limits_{i=1}^{n} V_1$ or $\binom{n}{2} - 1$ vertices of $\bigcup\limits_{i=1}^{n} V_2$ that correspond to permutations with disjoint fixes, along with v_{π_0}. In this case, all the vertices of $V(\overline{\Lambda^{*}}_{K_n}) - D'$ that dominates the only vertex in $\bigcup\limits_{i=1}^{n} V_i^{n-2}$ which is not dominated by D' can be other partite sets in this coalition partition. As there can be at most $2\rho(n-1) + \rho(n-2) + 1$ such partite sets, the order of the coalition partition of $\overline{\Lambda^{*}}_{K_n}$ in this case can at most be $2\rho(n-1) + \rho(n-2) + 2$.

Subcase 2b. Let $D' = \bigcup\limits_{i=1}^{n} \bigcup\limits_{k=\lceil \frac{n}{2} \rceil}^{n} V_i^k - \{v_{(1)(2)\dots(\lfloor \frac{n}{2} \rfloor+1)(\lfloor \frac{n}{2} \rfloor+1\, \lfloor \frac{n}{2} \rfloor+2 \dots n-1\, n)}\}$. Therefore, all the vertices of $V(\overline{\Lambda^{*}}_{K_n}) - D'$ that dominate $v_{(1)(2)\dots(\lfloor \frac{n}{2} \rfloor+1)(\lfloor \frac{n}{2} \rfloor+1\, \lfloor \frac{n}{2} \rfloor+2 \dots n-1\, n)}$ must be other partite sets in this coalition partition. As there are $\sum\limits_{k=1}^{\lceil \frac{n}{2} \rceil -1} \binom{\lceil \frac{n}{2} \rceil -1}{2} \rho(n-k)$ such vertices, $C(\overline{\Lambda^{*}}_{K_n}) \geq 2 + \sum\limits_{k=1}^{\lceil \frac{n}{2} \rceil -1} \binom{\lceil \frac{n}{2} \rceil -1}{2} \rho(n-k)$.

Subcase 2c. Let $D' = V_i - \{v_\pi\}$, for some $v_\pi \in V_i^k$, $1 \leq i \leq n$ and $1 \leq k \leq n-2$. As each V_i is a minimal dominating set of $\overline{\Lambda^{*}}_{K_n}$, all the vertices that dominate v_π can be singletons in this coalition partition of $\overline{\Lambda^{*}}_{K_n}$, including v_π. Thus, as the value of k decreases, the number of singletons increase. When $v_\pi \in V_i^1$, all the vertices in $V(\overline{\Lambda^{*}}_{K_n}) - (V_i \cup \{v_\pi\})$ can be singletons in this coalition partition, and hence, $C(\overline{\Lambda^{*}}_{K_n}) \geq n! - \rho(n) - (n-1)! + 2$.

This exhausts the possibilities of obtaining a coalition partition of $\overline{\Lambda^{*}}_{K_n}$, and it can be seen that the existence a coalition partition of $\overline{\Lambda^{*}}_{K_n}$, as mentioned in *Case* 1, and

Subcase 2a, *Subcase* 2b and *Subcase* 2c, are guaranteed. On comparing the values of $C(\overline{\Lambda^{*}}_{K_n})$ obtained in different cases, we get $C(\overline{\Lambda^{*}}_{K_n}) = n! - \rho(n) - (n-1)! + 2$. $\square$

An *independent coalition partition* σ of a graph G is a partition of $V(G)$ into independent sets such that every partite set $D \in \sigma$ is either a universal vertex of G or not a dominating set of G such that for any $D \in \sigma$ there exists a $D' \in \sigma$, whose union $D \cup D'$ is an independent dominating set of G, and the *independent coalition number* $IC(G)$ of G is the maximum possible order of an independent coalition partition of G. Recall that an *independent dominating set* of G is a dominating set of G which is independent (ref. [1]).

Theorem 2.3. *The graph* $\overline{\Lambda^{**}}_{K_n}$ *admits an independent coalition partition if and only if* $n = 3$ *and* $IC(\overline{\Lambda^{**}}_{K_3}) = 3$.

Proof. As $\overline{\Lambda^{**}}_{K_3} \cong K_3$, it is immediate that $IC(\overline{\Lambda^{**}}_{K_3}) = 3$. For $n \geq 4$, any subset of $V(\overline{\Lambda^{**}}_{K_n})$ is independent if it either consist of vertices that correspond to permutations whose fixes are pair-wise non-disjoint. Hence, any subset of V_i, for each $1 \leq i \leq n$, is an independent set of $\overline{\Lambda^{**}}_{K_n}$ such that $\bigcap\limits_{v_\pi \in V_i} \text{fix}(\pi) \neq \emptyset$, and $\bigcup\limits_{i=1}^{n} \bigcup\limits_{k=\lfloor \frac{n}{2} \rfloor + 1}^{n-2} V_i^k \cup V_1^{\frac{n}{2}}$ is an independent set of $\overline{\Lambda^{**}}_{K_n}$ such that $\text{fix}(\pi) \cap \text{fix}(\pi') \neq \emptyset$, for any two distinct vertices $v_\pi, v_{\pi'}$ in it.

Note that each V_i, $1 \leq i \leq n$, is a minimal independent dominating set and maximum independent set of $\overline{\Lambda^{**}}_{K_n}$. Also, the sets $\bigcup\limits_{i=1}^{n} \bigcup\limits_{k=\lfloor \frac{n}{2} \rfloor + 1}^{n-2} V_i^k$ and $\bigcup\limits_{i=1}^{n} \bigcup\limits_{k=\lfloor \frac{n}{2} \rfloor + 1}^{n-2} V_i^k \cup V_1^{\frac{n}{2}}$, are minimal independent dominating sets and maximal independent sets of $\overline{\Lambda^{**}}_{K_n}$, when n is odd, and n is even, respectively.

Assume that there exists an independent coalition partition of $\overline{\Lambda^{**}}_{K_n}$, $n \geq 4$. Hence, there exists at least n partite sets in this coalition partition of $\overline{\Lambda^{**}}_{K_n}$, where each of its partite set contains the vertices of V_i^1, for each $1 \leq i \leq n$. Let $\sigma'_{ind} = \left\{ D'_j : 1 \leq j \leq n \right\}$ be such a coalition partition of $\overline{\Lambda^{**}}_{K_n}$, $n \geq 4$.

The coalition partner of any partite set of σ'_{ind} that contains the vertices of V_i^1, for some $1 \leq i \leq n$, in σ'_{ind} can only contain the vertices of $\bigcup\limits_{k=2}^{n-2} V_i^k$, for the corresponding i values, owing to the fact that their union must be an independent set of $\overline{\Lambda^{**}}_{K_n}$. Hence, each V_i can be partitioned into at most two partite sets in any independent coalition partition of $\overline{\Lambda^{**}}_{K_n}$, which henceforth makes their union an independent dominating set of $\overline{\Lambda^{**}}_{K_n}$.

Let $V_1 = D'_1 \cup D'_2$. Hence, the partite sets $D'_j \in \sigma'_{ind}$, $3 \leq j \leq r$, must consist of the vertices $V(\overline{\Lambda^{**}}_{K_n}) - V_1$. As the vertices of $V_2 - V_1$ do not dominate the vertices of $V_1 \cap V_2$, the set $V_2 - V_1$ is an independent set, but not a dominating set of $\overline{\Lambda^{**}}_{K_n}$. Thus, let $V_2 - V_1 = D'_3$. Therefore, $D'_3 \in \sigma'_{ind}$ becomes an independent dominating set of $\overline{\Lambda^{**}}_{K_n}$ when it forms coalition with a partite set of σ'_{ind} consisting the vertices of $V_1 \cap V_2$. As $V_1 = D'_1 \cup D'_2$, make $D'_1 = V_1 \cap V_2$ and $D'_2 = V_1 - V_2$. Therefore,

D'_1, D'_2, D'_3 are independent sets of $\overline{\Lambda^{**}}_{K_n}$ that form coalition with one among them, to be an independent dominating set of $\overline{\Lambda^{**}}_{K_n}$.

As $n \geq 4$, $V_3 \neq \emptyset$, and hence, let $D'_4 = V_3 - (V_1 \cap V_2)$. This set becomes an independent dominating set of $\overline{\Lambda^{**}}_{K_n}$ when it forms a coalition with a partite set of σ'_{ind} consisting only the other vertices of V_3. That is, the coalition partner of D'_4 must contain the vertices of $V_1 \cap V_2 \cap V_3$, $(V_1 - V_2) \cap V_3$, and $(V_2 - V_1) \cap V_3$. A partite set of this nature exists only when D'_1, D'_2 and D'_3 are altered in some manner. However, altering the partite sets D'_1, D'_2, D'_3 makes them either non-independent sets of $\overline{\Lambda^{**}}_{K_n}$ or independent but non-dominating sets of $\overline{\Lambda^{**}}_{K_n}$ that cannot form a coalition in σ'_{ind}; a contradiction. Hence this proves the result. ☐

Proposition 2.1. *The graphs* $\overline{\Lambda^{*}}_{K_n}$ *and* $\overline{\Lambda}_{K_n}$ *admit an independent coalition partition if and only if* $n \leq 3$.

Proof. As $\overline{\Lambda^{*}}_{K_n} \cong \overline{\Lambda^{**}}_{K_n} \cup v_{\pi_0}$, it follows from Theorem 2.3 that $\overline{\Lambda^{*}}_{K_n}$ admits an independent coalition partition if and only if $n = 2, 3$. Also, as $\overline{\Lambda}_{K_n} \cong \overline{\Lambda^{*}}_{K_n} + K_{\rho(n)}$, $\overline{\Lambda}_{K_n}$ admits an independent coalition partition if and only if $\overline{\Lambda^{*}}_{K_n}$ does.

When $n = 1, 2$, $\overline{\Lambda}_{K_n} \cong K_n$ and $\overline{\Lambda^{*}}_{K_2} \cong K_1$. Hence, $IC(\overline{\Lambda^{*}}_{K_n})$ and $IC(\overline{\Lambda}_{K_n})$ is straightforward, in these cases. As $\overline{\Lambda^{*}}_{K_3} \cong K_3 \cup K_1$, a partition in which all the vertices of $\overline{\Lambda^{*}}_{K_3}$ and $\overline{\Lambda^{**}}_{K_3}$ are singletons form their independent coalition partitions, as any vertex that corresponds to a non-identity permutation with non-empty fix, forms a coalition with $\{v_{\pi_0}\}$, and hence, completing the proof. ☐

Corollary 2.1. *For* $n \leq 3$, $IC(\overline{\Lambda}_{K_n}) = n!$ *and* $IC(\overline{\Lambda^{*}}_{K_n}) = \begin{cases} n, & \text{if } n = 1, 2, \\ 4, & \text{if } n = 3. \end{cases}$

A partition σ of $V(G)$ is said to be a *paired coalition partition* of a graph G if every partite set of σ is not a paired dominating set of G and it forms a coalition with another partite set of σ, which is not a paired dominating set of G, to obtain a paired dominating set of G. The maximum order of a paired coalition partition of G is the *paired coalition number* $PC(G)$ of G. Recall that a *paired dominating set* of G is a dominating set of G that contains a perfect matching, not necessarily induced (see [22]).

Theorem 2.4. *For* $n \geq 4$,

$$PC(\overline{\Lambda^{**}}_{K_n}) = \begin{cases} n! - \rho(n) - n + 1, & \text{when } n \text{ is even}, \\ n! - \rho(n) - n, & \text{when } n \text{ is odd}. \end{cases}$$

Proof. For $\overline{\Lambda^{**}}_{K_n}$, $n \geq 4$, we investigate the paired coalition partitions in two cases, based on n being odd or even.

Case 1. When n is even, a minimum dominating set D of $\overline{\Lambda^{**}}_{K_n}$ that consists of $n-1$ vertices of $\bigcup\limits_{i=1}^{n} V_i^1$, one from each $1 \leq i \leq n$, is not a paired dominating set of the graph $\overline{\Lambda^{**}}_{K_n}$. In view of this, a partition $\sigma_p = \{D_1, D_2, \ldots, D_{n!-\rho(n)-n+1}\}$, of $V(\overline{\Lambda^{**}}_{K_n})$

such that D_1 is a minimum dominating set of $\overline{\Lambda^{**}}_{K_n}$ containing one vertex from each V_i^1, $1 \leq i \leq n-1$, and D_j, $2 \leq j \leq n! - \rho(n) - n + 1$, are singletons consisting of one vertex in $V(\overline{\Lambda^{**}}_{K_n}) - D_1$ becomes a paired coalition partition of $\overline{\Lambda^{**}}_{K_n}$.

If $PC(\overline{\Lambda^{**}}_{K_n}) > n! - \rho(n) - n + 1$, there should exist at least one more partite set in such a paired coalition partition of $\overline{\Lambda^{**}}_{K_n}$, which happens only when the non-singleton $D_1 \in \sigma_p$ is partitioned further. However, this reduces the number of singletons in this paired coalition partition of $\overline{\Lambda^{**}}_{K_n}$ by at least $\rho(n-1) \geq n$. Hence, the partition σ_p obtained above is a maximum paired coalition partition of $\overline{\Lambda^{**}}_{K_n}$ of order $n! - \rho(n) - n + 1$, when n is even.

Case 2. When n is odd, using the same strategy as in *Case* 1, make a non-singleton partite set of a paired coalition partition of $\overline{\Lambda^{**}}_{K_n}$ as a dominating set with n vertices; one from each V_i^1, $1 \leq i \leq n$, and all other vertices of $\overline{\Lambda^{**}}_{K_n}$ as singletons. This becomes a maximum paired coalition partition of $\overline{\Lambda^{**}}_{K_n}$, yielding, $PC(\overline{\Lambda^{**}}_{K_n}) = n! - \rho(n) - n$, when n is odd. ☐

Note that as the notion of paired domination is defined for graphs without isolated vertices, we do not investigate them for $\overline{\Lambda^{*}}_{K_n}$, and as $\overline{\Lambda}_{K_n}$ have universal vertices, the following result is immediate.

Corollary 2.2. *For any $n \geq 3$, $PC(\overline{\Lambda}_{K_n}) = n!$.*

A subset $D \subseteq V(G)$ is a *strong dominating set* of a graph G, if there exists a vertex $u \in D$, such that $uv \in E(G)$ and $d(u) \geq d(v)$, for all $v \in V(G) - D$, where $d(w)$, for any vertex $w \in V(G)$ denotes the degree of w in G. Partitioning the vertex set of a graph G such that every partite set is not a strong dominating set of G; but forms a coalition with another partite set in the partition, such that their union is a strong dominating set of G is called a *strong coalition partition* of G and the maximum order of such a partition of $V(G)$ is the *strong coalition number* of G, denoted by $SC(G)$ (ref. [11]).

Theorem 2.5. *For $n \geq 3$,*

$$SC(\overline{\Lambda^{**}}_{K_n}) = \begin{cases} 3, & \text{if } n = 3,\\ 8, & \text{if } n = 4,\\ 55, & \text{if } n = 5,\\ 3 + \left(\binom{n}{2} - \binom{\lceil \frac{n}{2} \rceil - 1}{2}\right)(3-n) - \lfloor \frac{n}{2} \rfloor & \\ + \sum\limits_{k=1}^{2} \binom{n}{k}\rho(n-k) + \sum\limits_{k=3}^{\lceil \frac{n}{2} \rceil - 1} \binom{\lceil \frac{n}{2} \rceil - 1}{k}\rho(n-k), & \text{if } n \geq 6. \end{cases}$$

Proof. For $n = 3$, the result is straightforward. In any strong dominating set of $\overline{\Lambda^{**}}_{K_n}$; $n \geq 4$, there must exist at least two vertices of $\bigcup\limits_{i=1}^{n} V_i^1$ that correspond to permutations with distinct fixes, as they have the highest degree in $\overline{\Lambda^{**}}_{K_n}$. Based on the values of n, the following cases are considered to prove the result.

Case 1. When $n \leq 5$. For $n = 4$, no maximum coalition partition of $\overline{\Lambda^{**}}_{K_4}$ is its strong coalition partition, as the vertices of $V_i^2 - V_{i'}$, for some $1 \leq i \neq i' \leq 4$, can neither be a singleton nor stand alone as a partite set in any maximum coalition partition of $\overline{\Lambda^{**}}_{K_n}$. Thus, $SC(\overline{\Lambda^{**}}_{K_4}) \leq 8$. It is a matter of immediate verification that the partition $\sigma_s = \{D'_1, D'_2, \ldots, D'_8\}$, where $D'_1 = \{v_{(1)(234)}, v_{(2)(134)}\}, D'_2 = \{v_{(1)(243)}, v_{(2)(143)}\}, D'_3 = \{v_{(3)(124)}\}, D'_4 = \{v_{(3)(142)}\}, D'_5 = \{v_{(4)(123)}\}, D'_6 = \{v_{(4)(132)}\}$, $D'_7 = \{v_{(3)(4)(12)}\}$, and $D'_8 = \bigcup\limits_{i=1}^{2} V_i^2$ is its strong coalition partition. Hence, $SC(\overline{\Lambda^{**}}_{K_4})$ $= 8$. When $n = 5$, it is evident that the maximum coalition partition of $\overline{\Lambda^{**}}_{K_5}$ given in Theorem 2.1 is also its maximum strong coalition partition.

Case 2. To prove the result for $n \geq 6$, consider the maximum coalition partition σ_c of $\overline{\Lambda^{**}}_{K_n}$ constructed in Theorem 2.1. Any singleton in σ_c whose coalition partner is one of the non-singleton partite sets of σ_c containing $n - 2$ vertices of $\bigcup\limits_{i=1}^{n} V_i^1$ forms coalition and becomes a strong dominating set of $\overline{\Lambda^{**}}_{K_n}$. Whereas, the partite sets of σ_c whose coalition partner is D_1 does not strongly dominate the vertices of $\bigcup\limits_{i=1}^{n} V_i^1$.

Therefore, for all such partite sets to form coalition with D_1 and become strong dominating sets of $\overline{\Lambda^{**}}_{K_n}$, D_1 must have at least two vertices of $\bigcup\limits_{i=1}^{n} V_i^1$, corresponding to permutations with distinct fixes. However, there exists at least $\rho(\lfloor \frac{n}{2} \rfloor + 1)$ singletons consisting of the vertices in $\bigcup\limits_{i=1}^{n} V_i^{\lceil \frac{n}{2} \rceil - 1} - \bigcup\limits_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1} V_i^{\lceil \frac{n}{2} \rceil - 1}$ whose coalition partner is D_1 in σ_c, that cannot strongly dominate any vertex from $\bigcup\limits_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1} \bigcup\limits_{k=1}^{\lceil \frac{n}{2} \rceil - 2} V_i^k$. Hence, for $D_1 \cup \{v_\pi\}$, for any $v_\pi \in \bigcup\limits_{i=1}^{n} V_i^{\lceil \frac{n}{2} \rceil - 1} - \bigcup\limits_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1} V_i^{\lceil \frac{n}{2} \rceil - 1}$, to be a strong dominating set of $\overline{\Lambda^{**}}_{K_n}$, one vertex from V_i^1, for each $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$, must be added to D_1.

This addition is possible as there exists $\rho(n-1) - \binom{n-1}{2} \geq 1$ singletons in σ_c, that has a vertex of V_i^1, for any $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$. Upon altering D_1 as D'_1 and removing $\lceil \frac{n}{2} \rceil - 1$ singletons of σ_c given in Theorem 2.1, the strong coalition partition σ_s of $\overline{\Lambda^{**}}_{K_n}$ is obtained. As $|\sigma_s| = C(\overline{\Lambda^{**}}_{K_n}) - \lceil \frac{n}{2} \rceil + 1$, $SC(\overline{\Lambda^{**}}_{K_n}) \geq C(\overline{\Lambda^{**}}_{K_n}) - \lceil \frac{n}{2} \rceil + 1$.

Owing to the definitions of coalition and strong coalition partitions of graphs, any strong coalition partition of $\overline{\Lambda^{**}}_{K_n}$, $n \geq 6$, of higher order can only be obtained by altering σ_s.

If a strong coalition partition of $\overline{\Lambda^{**}}_{K_n}$, $n \geq 6$ is obtained by altering σ_s such that the structure of the coalition partition σ_c is changed, the existence of such a strong coalition partition of $\overline{\Lambda^{**}}_{K_n}$, $n \geq 6$, of higher order contradicts Theorem 2.1. Hence, if σ_s has to be altered, only the newly added vertices of $D'_1 \in \sigma_s$ can be altered. However, if any of the vertices of D'_1 from V_i^1 is removed from it, the number of singletons in σ_s decreases at least by $\rho(\lfloor \frac{n}{2} \rfloor + 1)$. As $\rho(\lfloor \frac{n}{2} \rfloor + 1) \geq \lceil \frac{n}{2} \rceil - 1$, for all $n \geq 6$, D'_1 cannot be altered. This completes the proof. $\square$

Based on Theorem 2.1 and Theorem 2.5, the relation between the coalition number and strong coalition number of $\overline{\Lambda^{**}}_{K_n}$ is given in the following result.

Corollary 2.3. *For* $n \geq 6$, $SC(\overline{\Lambda^{**}}_{K_n}) = C(\overline{\Lambda^{**}}_{K_n}) - \lceil \frac{n}{2} \rceil + 1$.

Theorem 2.6. *For* $n \in \mathbb{N}$, $SC(\overline{\Lambda^{*}}_{K_n}) = \begin{cases} 3, & \textit{if } n = 3, \\ 2 + \sum\limits_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor}{k} \rho(n-k), & \textit{if } n \geq 4. \end{cases}$

Proof. As v_{π_0} is isolated in $\overline{\Lambda^{*}}_{K_n}$, the coalition partner of any partite set in any strong coalition partition of $\overline{\Lambda^{*}}_{K_n}$ is the partite set in the partition that contains v_{π_0}. As in Theorem 2.2, v_{π_0} can either be a singleton or a non-singleton in a strong coalition partition of $\overline{\Lambda^{*}}_{K_n}$.

Case 1. If v_{π_0} is a singleton in a strong coalition partition of $\overline{\Lambda^{*}}_{K_n}$, then all the remaining partite sets of this partition must be strong dominating sets of $\overline{\Lambda^{**}}_{K_n}$. Therefore, as mentioned in *Case* 1 of Theorem 2.2, at most $2\rho(n-1) + \rho(n-2) + 2$ partite sets can be there in this strong coalition partition of $\overline{\Lambda^{*}}_{K_n}$.

Case 2. Let D' be a non-singleton partite set of a strong coalition partition of $\overline{\Lambda^{*}}_{K_n}$, which contains v_{π_0}. Then, the following possibilities of obtaining a strong coalition partition of $\overline{\Lambda^{*}}_{K_n}$ arise.

Subcase 2a. If $D' = V_j - \{v_\pi\}$, where $v_\pi \in V_j^k$, for some $1 \leq j \leq n$ and $1 \leq k \leq n-2$, every other partite set in this strong coalition partition of $\overline{\Lambda^{*}}_{K_n}$ must contain vertex that strongly dominates v_π. Hence, every partite in this strong coalition partition of $\overline{\Lambda^{*}}_{K_n}$ must contain a vertex from $\bigcup\limits_{i=1}^{n} \bigcup\limits_{k'=1}^{k} V_i^{k'} - V_j$, and the order of such a partition of $V(\overline{\Lambda^{*}}_{K_n})$ is at most $2 + \sum\limits_{k^*=1}^{k} \binom{n-k}{k^*} \rho(n-k^*)$, where $k = |\operatorname{fix}(\pi)|$.

Subcase 2b. Let D' contain either $n-2$ vertices of $\bigcup\limits_{i=1}^{n} V_1$ or $\binom{n}{2} - 1$ vertices of $\bigcup\limits_{i=1}^{n} V_2$ that correspond to permutations with disjoint fixes, along with v_{π_0}. In this case, all the other partite sets in this coalition partition of $\overline{\Lambda^{*}}_{K_n}$ must contain the vertices of $V(\overline{\Lambda^{*}}_{K_n}) - D'$ that strongly dominates the only vertex in $\bigcup\limits_{i=1}^{n} V_i^{n-2}$ which is not dominated by D'. As there are $2\rho(n-1) + \rho(n-2) + 1$ such vertices, the order of this strong coalition partition of $\overline{\Lambda^{*}}_{K_n}$ can be at most $2\rho(n-1) + \rho(n-2) + 2$.

Subcase 2c. When $D' = \bigcup\limits_{i=1}^{n} \bigcup\limits_{k=\lceil \frac{n}{2} \rceil}^{n} V_i^k$, it is a dominating set of $\overline{\Lambda^{*}}_{K_n}$; whereas, not its strong dominating set. Therefore, this becomes a strong dominating set of $\overline{\Lambda^{*}}_{K_n}$, on coalition with the vertices that strongly dominate the vertices of $\bigcup\limits_{i=1}^{n} \bigcup\limits_{k=1}^{\lceil \frac{n}{2} \rceil - 1} V_i^k$. This implies that every other partite set in this coalition partition must contain at least two vertices of V_i^1 with disjoint fixes. Hence, at most $1 + \lfloor \frac{n\rho(n-1)}{2} \rfloor$ partite sets can be there in this strong coalition partition of $\overline{\Lambda^{*}}_{K_n}$.

In this case, let D' be modified, as follows. Along with $\bigcup\limits_{i=1}^{n} \bigcup\limits_{k=\lceil \frac{n}{2} \rceil}^{n} V_i^k$, let D' contain $\lceil \frac{n}{2} \rceil - 1$ vertices, say $v_{\pi_1}, v_{\pi_2}, \ldots, v_{\pi_{\lceil \frac{n}{2} \rceil - 1}}$, of $\bigcup\limits_{i=1}^{n} V_i^1$ that correspond to permutations with mutually disjoint fixes. Therefore, all the vertices of $\bigcup\limits_{i=1}^{n} \bigcup\limits_{k=1}^{\lceil \frac{n}{2} \rceil - 1} V_i^k$, except the ones in $\bigcup\limits_{i=1}^{n} V_i^{\lceil \frac{n}{2} \rceil - 1}$ that correspond to permutations whose fix is $\bigcup\limits_{j=1}^{\lceil \frac{n}{2} \rceil - 1} \mathrm{fix}(\pi_j)$ are strongly dominated by D'. Hence, all the other partite sets this coalition partition of $\overline{\Lambda^*}_{K_n}$ must contain at least one vertex that strongly dominates the vertices in $\bigcup\limits_{i=1}^{n} V_i^{\lceil \frac{n}{2} \rceil - 1}$ that correspond to permutations whose fix is $\bigcup\limits_{j=1}^{\lceil \frac{n}{2} \rceil - 1} \mathrm{fix}(\pi_j)$. Hence, the order of this strong coalition partition of $\overline{\Lambda^*}_{K_n}$ is at most $2 + \sum\limits_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor}{k} \rho(n-k)$.

As this exhausts all the possibilities to obtain a maximal strong coalition partition of $\overline{\Lambda^*}_{K_n}$, it is deduced that $C(\overline{\Lambda^*}_{K_n}) \leq 2 + \sum\limits_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor}{k} \rho(n-k)$.

Consider a partition $\sigma''_{st} = \{D_j : 1 \leq j \leq x'_{st}\}$, where $x'_{st} = 2 + \sum\limits_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor}{k} \rho(n-k)$, such that $D_1 = V_1 - \{v_{(1)(2)\ldots(\lfloor \frac{n}{2} \rfloor)(\lfloor \frac{n}{2} \rfloor + 1\, \lfloor \frac{n}{2} \rfloor + 2\, \ldots\, n-1\, n)}\}$, and D_j; $2 \leq j \leq x'_{st} - 1$, are singletons containing the vertices of $\bigcup\limits_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n} \bigcup\limits_{k=1}^{\lfloor \frac{n}{2} \rfloor} V_i^k - \bigcup\limits_{i=1}^{\lfloor \frac{n}{2} \rfloor} V_i$, and $D_{x'_{st}} = V(\overline{\Lambda^*}_{K_n}) - \bigcup\limits_{j=1}^{x'_{st}-1} D_j$. As it can be verified that σ''_{st} is a strong coalition partition of $\overline{\Lambda}_{K_n}$ as mentioned in *Subcase* 2a, having order $2 + \sum\limits_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor}{k} \rho(n-k)$, the proof is done. □

In view of the fact that $\overline{\Lambda}_{K_n}$ has $\rho(n)$ universal vertices and no vertex partition of the non-universal vertices of $\overline{\Lambda}_{K_n}$ can strongly dominate the universal vertices of $\overline{\Lambda}_{K_n}$, the following result is obtained.

Proposition 2.2. *For $n \in \mathbb{N}$, $\overline{\Lambda}_{K_n}$ admits a strong coalition partition if and only if $n \leq 2$ and $SC(\overline{\Lambda}_{K_n}) = n$, in this case.*

An illustration of the maximum coalition partitions of the graphs $\overline{\Lambda^{**}}_{K_n}$, $\overline{\Lambda^*}_{K_n}$ and $\overline{\Lambda}_{K_n}$ given in the above obtained results, is given in Table 1, for the value when $n = 4$.

Graph	Invariant	Maximum Partition
$\overline{\Lambda^{**}}_{K_4}$	$C(G) = 11$	$\Big\{\{v_{(1)(2\,3\,4)}, v_{(1)(2)}, v_{(1)(3)}, v_{(1)(4)}\}, \{v_{(1)(2\,4\,3)}\}, \{v_{(2)(1\,4\,3)}\}, \{v_{(2)(1\,3\,4)}\}, \{v_{(2)(3)}\}, \{v_{(2)(4)}\}, \{v_{(3)(1\,4\,2)}\}, \{v_{(3)(1\,2\,4)}\}, \{v_{(3)(4)}\}, \{v_{(4)(1\,3\,2)}\}, \{v_{(4)(1\,2\,3)}\}\Big\}$
	$PC(G) = 12$	$\Big\{\{v_{(1)(2\,3\,4)}, v_{(2)(1\,4\,3)}, v_{(3)(1\,4\,2)}\}, \{v_{(1)(2\,4\,3)}\}, \{v_{(1)(2)}\}, \{v_{(1)(3)}\}, \{v_{(1)(4)}\}, \{v_{(2)(3)}\}, \{v_{(2)(4)}\}, \{v_{(2)(1\,3\,4)}\}, \{v_{(3)(1\,2\,4)}\}, \{v_{(3)(4)}\}, \{v_{(4)(1\,3\,2)}\}, \{v_{(4)(1\,2\,3)}\}\Big\}$
	$SC(G) = 7$	$\Big\{\{v_{(1)(2\,3\,4)}, v_{(2)(1\,4\,3)}\}, \{v_{(1)(2\,4\,3)}, v_{(2)(1\,3\,4)}\}, \{v_{(3)(1\,4\,2)}\}, \{v_{(3)(1\,2\,4)}\}, \{v_{(4)(1\,3\,2)}\}, \{v_{(4)(1\,2\,3)}\}, \{v_{(1)(2)}, v_{(1)(3)}, v_{(1)(4)}, v_{(2)(3)}, v_{(2)(4)}, v_{(3)(4)}\}\Big\}$
$\overline{\Lambda^{*}}_{K_4}$	$C(G) = 11$	$\sigma = \Big\{\{v_{\pi_0}, v_{(1)(2\,3\,4)}, v_{(1)(2)}, v_{(1)(3)}, v_{(1)(4)}\}, \{v_{(1)(2\,4\,3)}\}, \{v_{(2)(1\,4\,3)}\}, \{v_{(2)(1\,3\,4)}\}, \{v_{(3)(1\,4\,2)}\}, \{v_{(3)(1\,2\,4)}\}, \{v_{(4)(1\,2\,3)}\}, \{v_{(4)(1\,3\,2)}\}, \{v_{(2)(3)}\}, \{v_{(2)(4)}, \{v_{(3)(4)}\}\Big\}$
	$PC(G)$	Does not admit
	$SC(G) = 7$	$\Big\{\{v_{\pi_0}, v_{(1)(2\,3\,4)}, v_{(1)(2\,4\,3)}, v_{(1)(3)}, v_{(1)(4)}\}\} \{v_{(3)(1\,4\,2)}\}, \{v_{(3)(1\,2\,4)}\}, \{v_{(4)(1\,2\,3)}\}, \{v_{(4)(1\,3\,2)}\}, \{v_{(3)(4)}\}, \{v_{(1)(2)}, v_{(2)(1\,4\,3)}, v_{(2)(1\,3\,4)}, \{v_{(2)(3)}\}, \{v_{(2)(4)}\}\Big\}$
$\overline{\Lambda}_{K_4}$	$C(G) = 20$	$\sigma \cup \Big\{\{v_{(1\,2\,3\,4)}\}, \{v_{(1\,2\,4\,3)}\}, \{v_{(1\,3\,2\,4)}\}, \{v_{(1\,3\,4\,2)}\}, \{v_{(1\,4\,3\,2)}\}, \{v_{(1\,4\,2\,3)}\}, \{v_{(1\,2)(3\,4)}\}, \{v_{(1\,3)(2\,4)}\}, \{v_{(1\,4)(2\,3)}\}\Big\}$
	$PC(G) = 24$	A partition in which all 24 vertices are of the graph are singletons
	$SC(G)$	Does not admit

Table 1: Illustration of different coalition partitions of $\overline{\Lambda^{**}}_{K_4}$, $\overline{\Lambda^{*}}_{K_4}$ and $\overline{\Lambda}_{K_4}$.

3. CONCLUSION

In this article, we have investigated coalition partitions of the n-inordinate invariant non-intersection graphs, and their subgraphs $\overline{\Lambda^{*}}_{K_n}$ and $\overline{\Lambda^{**}}_{K_n}$. As this study on the coalitions in an algebraic graph is novel in the area of algebraic graph theory, it opens another aspect of structural analysis to all the other algebraic graphs existing in the literature. Also, as these classes of graphs give different examples for the existence of graphs with different coalition properties, it leads to different prospects of comparative investigation between the different types of coalition partitions with respect to the structure of graphs, opening a fertile area for further research.

REFERENCES

[1] S. Alikhani, D. Bakhshesh, H. Golmohammadi and S. Klavzar, *On independent coalition in graphs and independent coalition graphs*, Discuss. Math. Graph Theory **45** (2025), 533–544. `https://doi.org/10.7151/dmgt.2543`

[2] T. W. Haynes, J. T. Hedetniemi, S. T. Hedetniemi, A. A. McRae and R. Mohan, *Introduction to coalitions in graphs*, AKCE Int. J. Graphs Combin. **17** (2020), 653–659. `https://doi.org/10.1080/09728600.2020.1832874`

[3] T. W. Haynes, S. T. Hedetniemi and M. A. Henning, *Topics in Domination in Graphs*, Vol. 64, Springer, 2020.

[4] I. N. Herstein, *Topics in Algebra*, John Wiley & Sons, New Jersey, 2006.

[5] C. L. Liu, *Introduction to Combinatorial Mathematics*, McGraw-Hill Inc., New York, 1968.

[6] S. Madhumitha and S. Naduvath, *Graphs defined on rings: A review*, Math. **11** (2023). `https://doi.org/10.3390/math11173643`

[7] S. Madhumitha and S. Naduvath, *Coloring of n-inordinate invariant intersection graphs*, J. Combin. Math. Combin. Comput. **123** (2024), 369–381. `https://doi.org/10.61091/jcmcc123-26`

[8] S. Madhumitha and S. Naduvath, *Graphs on groups in terms of the order of elements: A review*, Discrete Math. Algorithms Appl. **16** (2024), Article ID 2330003. `https://doi.org/10.1142/S1793830923300035`

[9] S. Madhumitha and S. Naduvath, *Centrality, entropy, and sensitivity analysis of n-inordinate invariant intersection graphs*, Discrete Math. Algorithms Appl. (2025). `https://doi.org/10.1142/S1793830925500831`

[10] S. Madhumitha and S. Naduvath, *Chromatic transversal domination in n-inordinate invariant intersection graphs*, Math. (Cluj) (2025) (accepted).

[11] S. Madhumitha and S. Naduvath, *Coalition partitions of n-inordinate invariant intersection graphs*, Discrete Math. Algorithms Appl. (2025). `https://doi.org/10.1142/S1793830925500739`

[12] S. Madhumitha and S. Naduvath, *Domination-related colorings of n-inordinate invariant intersection graphs*, Discrete Math. Algorithms Appl. (2025). `https://doi.org/10.1142/S1793830925500983`

[13] S. Madhumitha and S. Naduvath, *Dominator colorings of n-inordinate invariant intersection graphs*, Utilitas Math. **122** (2025), 81–92. `https://doi.org/10.61091/um122-06`

[14] S. Madhumitha and S. Naduvath, *Geodetic sets in n-inordinate invariant intersection graphs*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **74** (2025), 723–732. `https://doi.org/10.31801/cfsuasmas.1620815`

[15] S. Madhumitha and S. Naduvath, *Invariant intersection graph of a graph*, J. Discrete Math. Sci. Cryptogr. **28** (2025), 2323–2335. `https://doi.org/10.47974/JDMSC-2235`

[16] S. Madhumitha and S. Naduvath, *Resolvability in n-inordinate invariant intersection graphs*, Palestine J. Math. (2025) (accepted).

[17] S. Madhumitha and S. Naduvath, *Color connections in n-inordinate invariant intersection graphs*, in: *Trends in Mathematics: Algebra, Analysis, and Associated Topics*, Springer, 2026, 151–163. `https://doi.org/10.1007/978-3-032-04418-1_13`

[18] S. Madhumitha and S. Naduvath, *A study on invariant intersection graph of a graph*, Asian-European J. Math. (2026). `https://doi.org/10.1142/S1793557126500129`

[19] S. Madhumitha and S. Naduvath, *Certain variants of domatic partitions of n-inordinate invariant intersection graphs*, Asian-European J. Math. **19** (2026), Article ID 2540016. `https://doi.org/10.1142/S1793557125400169`

[20] S. Madhumitha and S. Naduvath, *Domination and colorings in n-inordinate invariant intersection graphs*, Proceedings of IC-CAMSTIA-2025, Springer, 2025.

[21] S. Madhumitha, S. Naduvath and V. S. Prebhath, *n-inordinate invariant intersection graphs and their derived graphs*, J. Appl. Math. Comput. (submitted).

[22] M. R. Samadzadeh, D. A. Mojdeh and R. Nadimi, *Paired coalition in graphs*, AKCE Int. J. Graphs Combin. (2024), 1–12. `https://doi.org/10.1080/09728600.2024.2389070`

[23] D. B. West, *Introduction to Graph Theory*, Prentice Hall of India, New Delhi, 2001.

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