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LIE POINT SYMMETRIES, HAMILTONIAN EQUATIONS AND CONSERVATION LAWS OF THE GEODESICS ON A SCHWARZSCHILD BLACK HOLE

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ABSTRACT. Classification of Schwarzschild geodesics via group analysis of Lie point symmetries is considered. Lie's symmetry method of differential equations (DEs) is applied to the system of Schwarzschild geodesics to classify geodesic curves. In this method Lie algebra of symmetries will be studied and some useful results in physics such as Hamiltonian equations and conservation laws are obtained.

1. MATHEMATICAL FORMULATION

That the Schwarzschild geometry is relevant to gravitational collapse follows from the *Birkhoff's (1923) theorem: let the geometry of a given region of space time* (t, r, θ, φ) *be spherically symmetric, and* $\sqrt{g} = e^{3\alpha}$ *be a solution to Einstein field equation in vacuum. Then the geometry is necessarily a piece of Schwarzschild geometry* [12]. The external of any spherically neutral, spherical star satisfies the conditions of Birkhoff's theorem, whether the star is static, vibrating, or collapsing, therefore the external field must be a piece of the Schwarzschild geometry.

Birkhoff's theorem is easily understood on physical grounds. Consider an equlibrium configuration that is unstable against gravitational collapse and that, like all equlibrium configurations, has the Schwarzschild geometry as its external gravitational field. Perturb this equilibrium configuration is spherically symmetric way, so that it begins to collapse radially. The perturbation and subsequent collapse cannot affect the external gravitational field so lang as exact spherical symmetry is maintained. Just as Maxwell's laws prohibit monopole electromagnetic waves, so Einsteins's laws

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prohibit monopole gravitational waves. There is no possible way for any gravitational influence of the radial collapse to propagate outward.

Not only the Birkhoff's theorem easy to understand, but it is also fairly easy to prove. Consider a spherical region of spacetime. Spherical symmetry alone is sufficient to guarantee that

(i) the set of all equivalent points s under rotational Lie group SO(3) is a two-dimensional submanifold of a four-dimensional manifold endowed by spacetime coordinate chart (except for center pointsa, where s is zero-dimensional);
(ii) the metric on s is that of standard 2-sphere. Then on s one will have $ds^2 = R^2(s)d\Omega^2$, where $d\Omega^2$ is the standard metric of a unit sphere, and $2\pi R$ is the circumference of s;
(iii) there is a spherically symmetric 4-velocity field $\mathbf{u}$, defined so that if $\mathcal{C}(\tau)$ is one trajectory of $\mathbf{u}$ with $\mathbf{u}=d/d\tau$, then each rotated curve $\overline{\mathcal{C}}$ under SO(3) must also be a trajectory of $\mathbf{u}$, and thus to guarantee that one can introduce Schwarzschild coordinates

$$ds^2 = e^{2\eta}dt^2 - e^{2\Lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1.1}$$

where $\eta = \eta(t,r)$ and $\Lambda = \Lambda(t,r)$.

Impose Einstein's vacuum field equation on the metric (1.1), using the orthonormal components of the Einstein tensor we have

$$G_{\hat{t}\hat{t}} = r^{-2}(1 - e^{-2\Lambda}) + 2r^{-1}\Lambda_r e^{-2\Lambda} = 0, \tag{1.2}$$

$$G_{\hat{t}\hat{r}} = G_{\hat{r}\hat{t}} = 2r^{-1}\Lambda_t e^{-(\Lambda+\eta)} = 0, \tag{1.3}$$

$$G_{\hat{r}\hat{r}} = 2r^{-1}\eta_r e^{-2\Lambda} + r^{-2}(e^{-2\Lambda - 1}) = 0, \tag{1.4}$$

$$\begin{aligned} G_{\hat{\theta}\hat{\theta}} = G_{\hat{\varphi}\hat{\varphi}} = & (\eta_{rr} + \eta_r^2 - \eta_r\Lambda_r + r^{-1}\eta_r - r^{-1}\Lambda_r)e^{-2\Lambda} \\ & - (\Lambda_{tt} + \Lambda_t^2 - \Lambda_t\eta_t)e^{-2\eta} = 0. \end{aligned} \tag{1.5}$$

Equation (1.3) guarantees that Λ is a function of r only and equation (1.2) then guarantees that Λ has the same form as for the Schwarzschild metric:

$$\Lambda = -\frac{1}{2}\ln\left|1 - \frac{2M}{r}\right|. \tag{1.6}$$

Equations (1.4) and (1.5) then become two equivalent equations for $\eta(t,r)$-equivalent by virtue of the Bianchi identity, $\nabla \cdot \mathbf{G} = 0$-whose solution is

$$\eta = \frac{1}{2}\ln\left|1 - \frac{2M}{r}\right| + f(t), \tag{1.7}$$

where f is an arbitrary function. Put expression (1.6) and (1.7) into the metric (1.1); thereby obtain

$$ds^2 = e^{2f(t)}\left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{1 - 2M/r} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

Then redefine the time coordinate

$$t_{\text{new}} = \int e^{f(t)} dt,$$

and thereby bring the line element into the Schwarzschild form

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{1 - 2M/r} - r^2(d\theta^2 + \sin^2\theta\varphi^2). \tag{1.8}$$

The Schwarzschild spacetime geometry (1.8) appears to behave badly near $r = 2M$, there g_{tt} becomes zero, and g_{rr} becomes infinite. However, one cannot be sure without careful study whether this pathology in the line element is due to a phathology of the (t, r, θ, φ) coordinate system near $r = 2M$. (As an example of a coordinate-induced pathology, consider the neighborhood of $\theta = 0$ on one of the invariant spheres, t =const and r =const. Then $g_{\varphi\varphi}$ becomes zero because that coordinates system behaves badly, however, the intrinsic, coordinate-independent geometry of the sphere is well-behaved there).

The worrisome region of the Schwarzschild geometry, $r = 2M$, is called the "gravitational radius" or the "Schwarzschild radius" or the "Schwarzschild surface" or the "Schwarzschild horizon" or the "Schwarzschild sphere". It is also called the "Schwarzschild singularity" in some of older litarature, but that is misnomer, since, as will be shown, the space time geometry is not singular there.

A Schwarzschild black hole or static black hole is a black hole that has no charge or angular momentum. A Schwarzschild black hole has a Schwarzschild metric, and cannot be distinguished from any other Schwarzschild black hole except by its mass. The Schwarzschild black hole is characterized by a surrounding spherical surface, called the event horizon, which is situated at the Schwarzschild radius, often called the radius of a black hole. Any non-rotating and non-charged mass that is smaller than its Schwarzschild radius forms a black hole. The solution of the Einstein field equations is valid for any mass M, so in principle (according to general relativity theory) a Schwarzschild black hole of any mass could exist if conditions became sufficiently favorable to allow for its formation.

The structure of the paper is the following. Section 1, is devoted to the mathematical formulation of the Schwarzschild geometry for using the results in the main idea, the geodesics of the black hole as an Riemannian geodesics are given in this section. In section 2, the method for finding Lie point symmetries are presented, classification of Lie subalgebras up to an inner automorphism and classification of geodesics due to Lie point symmetries is done in this section. The third section is specified to study some classical similarity solution of the geodesics. Hamiltonian equations and Hamiltonian symmetry symmetry groups are found in forth and fifth and the last section gives some local conservation laws for the system of geodesics.

1.1. **Geodesics of Schwarzschild sphere.** In differential geometry if we have a spacetime coordinate such as $\mathbf{x} = (x^1(\tau), \ldots, x^n(\tau))$, the geodesic equation [10], for its world line is:

$$\ddot{x}^\alpha + \sum_{\alpha=1}^{n} \Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0, \quad \mu,\nu = 1,...,n, \tag{1.9}$$

where $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol of the second type obtained by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\sum_{\ell=1}^{n} g^{\alpha\ell}\left(\frac{\partial g_{\nu\ell}}{\partial x^\mu} + \frac{\partial g_{\mu\ell}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\ell}\right),$$

for Riemannian metric $g_{\mu\nu}$.

The nonzero components of the Christoffel symbols are:

$$\Gamma^1_{12} = -\Gamma^2_{22} = -\frac{a}{2r(a-r)}, \quad \Gamma^2_{11} = -\frac{a(a-r)}{2r^3}, \quad \Gamma^2_{33} = a - r,$$
$$\Gamma^2_{44} = (a-r)\sin^2\theta, \quad \Gamma^3_{23} = \Gamma^4_{24} = \frac{1}{r}, \quad \Gamma^3_{44} = -\frac{1}{2}\sin 2\theta, \quad \Gamma^4_{34} = \cot\theta.$$

By substituting these symbols in (1.9) we obtain geodesics system for metric (1.8) such as:

$$\begin{cases} t''(\tau) - \frac{a}{r(a-r)}t'(\tau)r'(\tau) = 0, \\ r''(\tau) + \frac{a}{2r(a-r)}r'(\tau)^2 + (a-r)\left[\theta'(\tau)^2 + \sin^2\theta\varphi'(\tau)^2 - \frac{a}{2r^3}t'(\tau)^2\right] = 0, \\ \theta''(\tau) + \frac{2}{r}r'(\tau)\theta'(\tau) - \frac{1}{2}\sin 2\theta\varphi'(\tau)^2 = 0, \\ \varphi''(\tau) + \frac{2}{r}r'(\tau)\varphi'(\tau) + 2\cot\theta\theta'(\tau)\varphi'(\tau) = 0. \end{cases}$$

2. Lie Point Symmetries

Symmetry plays a very important role in various fields of nature. As is known to all, Lie method is an effective method and a large number of equations [7] are solved with the aid of this method. There are still many authors using this method to find the exact solutions [14, 15] of non-linear DEs. It is also a powerful tool for finding exact solutions of non-linear problems [15,16]. One of the most important application of symmetry's method is the reducing systems of DEs, i.e., finding equivalent systems of DEs of simpler form, that is called reduction. This method provides a systematic computational algorithm for determining a large classes of special solutions. The solutions of the obtained equivalent system will correspond to solutions of the original system. Many examples of applications to physical problems have been demonstrated in a huge number of papers and a lot of excellent books. The general procedure to obtain Lie symmetries of differential equations, and their applications to find analytic solutions of the equations are described in detail in several monographs on the subject (e.g. [7, 14, 15]) and in numerous papers in the literature (e.g. [2, 5, 6, 13]).

Nowadays the group theory of DEs is extended to DEs of fractional order. This subject is the rapidly growing field of research. In recent years, fractional order DEs have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasicity, biology, probability, mathematical physics and engineering [11, 17]. The same effective approach for investigating and constructing solutions of fractional DEs is the group analysis based on the fractional calculas such as Riemann-Liouville, Caputo and etc. methods. Here the calculations are different from the classical method which is the main purpose of the present paper, but the obtained results are more interesting and newer [9, 18]. We will discause the classical group theory method for DEs in the sequel.

Consider a system of DE (PDE or ODE) in the dependent variables $u^\alpha(1 \le \alpha \le m)$ and dependent variables $x^i(1 \le i \le n)$ of the form:

$$\Delta^s(x^i, u^\alpha, u_i^\alpha, u_{ij}^\alpha, \ldots) = 0, \quad 1 \le s \le k, \tag{2.1}$$

where the subscripts denote partial derivatives (e.g. $u_i^\alpha = \partial u^\alpha/\partial x^i$). To determine continuous symmetries of (2.1), it is useful to consider infinitesimal Lie transformations of the form:

$$\tilde{x}^i = x^i + \varepsilon\xi^i + O(\varepsilon^2), \quad \tilde{u}^\alpha = u^\alpha + \varepsilon\eta^\alpha + O(\varepsilon^2), \tag{2.2}$$

that leave the equation system invariant to $O(\varepsilon^2)$. Lie point symmetries correspond to the case where the infinitesimal generators $\xi^i = \xi^i(x^i, u^\alpha)$ and $\eta^\alpha = \eta^\alpha(x^i, u^\alpha)$ depend only on the x^i and the u^α and not on the derivatives or integrals of the u^α. Generalized Lie symmetries are obtained in the case when the transformations (2.2) also depend on the derivatives or integrals of the u^α.

The infinitesimal transformations for the first and second derivatives to $O(\varepsilon^2)$ are given by the prolongation formulae:

$$\tilde{u}_i^\alpha = u_i^\alpha + \varepsilon\zeta_i^\alpha, \quad \tilde{u}_{ij}^\alpha = u_{ij}^\alpha + \varepsilon\zeta_{ij}^\alpha,$$

where

$$\zeta_i^\alpha = D_i\hat{\eta}^\alpha + \xi^s u_{si}^\alpha, \quad \zeta_{ij}^\alpha = D_iD_j\hat{\eta}^\alpha + \xi^s u_{sij}^\alpha. \tag{2.3}$$

Here

$$\hat{\eta}^\alpha = \eta^\alpha - \xi^s u_s^\alpha, \tag{2.4}$$

corresponds to the canonical Lie transformation for which $\tilde{x}^i = x^i$ and $\tilde{u}^\alpha = u^\alpha + \varepsilon\hat{\eta}^\alpha$. The symbol D_i in (2.3) denotes the total derivative operator with respect to x^i. Similar formulae to (2.3) apply for the transformation of the higher order derivatives.

The condition for invariance of the DE system (2.1) to $O(\varepsilon^2)$ under the Lie transformation (2.2) can be expressed in the form:

$$\mathcal{L}_{\mathbf{v}}\Delta^s \equiv \tilde{\mathbf{v}}(\Delta^s) = 0 \quad \text{whenever} \quad \Delta^s = 0, 1 \le s \le k, \tag{2.5}$$

where

$$\tilde{\mathbf{v}} = \mathbf{v} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} + \cdots$$

is the prolongation of the vector field

$$\mathbf{v} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha},$$

associated with the infinitesimal transformation (2.2). The symbol $\mathcal{L}_{\mathbf{v}}\Delta^s$ in (2.5) denotes the Lie derivative of Δ^s with respect to the vector field $\mathbf{v}$ (i.e., $\mathcal{L}_{\mathbf{v}}\Delta^s = \frac{\mathrm{d}\Delta^s}{\mathrm{d}\varepsilon}|_{\varepsilon=0}$).

The Lie symmetries of the Schwarzschild geodesics system (1.10) for t, r, θ and φ can be found by solving the Lie determining equation (2.5) for the infinitesimal generators of the Lie group. Below we first write down the Lie determining equations that correspond to the point Lie group. the point Lie algebra system is briefly described, and the symmetries are used to obtain some results for the solutions of system (1.10).

The infinitesimal Lie transformations for the system (1.10) are of the form:

(2.6)
$$\tilde{\tau} = \tau + \varepsilon\xi^{\tau}, \qquad \tilde{t} = t + \varepsilon\eta^{t}, \qquad \tilde{r} = r + \varepsilon\eta^{r}, \qquad \tilde{\theta} = \theta + \varepsilon\eta^{\theta}, \qquad \tilde{\varphi} = \varphi + \varepsilon\eta^{\varphi}.$$

The corresponding canonical symmetry generators $\hat{\eta}^t, \hat{\eta}^r, \hat{\eta}^\theta$ and $\hat{\eta}^\varphi$ are given by the formulae analogous to (2.4). Thus

$$\hat{\eta}^{\alpha} = \eta^{\eta} - \xi^{\tau}\eta_{\tau},$$

relates the canonical symmetry generator $\hat{\eta}^\alpha$ to η^α, where α can be any of the dependent variables t, r, θ and φ.

The Lie determining equations (2.5) for the infinitesimal generators of the system (1.10) can be written in the form:

$$\begin{array}{lll}
\xi_t = 0, & \xi_r = 0, & \xi_\theta = 0, \\
\xi_{\tau\tau} = 0, & \eta_{1\tau} = 0, & \eta_{1t} = 0, \\
\eta_2 = 0, & \eta_{3\tau} = 0, & \eta_{3t} = 0, \\
\eta_{3\theta} = 0, & \eta_{3\varphi} = 0, & \eta_{1\theta\theta} = -\eta_{1\theta}\cot\theta, \\
\eta_{1\varphi\varphi} = \frac{1}{2}\sin 2\theta\eta_{1\theta}, & \xi_\varphi = 0, & \eta_{1r} = 0, \\
\eta_{3r} = 0, & \eta_{1\theta\varphi} = -\frac{1}{2}\sec 2\theta\eta_{1\varphi}, &
\end{array} \tag{2.7}$$

for the vector field

$$\mathbf{v} = \xi\frac{\partial}{\partial\tau} + \eta_1\frac{\partial}{\partial t} + \eta_2\frac{\partial}{\partial r} + \eta_3\frac{\partial}{\partial\theta} + \eta_4\frac{\partial}{\partial\varphi}.$$

Solving the system (2.7) we obtain:

$$\begin{aligned}
&\xi = C_4\tau + C_5, \quad \eta_1 = C_3 + (C_1\sin\varphi + C_2\cos\varphi)\cot\theta, \\
&\eta_2 = 0, \quad \eta_3 = C_6, \quad \eta_4 = -C_1\cos\varphi + C_2\sin\varphi.
\end{aligned}$$

The general vector field $\mathbf{v}$ in the point Lie algebra corresponding to the transformations (2.6) can be written in the form:

$$\mathbf{v} = \sum_{i=1}^{6} a_i\mathbf{v}_i,$$

Table 1. Commutators Table of $\mathcal{G}$

$[\mathbf{v}_i, \mathbf{v}_j]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_1$	0	0	0	0	0	0
$\mathbf{v}_2$	0	0	0	0	0	0
$\mathbf{v}_3$	$-\mathbf{v}_1$	0	0	0	$-\mathbf{v}_6$	$\mathbf{v}_5$
$\mathbf{v}_4$	0	0	0	0	0	0
$\mathbf{v}_5$	0	0	$\mathbf{v}_6$	0	0	$-\mathbf{v}_3$
$\mathbf{v}_6$	0	0	$-\mathbf{v}_5$	0	$\mathbf{v}_3$	0

where the basis vector fields $\{\mathbf{v}_i : 1 \leq i \leq 6\}$ are

$$\begin{aligned} \mathbf{v}_1 &= \frac{\partial}{\partial \tau}, \quad \mathbf{v}_2 = \frac{\partial}{\partial t}, \quad \mathbf{v}_3 = \frac{\partial}{\partial \varphi}, \quad \mathbf{v}_4 = \tau \frac{\partial}{\partial \tau}, \\ \mathbf{v}_5 &= \sin\varphi \frac{\partial}{\partial \theta} + \cot\theta \cos\varphi \frac{\partial}{\partial \varphi}, \quad \mathbf{v}_6 = -\cos\varphi \frac{\partial}{\partial \theta} + \cot\theta \sin\varphi \frac{\partial}{\partial \varphi}. \end{aligned} \tag{2.8}$$

The commutator table of the Lie algebra $\mathcal{G}$ spanned by the vector fields $\mathbf{v}_i$'s are given in Table 1.

A straight forward calculation shows that if $\mathbf{u} = (t, r, \theta, \varphi)$ be a geodesic curve of metric (1.8), then so are:

$$\begin{aligned} \tilde{\mathbf{u}}(\tau) &= \mathbf{u}(t(\tau+\varepsilon), r(\tau+\varepsilon), \theta(\tau+\varepsilon), \varphi(\tau+\varepsilon)), \\ \tilde{\mathbf{u}}(\tau) &= \mathbf{u}(t(\tau)+\varepsilon, r(\tau), \theta(\tau), \varphi(\tau)), \\ \tilde{\mathbf{u}}(\tau) &= \mathbf{u}(t(\tau), r(\tau), \theta(\tau), \varphi(\tau)+\varepsilon), \\ \tilde{\mathbf{u}}(\tau) &= \mathbf{u}(t(e^{\varepsilon}\tau), r(e^{\varepsilon}\tau), \theta(e^{\varepsilon}\tau), \varphi(e^{\varepsilon}\tau)), \\ \tilde{\mathbf{u}}(\tau) &= \mathbf{u}(t(\tau), r(\tau), \varepsilon \sin\varphi(\tau), \arcsin(\varepsilon\theta(\tau)) + \varepsilon\sqrt{1-\varepsilon^2\theta^2(\tau)}\cot\theta(\tau)), \\ \tilde{\mathbf{u}}(\tau) &= \mathbf{u}(t(\tau), r(\tau), -\cos\varphi(\tau), \arccos(-\varepsilon\theta(\tau)) + \varepsilon\sqrt{1-\varepsilon^2\varphi^2(\tau)}\cot\theta(\tau)). \end{aligned}$$

The first and forth curve demonstrate the time, radius, colatitude and longitude angle invariance of the system, second and third curves show genuinely local group of transformations.

We know that evaluation of the flow of vector fields in $\mathcal{G}$ serves to define the exponential map $\exp : \mathcal{G} \to G$. Since $\exp(0) = e$, $d\exp(0) = I_d$, the exponential map defines a local diffeomorphism in a neighborhood of $0 \in \mathcal{G}$. Consequently, all Lie groups having the same Lie algebra look locally the same in a neighborhood of the identity; only the global topological properties are different. Globally, the exponential map is not necessarily one-to-one nor onto. However, if a Lie group is connected, it can be completely recovered by successive exponentiations.

The most general one parameter group of symmetries is obtained by considering a general linear combination $c_1\mathbf{v}_1 + \cdots + c_6\mathbf{v}_6$ of the given vector fields; the explicit formulae for the group transformations are very complicated. In particular if g is near the identity, it can be represented uniquely in the form

$$g = \exp(\varepsilon_6 \mathbf{v}_6) \circ \cdots \circ \exp(\varepsilon_1 \mathbf{v}_1). \tag{2.9}$$

For instance if $\varepsilon_1 = \varepsilon_4 = 0$ then the most general Lie group action with respect to (2.9) is

$$g = \Bigg(t + \varepsilon_2, r, \varepsilon_5 - \cos\left(\arcsin(\varepsilon_5 \theta) + \varepsilon_5 \sqrt{1 - \varepsilon_5^2 \theta^2} \cot\theta \right), \sqrt{1 - \varepsilon_5^2 \varepsilon_6^2 \sin^2\varphi} + \varepsilon_6 \sqrt{1 - \varepsilon_6 \left[\arcsin(\varepsilon_5 \theta) + \varepsilon_5 \sqrt{1 - \varepsilon_5 \theta^2} \cot\theta \right]^2} \cot(\varepsilon_5 \sin\varphi) + \varepsilon_3 \Bigg).$$

2.1. **Classification of Subalgebras.** As is well known, the theoretical Lie group method plays an important role for finding exact solutions and performing symmetry reductions of DE. Since any linear combination of infinitesimal generators is also an infinitesimal generator, there are always infinitely many different symmetry subgroups for the DE. So, a mean of determining which subgroups would give essentially different types of solutions is necessary and significant for a complete understanding of the invariant solutions. As any transformation in the full symmetry group maps a solution to another solution, it is sufficient to find invariant solutions which are not related by transformations in the full symmetry group, this has led to the concept of an optimal system [9]. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. This problem is attacked by the naive approach of taking a general element in the Lie algebra and subjecting it to various adjoint transformations so as to simplify it as much as possible. The idea of using the adjoint representation for classifying group-invariant solutions is due to [15, 16].

Optimal system of a Lie algebra is equivalent to find nonessentially different invariant solutions which are not in a same orbit of adjoint actions. It means that the problem of finding invariant solutions under a group action G or its subgroups is reduced to problem of constructing optimal system of its corresponding subalgebras. As a result if the optimal system of subalgebras is found this optimality is meant that it is the smallest subspace in the set of solutions possesing the following property: Any invariant solution which can be found from any subgroup of G; is contained in one of the orbit of $G-$action, i.e., optimal system.

The problem of finding optimal system is divided to two method; a) using the generators of Lie algebra directly and b) using the ideal of Lie algebra. The first method is processing in the sequel. But second method has two levels. Finding optimal system from ideals and extend it on the whole Lie algebra. It means that suppose $\mathcal{G}$ is a subalgebra spanned by n vector fields $X_1, \ldots, X_n$ with m ideals $\mathcal{G}_1, \ldots, \mathcal{G}_m$. This method is starting by selecting $X_1 \in \mathcal{G}_1$ and we introduce it a member of the optimal system. Next by selecting $X_2 \in \mathcal{G}_2$ where X_2 is not in $\mathcal{G}_1$. Thus, $X_1 + aX_2$ is another member of optimal system and if it is possible we can normalize the coefficient a by adjoint action. This procedure will continue until the las ideal. Finally we normalize the linear combination of obtained members with other generators of $\mathcal{G}$ that are not in any ideals with adjoint action. The last level gives us the one-dimensional optimal system.

2.1.1. *One-dimensional optimal system.* The adjoint action is given by the Lie series

$$\mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i)\mathbf{v}_j) = \mathbf{v}_j - \varepsilon[\mathbf{v}_i,\mathbf{v}_j] + \frac{\varepsilon^2}{2}[\mathbf{v}_i,[\mathbf{v}_i,\mathbf{v}_j]] - \cdots\,,$$

where $[\mathbf{v}_i,\mathbf{v}_j]$ is the commutator for the Lie algebra, ε is a parameter, and $i,j = 1,2,\ldots,6$. Let $F_i^\varepsilon : \mathcal{G} \to \mathcal{G}$ defined by $\mathbf{v} \mapsto \mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i)\mathbf{v})$ is a linear map, for $i = 1,2,\ldots,6$. The matrices M_i^ε of F_i^ε, $i = 1,2,\ldots,6$, with respect to basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4,\mathbf{v}_5,\mathbf{v}_6\}$ are given by

$$M_1^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ -\varepsilon & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M_2^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_3^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & \cos\varepsilon & \sin\varepsilon\\ 0 & 0 & 0 & 0 & -\sin\varepsilon & \cos\varepsilon \end{pmatrix}, \qquad M_4^\varepsilon = \begin{pmatrix} e^\varepsilon & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_5^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & \cos\varepsilon & 0 & 0 & -\sin\varepsilon\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & -\sin\varepsilon & 0 & 0 & \cos\varepsilon \end{pmatrix}, \qquad M_6^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & \cos\varepsilon & 0 & 0 & \sin\varepsilon\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -\sin\varepsilon & 0 & \cos\varepsilon & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By acting these matrices on a vector field $\mathbf{v}$ alternatively we can show that a one-dimensional optimal system of $\mathcal{G}$ is given by:

$$\mathbf{v}_2,\quad \mathbf{v}_6,\quad a_1\mathbf{v}_1 + a_2\mathbf{v}_2,\quad a_1\mathbf{v}_2 + a_2\mathbf{v}_5,\quad a_1\mathbf{v}_2 + a_2\mathbf{v}_3 + a_3\mathbf{v}_4, \tag{2.10}$$

where a_1, a_2 and a_3 are real constants. The process above begins with the selection of vector $X = \sum_i a_i\mathbf{v}_i$ and its image under M_i^ε, obtained by the adjoint automorphism. If M_i^ε is the matrix of the automorphism M^ε in the basis $\{\mathbf{v}_i\}$, then the components on the image of X in the basis are given by an equation in the form of

$$M_{ij}^\varepsilon a_j,\quad j = 1,2,\ldots,6. \tag{2.11}$$

The next step is the selection of values of the parameter M_i^ε, on which this automorphism depends, to achieve the maximum possible simplification of the set of equations (2.11). This permits the choice of simplest representative of class of similar algebras to which the element X belongs. Usually, this means choosing the maximum possible number of null values for these components.

2.1.2. *Two-dimensional optimal system.* Next step is to construct two-dimensional optimal system, i.e., classification of two-dimensional subalgebras of $\mathcal{G}$. The process is by selecting one of the vector fields in (2.10), say, any vector field of (2.10). Let us consider X_1 (or X_i, $i = 2, 3, 4, 5, 6$). Corresponding to it, a vector field $X = a_1\mathbf{v}_1 + \cdots + a_4\mathbf{v}_6$, where a_i's are smooth functions of $(\tau, t, r, \theta, \varphi)$ is chosen, so we must have

$$[X_1, X] = \lambda X_1 + \mu X. \tag{2.12}$$

Equation (2.12) leads us to the system

$$C^i_{jk}\alpha_j a_k = \lambda a_i + \mu\alpha_i, \quad i = 1, \ldots, 6. \tag{2.13}$$

The solutions of the system (2.13), give one of the two-dimensional generator and the second generator is X_1 or, X_i, $i = 2, 3, 4, 5, 6$ if selected. After the construction of all two-dimensional subalgebras, for every vector fields of (2.10), they need to be simplified by the action of adjoint matrices in the manner analogous to the way of one-dimensional optimal system. Thus the two-dimensional optimal system of $\mathcal{G}$ has three classes of $\mathcal{G}$'s members combinations such as

$$\begin{aligned} &\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2, \beta_1\mathbf{v}_3 + \beta_2\mathbf{v}_4 + \beta_3\mathbf{v}_5 + \beta_4\mathbf{v}_6,\\ &\alpha_1\mathbf{v}_2 + \alpha_2\mathbf{v}_5, \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_4 + \beta_3\mathbf{v}_6,\\ &\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_6, \beta_1\mathbf{v}_2 + \beta_2\mathbf{v}_2 + \beta_3\mathbf{v}_4. \end{aligned} \tag{2.14}$$

2.1.3. *Three-dimensional optimal system.* This system can be developed by the method of expansion of two-dimensional optimal system. For this take any two-dimensional subalgebras of (2.14), let us consider the first two vector fields of (2.14), and call them Y_1 and Y_2, thus, we have a subalgebra with basis $\{Y_1, Y_2\}$, find a vector field $Y = a_1\mathbf{v}_1 + \cdots + a_6\mathbf{v}_6$, where a_i's are smooth functions of $(\tau, t, r, \theta, \varphi)$, such the triple $\{Y_1, Y_2, Y\}$ generates a basis of a three-dimensional algebra. For that it is necessary an sufficient that the vector field Y satisfies the equations

$$[Y_1, Y] = \lambda_1 Y + \mu_1 Y_1 + \nu_1 Y_2, \quad [Y_2, Y] = \lambda_2 Y + \mu_2 Y_1 + \nu_2 Y_2, \tag{2.15}$$

and following from (2.15), we obtain the system

$$C^i_{jk}\beta^j_r a_k = \lambda_1 a_i + \mu_1\beta^i_r + \nu_1\beta^i_s, \quad C^i_{jk}\beta^j_s a_k = \lambda_2 a_i + \mu_2\beta^i_r + \nu_2\beta^i_s. \tag{2.16}$$

The solutions of system (2.16) is linearly independent of $\{Y_1, Y_2\}$ and give a three-dimensional subalgebra. This process is used for the another two couple vector fields of (2.14).

Consequently the three-dimensional optimal system of $\mathcal{G}$ is given by

$$\begin{aligned}&\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4,\\&\mathbf{v}_2, \mathbf{v}_3, \alpha_1\mathbf{v}_4 + \alpha_2\mathbf{v}_5,\\&\mathbf{v}_1, \mathbf{v}_2, \alpha_1\mathbf{v}_3 + \alpha_2\mathbf{v}_4 + \alpha_3\mathbf{v}_5 + \alpha_4\mathbf{v}_6.\end{aligned} \tag{2.17}$$

2.1.4. *Four-dimensional optimal system.* This system can be developed by the method of expansion of three-dimensional optimal system. For this take any three-dimensional subalgebras of (2.17), let us consider the first three vector fields of (2.17), and call them Y_1, Y_2 and Y_3, thus, we have a subalgebra with basis $\{Y_1, Y_2, Y_3\}$, find a vector field $Y = a_1\mathbf{v}_1 + \cdots + a_6\mathbf{v}_6$, where a_i's are smooth functions of $(\tau, t, r, \theta, \varphi)$, such the quadruple $\{Y_1, Y_2, Y_3, Y\}$ generates a basis of a three-dimensional algebra. For that it is necessary an sufficient that the vector field Y satisfies the equations

$$\begin{aligned}[Y_1, Y] &= \lambda_1 Y + \mu_1 Y_1 + \nu_1 Y_2 + \gamma_1 Y_3,\\ [Y_2, Y] &= \lambda_2 Y + \mu_2 Y_1 + \nu_2 Y_2 + \gamma_2 Y_3,\\ [Y_3, Y] &= \lambda_3 Y + \mu_3 Y_1 + \nu_3 Y_2 + \gamma_3 Y_3,\end{aligned} \tag{2.18}$$

and following from (2.18), we obtain the system

$$\begin{aligned}C^i_{jk}\beta^j_r a_k &= \lambda_1 a_i + \mu_1\beta^i_r + \nu_1\beta^i_s + \gamma_1\beta^i_l,\\ C^i_{jk}\beta^j_s a_k &= \lambda_2 a_i + \mu_2\beta^i_r + \nu_2\beta^i_s + \gamma_2\beta^i_l,\\ C^i_{jk}\beta^j_l a_k &= \lambda_3 a_i + \mu_3\beta^i_r + \nu_3\beta^i_s + \gamma_3\beta^i_l.\end{aligned} \tag{2.19}$$

The solutions of system (2.19) is linearly independent of $\{Y_1, Y_2, Y_3\}$ and give a four-dimensional subalgebra. This process is used for the another triple vector fields of (2.17).

Consequently the four-dimensional optimal system of $\mathcal{G}$ is given by

$$\begin{aligned}&\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5, \mathbf{v}_6,\\&\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4, \alpha_1\mathbf{v}_3 + \alpha_2\mathbf{v}_5,\\&\mathbf{v}_3, \mathbf{v}_5, \mathbf{v}_6, \alpha_1\mathbf{v}_2 + \alpha_2\mathbf{v}_4.\end{aligned} \tag{2.20}$$

2.1.5. *Five-dimensional optimal system.* A similar method such as another $n-$ dimensional optimal system for vector fields (2.20), leads us to find all five-dimensional subalgebras of $\mathcal{G}$. By omitting calculations, we are only giving five-dimensional subalgebras in the sequel;

$$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5, \mathbf{v}_6, \quad \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \quad \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6.$$

All previous calculations lead to the Table 2, 3 and 4 for the optimal system of $\mathcal{G}$.

TABLE 2. Optimal system of subalgebras

dimension	1	2
subalgebras	$\langle \mathbf{v}_2 \rangle$	$\langle \alpha_1\mathbf{v}_2 + \alpha_2\mathbf{v}_5, \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_4 + \beta_3\mathbf{v}_6 \rangle$
	$\langle \mathbf{v}_6 \rangle$	$\langle \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_6, \beta_1\mathbf{v}_2 + \beta_2\mathbf{v}_2 + \beta_3\mathbf{v}_4 \rangle$
	$\langle a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \rangle$	$\langle \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2, \beta_1\mathbf{v}_3 + \beta_2\mathbf{v}_4 + \beta_3\mathbf{v}_5 + \beta_4\mathbf{v}_6 \rangle$
	$\langle a_1\mathbf{v}_2 + a_2\mathbf{v}_5 \rangle$	
	$\langle a_1\mathbf{v}_2 + a_2\mathbf{v}_3 + a_3\mathbf{v}_4 \rangle$	

TABLE 3. Optimal system of subalgebras

dimension	3	4
subalgebras	$\langle \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4 \rangle$	$\langle \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5, \mathbf{v}_6 \rangle$
	$\langle \mathbf{v}_2, \mathbf{v}_3, \alpha_1\mathbf{v}_4 + \alpha_2\mathbf{v}_5 \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4, \alpha_1\mathbf{v}_3 + \alpha_2\mathbf{v}_5 \rangle$
	$\langle \mathbf{v}_1, \mathbf{v}_2, \alpha_1\mathbf{v}_3 + \alpha_2\mathbf{v}_4 + \alpha_3\mathbf{v}_5 + \alpha_4\mathbf{v}_6 \rangle$	$\langle \mathbf{v}_3, \mathbf{v}_5, \mathbf{v}_6, \alpha_1\mathbf{v}_2 + \alpha_2\mathbf{v}_4 \rangle$

TABLE 4. Optimal system of subalgebras

dimension	5	6
subalgebras	$\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5, \mathbf{v}_6 \rangle$	$\mathcal{G}$
	$\langle \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \rangle$	
	$\langle \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \rangle$	

2.2. **Classification of similarity solution.** An optimal system of $n-$parameter similarity solutions to a system of DE is a collection of solutions $\mathbf{u} = f(\mathbf{x})$ with the following properties.

i) Each solution in the list is invariant under some $n-$parameter symmetry group of the system of differential equations.
ii) If $\mathbf{u} = \widetilde{f}(\mathbf{x})$ is any other solution invariant under an $n-$parameter symmetry group, then there is a further symmetry g of the system which maps $\widetilde{f}$ to a solution $f = g \cdot \widetilde{f}$ on the list.

Consequently, there is an one-to-one corresponding between optimal system and similarity solutions. One say that if G be a symmetry group of a system of PDE and $\{\Theta_\alpha\}$ be an optimal system on n-parameter subgroups of G. Then the collection of all Θ_α-invariant solutions, for Θ_α in the optimal system, forms an optimal system of n-parameter similarity solutions to the mentioned system.

According to our optimal system of symmetry group of the system (1.10), finding n-parameter of similarity solutions we need only find those for n-parameter subgroups. In the next section we give some similarity solutions for the system (1.10) by using one-dimensional optimal system.

3. Classical Similarity Solutions

Lie point symmetries of the system (1.10) are determined by solving the overdetermined Lie determining equation (2.5) for the infinitesimal generators $\{\xi_i : 1 \leq i \leq n\}$ and the $\{\eta^\alpha : 1 \leq \alpha \leq m\}$. Classical similarity solutions of (1.10) are obtained by requiring the solution surfaces for the u^α are map onto the same set of surfaces, in the sense that $u'^\alpha(\mathbf{x}) = u^\alpha(\mathbf{x}')$. These conditions, to $O(\varepsilon^2)$ yield the first order PDE

$$\xi^i u_i^\alpha = \eta^\alpha,$$

with characteristics (2.4) being given by the group trajectories:

$$\frac{dx^i}{d\varepsilon} = \xi^i \quad \text{and} \quad \frac{du^\alpha}{d\varepsilon} = \eta^\alpha, \quad 1 \leq i \leq n, 1 \leq \alpha \leq m.$$

Integration of the group trajectories yields the invariants of the point Lie group admitted by the system, and these may be used to construct the classical similarity solutions of the system (1.10).

In this section, we obtain some classical similarity solutions [1,3] of the Schwarzschild geodesics system (1.10). The most useful solutions are the traveling wave solutions associated with the space and time translation symmetries. Another solutions are given in the sequel.

3.1. **Similarity solution corresponding to $a_1\mathbf{v}_2 + a_2\mathbf{v}_3 + a_3\mathbf{v}_4$.** The general Lie point symmetry operator X corresponding to $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ is

$$X = a_1\frac{\partial}{\partial t} + a_2\frac{\partial}{\partial \varphi} + a_3\tau\frac{\partial}{\partial \tau},$$

where $(\tau, t, r, \theta, \varphi)$ are the basis variables.

The classical similarity solutions of the system (1.10) for this operator are obtained by integrating the group trajectories

$$\frac{d\tau}{d\varepsilon} = a_3\tau, \quad \frac{dt}{d\varepsilon} = a_1, \quad \frac{d\varphi}{d\varepsilon} = a_2, \quad \frac{dr}{d\varepsilon} = \frac{d\theta}{d\varepsilon} = 0,$$

where ε is a parameter along the trajectories. Integration of the above system yields the integrals:

$$\varphi - \alpha t = I_1, \quad \ln|\tau| - \beta t = I_2, \quad r = I_3, \quad \theta = I_4,$$

for the group invariants $\{I_1, I_2, I_3, I_4\}$, where

$$\alpha = \frac{a_2}{a_1}, \quad \beta = \frac{a_3}{a_1},$$

(we implicity assume $a_1 \neq 0$). Thus, we obtain classical similarity solution:

$$t = \frac{\ln|\tau| - \nu}{\beta},$$

where ν is a constant.

3.2. **Similarity solution corresponding to $a_1\mathbf{v}_1 + a_2\mathbf{v}_2$.** The general Lie point symmetry for these vector fields is:

$$X = a_1\frac{\partial}{\partial\tau} + a_2\frac{\partial}{\partial t},$$

where $(\tau, t, r, \theta, \varphi)$ are the basis variables. By similar way to find the classical similarity solution we should integrate the group trajectories

$$\frac{d\tau}{d\varepsilon} = a_1, \quad \frac{dt}{d\varepsilon} = a_2, \quad \frac{dr}{d\varepsilon} = \frac{d\varphi}{d\varepsilon} = \frac{d\theta}{d\varepsilon} = 0,$$

where ε is a parameter for trajectories. By integrating this system we have:

$$t - \alpha\tau = I_1, \quad r = I_2, \quad \theta = I_3, \quad \varphi = I_4,$$

for the group invariant $\{I_1, I_2, I_3, I_4\}$ and

$$\alpha = \frac{a_2}{a_1}, \quad a_1 \neq 0.$$

Thus, in this case

$$t = \alpha t + \nu,$$

is the similarity solution and wave traveling similarity variable where ν is an arbitrary constant.

3.3. **Similarity solution corresponding to $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_6\mathbf{v}_6$.** The general Lie point symmetry for this case is:

$$X = a_1\frac{\partial}{\partial\tau} + a_2\frac{\partial}{\partial t} + a_3\Big(-\cos\varphi\frac{\partial}{\partial\varphi}\Big) + a_4\cot\theta\sin\varphi\frac{\partial}{\partial\theta}.$$

By integrating the group trajectories:

$$\frac{d\tau}{d\varepsilon} = a_1, \quad \frac{dt}{d\varepsilon} = a_2, \quad \frac{dr}{d\varepsilon} = 0, \quad \frac{d\theta}{d\varepsilon} = -a_3\cos\varphi, \quad \frac{d\varphi}{d\varepsilon} = a_4\cot\theta\cos\varphi,$$

where ε is a parameter for trajectories. Integration of this system yields the similarity variables:

$$t - \alpha t = I_1, \quad \theta + \beta\tau\cos\varphi = I_2, \quad \ln|\csc\varphi - \cot\varphi| - a_4\cot\theta = I_3, \quad r = I_4,$$

for the group invariant $\{I_1, I_2, I_3, I_4\}$ where

$$\alpha = \frac{a_2}{a_1}, \quad \beta = -\frac{a_3}{a_1},$$

for $a_1 \neq 0$. Thus, in this case

$$\ln|\csc\varphi - \cot\varphi| - a_4\cot(\beta\tau + c) = \tilde{c},$$

is the similarity solution with two constant parameter c and $\tilde{c}$.

3.4. **Similarity solution corresponding** $a_1\mathbf{v}_2 + a_2\mathbf{v}_5$. The general Lie point symmetry for case above is:

$$X = a_1\frac{\partial}{\partial t} + a_2 \sin\varphi\frac{\partial}{\partial \theta} + a_3\cos\varphi\cot\theta\frac{\partial}{\partial\varphi}.$$

By integrating the group trajectories:

$$\frac{d\tau}{d\varepsilon} = 0, \quad \frac{dt}{d\varepsilon} = a_1, \quad \frac{dr}{d\varepsilon} = 0, \quad \frac{d\theta}{d\varepsilon} = a_2\sin\varphi, \quad \frac{d\varphi}{d\varepsilon} = a_3\cot\theta\cos\varphi,$$

where ε is a parameter for trajectories. Integration of this system yields the similarity variables:

$$t - \alpha t = I_1, \quad \theta + \alpha t = I_2, \quad \ln|\sec\varphi + \tan\varphi| - \beta\cot\theta = I_3, \quad r = I_4,$$

for the group invariant $\{I_1, I_2, I_3, I_4\}$ where

$$\alpha = \frac{a_2}{a_1}, \quad \beta = \frac{a_3}{a_1},$$

for $a_1 \neq 0$. Thus, in this case

$$\ln|\sec\varphi + \tan\varphi| - \alpha\cot(\beta\tau + c) = \tilde{c},$$

is the similarity solution with two constant parameter c and $\tilde{c}$.

4. Hamiltonian Equations

Consider the six-dimensional Lie algebra $\mathcal{G}$, (2.8), of the symmetry group for Schwarzschild geodesics. Let $\omega_1, \ldots, \omega_6$ be a dual basis for $\mathcal{G}^*$ and $u = u^1\omega_1 + \cdots + u^6\omega_6$ a typical point therein. If $F : \mathcal{G}^* \to \mathbb{R}$, then its gradient is the vector

$$\nabla F = \frac{\partial F}{\partial u^1}\mathbf{v}_1 + \cdots + \frac{\partial F}{\partial u^6}\mathbf{v}_6.$$

Thus the Lie-Poisson bracket on $\mathcal{G}^*$ is

$$\begin{aligned}\{F, H\} =& u^1\left(\frac{\partial F}{\partial u^1}\frac{\partial H}{\partial u^3} - \frac{\partial F}{\partial u^3}\frac{\partial H}{\partial u^1}\right) + u^5\left(\frac{\partial F}{\partial u^3}\frac{\partial H}{\partial u^6} - \frac{\partial F}{\partial u^6}\frac{\partial H}{\partial u^3}\right)\\ &+ u^6\left(\frac{\partial F}{\partial u^3}\frac{\partial H}{\partial u^5} - \frac{\partial F}{\partial u^5}\frac{\partial H}{\partial u^3}\right).\end{aligned}$$

The structure matrix $\mathbf{J}(u) = (J^{ij}(u))$, where $J^{ij} = \{u^i, u^j\}$ is given by

$$\mathbf{J}(u) = \begin{pmatrix} 0 & 0 & u^1 & 0 & -u^6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -u^1 & 0 & 0 & 0 & 0 & -u^5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ u^6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -u^5 & 0 & 0 & 0 \end{pmatrix}.$$

Hamilton's equations corresponding to the Hamiltonian function $H(u)$ are therefore

$$\frac{du^i}{dt} = \mathbf{J}(u)\nabla H(u). \tag{4.1}$$

For example, if

$$H(u) = \sum_{i=1}^{6} \frac{(u^i)^2}{2I_i}, \tag{4.2}$$

where $I_i's$ are certain constants, then Hamilton's equations become the equations of a rigid body

$$\begin{aligned} \frac{du^1}{d\varepsilon} &= \frac{u^1u^3}{I_3} - \frac{u^5u^6}{I_5}, \quad \frac{du^3}{d\varepsilon} = -\frac{(u^1)^2}{I_1} - \frac{u^5u^6}{I_6}, \\ \frac{du^5}{d\varepsilon} &= \frac{u^1u^6}{I_1}, \quad \frac{du^6}{d\varepsilon} = -\frac{u^3u^5}{I_3}, \end{aligned}$$

in which $(I_1, \dots, I_6)$ are the moments of inertia about the coordinate axes and $u^1, \dots, u^6$ the correspopnding body angular momenta. (The angular velocities are $\omega^i = u^i/I_i$). The Hamiltonian function is the kinetic energy of the body.

4.1. **Hamiltonian Symmetry Group.** In this section we will find the Hamiltonian symmetry generator for equations (4.1).

Lemma 4.1. *The Hamiltonian vector field associated with $H(u)$ has the form*

$$\boldsymbol{v}_H = \sum_{i,j=1}^{m} J^{ij}(u)\frac{\partial H}{\partial u^j}\frac{\partial}{\partial u^i}. \tag{4.3}$$

Consider a system of ODE in Hamiltonian form

$$\frac{du}{dt} = \mathbf{J}(u)\nabla H(u,t), \tag{4.4}$$

where $H(u,t)$ is a Hamiltonian funtion and $\mathbf{J}(u)$ is the structure matrix determining the Poisson bracket.

Theorem 4.1. *A function $P(u,t)$ is a first integral for the Hamiltonian system* (4.4) *if and only if*

$$\frac{\partial P}{\partial t} + \{P, H\} = 0,$$

for all u, t. In particular, a time-independent function $P(u)$ is a first integral if and only if $\{P, H\} = 0$ everywhere.

Corollary 4.1. *If $u_t = \boldsymbol{J}\nabla H$ is any Hamiltonian system with time-independent Hamiltonian funtion $H(u)$, then $H(u)$ itself is automatically a first integral.*

For a Hamiltonian system symmetry groups are one-parameter Hamiltonian symmetry groups whoese infinitesimal generators (in evolutionary form) are Hamiltonian vector fields. The comming lemma shows that any first integral leads to such a symmetry group.

Lemma 4.2. *Let $P(u, t)$ be a first integral of a Hamiltonian system. Then the Hamiltonian vectir field $\boldsymbol{v}_P$ determined by P generates a one-parameter symmetry group of the system.*

Generally there is not a one-to-one correspondence between Hamiltonian vector fields and their corresponding Hamiltonian function. More generally, we can add any time-dependent function $C(u, t)$ (meaning that for each fixed t, C is a time-independent function) to a given function P without changing the form of its Hamiltonian vector field. Once we recognize the possibility of modifying the function determining a Hamiltonian symmetry group, we can readily prove a converse to the preceding proposition. This forms the Hamiltonian version of Noether's theorem.

Theorem 4.2. *A vector $\boldsymbol{w}$ generates a Hamiltonian symmetry group of a Hamiltonian system of ODE if and only if there exists a first integral $P(u, t)$ so that $\boldsymbol{w} = \boldsymbol{v}_P$ is the corresponding Hamiltonian vector field. A second function $\tilde{P}(u, t)$ determines the same Hamiltonian symmetry if and only if $\tilde{P} = P + C$ for some time-dependent function $C(u, t)$.*

Using this theorem and equation (4.3) we conclude that the Hamiltonian symmetry group for the system (1.10) corresponding to the Hamiltonian function (4.2) is given by

$$\begin{aligned}\mathbf{v}_H = {} & \left(\frac{u^1 u^3}{I_3} - \frac{u^5 u^6}{I_5}\right)\frac{\partial}{\partial u^1} + \left(\left(u^1\right)^2 I_1 - \frac{u^5 u^6}{I_6}\right)\frac{\partial}{\partial u^3} + \left(\frac{u^1 u^6}{I_1}\right)\frac{\partial}{\partial u^5} \\ & - \left(\frac{u^3 u^5}{I_3}\right)\frac{\partial}{\partial u^6}.\end{aligned}$$

5. Noether Symmetries

The main significant of variational symmetries is celbrated Noether's theorem [4]. According to this theorem there is a procedure which relates the constants of the motion of the given Lagrangian system to its symmetry transformations [15]. Lie symmetries of the system of the geodesic equations for a spacetime yield conserved quantities but there are also non-Noether symmetries which have not any relationship to conservation laws and hence are of no interest of our purpose. To compute symmetries of a system of geodesic equation is so complicated, as it involves the second prolongation of infinitesimal generators. On the other hand, the symmetries of a Lagrangian yield directly the conserved quantities which are of our interest, and it is wotrh mentioning that here only the first prolongation of the corresponding infinitesimal generators is required.

Noether symmetries, or symmetries of a Lagrangian, are defined as follows: consider a vector field

$$\mathbf{v} = \xi(\tau, x^\alpha)\frac{\partial}{\partial \tau} + \eta^\beta(\tau, x^\alpha)\frac{\partial}{\partial x^\beta}, \tag{5.1}$$

where $\alpha, \beta = 1, 2, 3, 4$. The first prolongation of the vector field (5.1) defined on the real parameter fiber bundle over the tangent bundle of the manifold, is expressed such as

$$\mathbf{v}^{(1)} = \mathbf{v} + \left(\frac{\partial \eta^\beta}{\partial \tau} + \frac{\partial \eta^\beta}{\partial x^\alpha} - \frac{\partial \xi}{\partial \tau}\frac{\partial x^\beta}{\partial \tau} - \frac{\partial \xi}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial \tau}\frac{\partial x^\beta}{\partial \tau}\right)\frac{\partial}{\partial x'^\beta},$$

where $x'^\beta = \partial x^\beta / \partial \tau$.

Since the geodesic equations are second order ODE, one generally takes first order Lagrangian. Particulary, we take $L(\tau, x^\alpha, x'^\alpha)$ then we obtain a set of second ODE

$$x''^\alpha = g(\tau, x^\alpha, x'^\alpha). \tag{5.2}$$

The vector field (5.1) is called the *Noether point symmetry* of this Lagrangian if there exist a gauge function, $\Phi(\tau, x^\alpha)$, such thet the identity

$$\mathbf{v}^{(1)}(L) + (D_\tau \xi)L = D_\tau \Phi, \tag{5.3}$$

holds for the total differential operator

$$D_\tau = \frac{\partial}{\partial \tau} + x'^\alpha \frac{\partial}{\partial x^\alpha}.$$

Consider a Lagrangian minimizing the arc lengh τ which results the system (1.10), as the Euler-Lagrange equations,

$$L[x^\alpha, x'^\alpha] = g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}, \tag{5.4}$$

for the Schwarzschild metric (1.1). Thus, the equation (5.4) becomes as:

$$L = \left(1 - \frac{2M}{r}\right)t''^2 - \left(1 - \frac{2M}{R}\right)^{-1} r''^2 - r^2\theta'^2 - r^2\sin^2\theta\varphi'^2. \tag{5.5}$$

Equations (5.3) including (5.5), yields defining equations for the six uknown functions ξ, η^α and Φ. thus, the Lie algebra of Noether symmetries is spanned by the following infinitesimal generators

$$\begin{aligned}
&\mathbf{v}_1 = \frac{\partial}{\partial \tau}, \quad \mathbf{v}_2 = \frac{\partial}{\partial t}, \quad \mathbf{v}_3 = \frac{\partial}{\partial \varphi}, \\
&\mathbf{v}_4 = \sin\varphi\frac{\partial}{\partial \theta} + \cot\theta\cos\varphi\frac{\partial}{\partial \varphi}, \quad \mathbf{v}_5 = -\cos\varphi\frac{\partial}{\partial \theta} + \cot\theta\sin\varphi\frac{\partial}{\partial \varphi}.
\end{aligned}$$

6. Conservation Laws

A conservation law of a non-degenerate system od DE is a divergence expression [1, 4], that vanishes on all solutions of the given system. In general, any such non-trivial expression that yields a local conservation law of the system arises from a linear combination formed local multipliers (characteristics) with each DE in the system, where the multipliers depend on the independent and dependent variables as well as at most a finite number of the dependent variables of the given system of DE. It turns out that a divergence expression depending on independent variables, dependent variables and their derivatives to some finite order is annihilated by the Euler operators associated with each of its dependent variables; conversely, if the Euler operators, associated with each dependent variable in an expression involving independent variables, dependent variables and their derivatives to some finite order, annihilated the expression, then the expression is a divergence expression. From this it follows that a given system of DE has a local conservation laws if and only if there exist a set of local multipliers whoese scalar product with each DE in system is identically annihilated without restricting the dependent variables in the scalar product to solution of the system, i.e., the independent variables, as well as each of their derivatives, are treated as arbitrary functions.

There are several method for constructing conservation laws of a system of DE. One of the most interesting systematic method for determinig conservation laws is related to Emmy Noether. She showed that for those PDE systems which are self adjoint, conservation laws arise from variational symmetries, i.e., symmetries which preserve the action integral [6, 8]. However, since the most of PDE systems arising in application are not self adjoint, thus the applicable of Noether's theorem is so limited. But there is a general systematic and computational mathod called direct method is applicable to all DE systems with no any special property. In this section we compute the conservation laws of the system (1.10) via Noether's method.

6.1. **Conseravtion Laws via Noether's Method.** Consider an $n-$th order DE system $\Delta^s(x^i, u^\alpha, u^\alpha_{(1)}, \dots, u^\alpha_{(n)}) = 0$ defined such as (2.1). A conservation law of this system is in the form of $D_i(\Lambda^i) = 0$, on its soloutions, where D_i is the total derivative with respect to x^i. The $p-$tuple $\Lambda = (\Lambda^1, \dots, \Lambda^p)$ is called the conserved vector of the system. Using a significant theorem we can find the conserved vectors of a given system of DE vie Noether symmetries.

Theorem 6.1. *If* $\mathbf{v}$ *is a Noether point symmetry corresponding to a Lagrangian* $L(\tau, x^\alpha, x'^\alpha)$*, then*

$$\Lambda = \xi L + (\eta^\alpha - x'^\alpha \xi)\frac{\partial}{\partial x'^\alpha} - \Phi,$$

is a first integral of (5.2) *corresponding to* $\mathbf{v}$*, where* $\Phi = \Phi(\tau, x^\alpha)$ *is the gauge function.*

Applying this theorem gives gives all the conservede vectors corresponding to Noether symmetries. The results are comming in the Table 5.

TABLE 5. Conservation laws arised from Noether symmetrieas

Noether symmetry	conserved vectors
$\mathbf{v}_1 = \frac{\partial}{\partial \tau}$	$\Lambda^1 = \left(1 - \frac{2M}{r}\right) t'^2 - \left(1 - \frac{2M}{r}\right)^{-1} r'^2 - r^2\theta'^2 - t^2 \sin^2\theta\varphi'^2$
$\mathbf{v}_2 = \frac{\partial}{\partial t}$	$\Lambda^2 = 2\left(1 - \frac{2M}{r}\right) t'$
$\mathbf{v}_2 = \frac{\partial}{\partial \varphi}$	$\Lambda^3 = 2r^2 \sin^2\theta\varphi'$
$\mathbf{v}_4 = -\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}$	$\Lambda^4 = -2r^2\cos\varphi\theta' + 2r^2\cot\theta\cos\varphi\sin^2\theta\varphi'$
$\mathbf{v}_5 = \sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}$	$\Lambda^5 = 2r^2\sin\varphi\theta' + 2r^2\cot\theta\cos\varphi\sin^2\theta\varphi'$

6.2. **Lie point symmetries and conservation laws.** In this subsection we show if any DE system such as (2.1) maps to DE system

$$\Gamma^s(x, u^\alpha, u^\alpha_{(1)}, \dots, u^\alpha_{(n)}) = 0, \tag{6.1}$$

by an invertible transformation, then any conservation laws of $\Delta_\nu(x, u^{(n)})$ maps to a conservation laws of $\Gamma_\nu(x, u^{(n)})$. When this transformation is a symmetry of system Δ then, the corresponding conservation laws is a conservation laws of Γ.

Consider the system (2.1), let

$$\Delta_\nu[U] = \Delta_\nu(x, U, \partial U, \dots, \partial^n U) = 0, \quad \nu = 1, \dots, \ell, \tag{6.2}$$

where $U(x) = (U^1(x), \dots, U^q(x))$ is a solution of the system (6.2). Consider an invertible point transformation

$$\begin{aligned} x^i =& x^i(z, W), \quad i = 1, \dots, p, \\ U^\alpha =& U^\alpha(z, W), \quad \alpha = 1, \dots, q, \end{aligned} \tag{6.3}$$

where $U(x) = (U^1(x), \dots, U^q(x))$, $z = (z^1, \dots, z^q)$ and $W(z) = (W^1(z), \dots, W^q(z))$.

Under the transformation (6.3) and its prolongation, any function $\Delta_\nu[U]$ maps to a function

$$\Gamma_\nu[W] = \Gamma_\nu(z, W, \partial W, \dots, \partial^n W).$$

In a special case $\Gamma_\nu[W] = \Delta_\nu[U]$, the components $x, U, \partial U, \dots, \partial^n U$ is written in the form of components $z, W, \partial W, \dots, \partial^n W$ in (6.3). If $U(x) = u(x)$ is a solution of the system (6.2), then $W(z) = w(z)$ is a solution of the system (6.1) in the form of

$$\Gamma_\nu[w] = \Gamma_\nu(z, w, \partial w, \dots, \partial^n w) = 0, \quad \nu = 1, \dots, \ell,$$

with p-independent variables $z = (z^1, \dots, z^p)$ and q-dependent variables $w = (w^1, \dots, w^q)$.

Theorem 6.2. *Suppose $D_i\Phi^i[U] = 0$ is a conservation laws of system DE systm* (6.2)*. There are some functions $\{\Psi^i[W]\}_{i=1}^p$ under transformations* (6.3) *such that*

$$\mathbf{J}[W] D_i\Phi^i[U] = \tilde{D}_i\Psi^i[W],$$

holds when $\Psi^i[W]$ *clearly determined in terms of a determinant which is replaced by the i-th column of the Jaccobian determinant*

$$\mathbf{J}[W] = \frac{D(x^1, \ldots, x^p)}{D(z^1, \ldots, z^q)},$$

with the column $\begin{pmatrix} \Phi^1[u] \\ \vdots \\ \Phi^p[U] \end{pmatrix}$, *which is*

$$\Psi^1[u] = \det \begin{pmatrix} \Phi^1[\tilde{u}] & D_2\tilde{x}^1 & \cdots & D_p\tilde{x}^1 \\ \Phi^2[\tilde{u}] & D_2\tilde{x}^2 & \cdots & D_p\tilde{x}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi^p[\tilde{u}] & D_2\tilde{x}^n & \cdots & D_p\tilde{x}^p \end{pmatrix},$$

$$\vdots \tag{6.4}$$

$$\Psi^p[u] = \det \begin{pmatrix} D_1\tilde{x}^1 & \cdots & D_{p-1}\tilde{x}^1 & \Phi^1[\tilde{u}] \\ D_1\tilde{x}^2 & \cdots & D_{p-1}\tilde{x}^2 & \Phi^2[\tilde{u}] \\ \vdots & \ddots & \vdots & \vdots \\ D_1\tilde{x}^p & \cdots & D_{p-1}\tilde{x}^p & \Phi^p[\tilde{u}] \end{pmatrix}.$$

Let us consider the invertible transformations (6.3) is a symmetry of system (6.2). Then, there are smooth functions $A^\nu_\tau[W]$ such that:

$$\Delta_\nu[U] = \Gamma_\nu[W] = A^\nu_\tau[W]\Delta_\tau[U].$$

Corollary 6.1. *If a point transformation* $(x, u) \mapsto (\tilde{x}(x, y), \tilde{u}(x, u))$ *be a symmetry of system* (6.2)*, then, a conservation law* $D_i\Phi^i[u] = 0$ *leads to a conservation law* $D_i\Psi^i[u] = 0$ *such that* (6.4) *holds for every conservation laws.*

This corollary shows that the action of a symmetry transformation of system (6.2) on a conservation laws $D_i\Phi^i[u] = 0$ leads us to a new conservation laws $D_i\Psi^i[u] = 0$.

Theorem 6.3. *Suppose the point transformation* (6.3) *is a symmetry of system* (6.2)*. If* $\{\Lambda_\nu[U]\}_{\nu=1}^{\ell}$ *be a set of conservation laws multipliers with conservation laws* $D_i\Phi^i[u]$*, then*

$$\tilde{\Lambda}_\tau[W]\Delta_\tau[W] = \tilde{D}_i\Psi^i[W],$$

where

$$\tilde{\Lambda}_\tau[W] = \mathbf{J}[W]A^\nu_\tau[W]\Lambda_\nu[U(z, W)], \quad \tau = 1, \ldots, \ell.$$

Corollary 6.2. *The set of multipliers* $\{\tilde{\Lambda}_\nu[U]\}_{\nu=1}^{\ell}$ *generates new conservation laws for system* (6.2) *if and only if it is a linear independent set on the solutions* $U(x) = u(x)$.

The main result of these section is, we can act point symmetries on the obtained conservation laws for finding new conservation laws. Now we apply it to find new conservation laws for system (1.10). Thus the functional independent conserved vectors are:

$$
\begin{aligned}
\Phi^1 =& 4\left(r^{-2}r't'^2 + \left(1 - \frac{2M}{r}t't''\right)\right),\\
\Phi^2 =& 2t\sin^2\theta\varphi'^2,\\
\Phi^3 =& t^2\sin 2\theta\sin\varphi\varphi'^2,\\
\Phi^4 =& t^2\sin 2\theta\cos\varphi\varphi'^2,\\
\Phi^5 =& r^2\sin 2\theta\sin\varphi\varphi',\\
\Phi^6 =& r^2\sin 2\theta\cos\varphi\varphi',\\
\Phi^7 =& 2r^2\sin\varphi\theta' + r^2\sin 2\theta\sin\varphi\varphi',\\
\Phi^8 =& 2r^2\cos\varphi - r^2\sin 2\theta\sin\varphi\varphi',\\
\Phi^9 =& 4r(\sin\theta\varphi'r' + r\cos\theta\varphi'\theta' + r\sin\theta\varphi'')\sin\theta\varphi',\\
\Phi^{10} =& \sin 2\varphi\cos 2\theta r^2\varphi' + r^2\cot\theta\cos\varphi(2\sin\varphi\theta' - \sin 2\theta\sin\varphi\varphi'),\\
\Phi^{11} =& -2r^2\cos 2\theta\cos^2\varphi + r^2(2\sin\varphi\theta' - \sin 2\theta\sin\varphi\varphi'),\\
\Phi^{12} =& -4r^2\cos 2\theta\cos^2\varphi_2 r^2\cot\theta\cos\varphi(\cos\varphi\theta' - \sin 2\theta\sin\varphi\varphi'),\\
\Phi^{13} =& -4r^2\cos 2\theta\cos^2\varphi + 2r^2\cot\theta\cos\varphi(\cos\varphi\theta' - \sin 2\theta\sin\varphi\varphi'),\\
\Phi^{14} =& \frac{2M}{r}r't'' + 2\left(1 - \frac{2M}{r}\right)t't'' + 2Mr^{-2}\left(1 - \frac{2M}{r}\right)^{-2}r'^3 - 2\left(1 - \frac{2M}{r}\right)^{-1}r'r''\\
& - 2rr'\theta'^2 - 2r^2\theta'\theta'' - 2t\sin^2\theta t'\varphi'^2 - 2t^2\sin^2\theta\cos\theta\theta'\varphi'^2 - 2t^2\sin^\theta\varphi'\varphi'',\\
\Phi^{15} =& -4r\cos^2\varphi r'\theta' + r^2\sin 2\varphi\theta'\varphi' - 4r^2\cos^\varphi\theta'' + 4r^2\cot\theta\cos 2\varphi\theta'\varphi'\\
& - 4r^2\sin^2\varphi\cos^3\varphi\varphi' + 2r^2\sin 2\theta\cot^2\varphi\cos^2\varphi\theta'\varphi' + \frac{1}{2}r^2\sin^2 2\theta\cos^2\varphi'',\\
\Phi^{16} =& -4r\sin^2\varphi r'\theta' + r^2\sin 2\varphi\theta'\varphi' + 2r^2\cos^2\varphi\theta'' + 4r^2\cot\theta\cos^\varphi\theta'\varphi'\\
& - 4r^2\sin^3\varphi\cos^2\varphi\varphi' + 2r^2\sin 2\theta\cot^2\varphi\cos^2\varphi\theta'\varphi' + \frac{1}{2}r^2\sin^2 2\theta\cos^2\varphi''.
\end{aligned}
$$

7. Conlusion

The Lie theory has a vast applications to study physical phenomena. This theory gives us so much useful devices for analyzing DEs, which qualify a lot of natural occurrences. In this paper we used Lie theory to study one of the cosmological happening which incidents in a special kind of black hole. For instance we classified geodesics of metric (1.8), by using the symmetries of geodesics system of differential equations, this allows us to have some interpretations for motions on geodesics curve in Schwarzschild black hole which could be useful for cosmologists. Classification of subalgebras helps us to find similarity solutions and exact solutions (if it is possible) for the system (1.10) and etc. Hamiltonian equations and conservatiopns laws via Noether's theorem are found in fourth and fifth sections respectively. Here we should thanks to prof. Cheviakov for the GeM, a useful package for Maple and symmetry computations.

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