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GROWTH ESTIMATES FOR CERTAIN CLASS OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we establish some lower bound estimates for a certain class of rational functions on a disk with prescribed poles and restricted zeros. The results obtained strengthen some known results for rational functions and in turn produce generalizations of some polynomial inequalities as well.

1. Introduction

Let $\mathbb{P}_n$ be the set of all complex polynomials $f(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n. Then it is well known that

$$\max_{|z|=1} |f'(z)| \leq n \max_{|z|=1} |f(z)| \tag{1.1}$$

and

$$\max_{|z|=\rho} |f(z)| \leq \rho^n \max_{|z|=1} |f(z)|, \quad \rho \geq 1. \tag{1.2}$$

The above inequalities are known as Bernstein inequalities, and have been the starting point for a considerable literature in approximation theory. Several papers and research monographs have been written on this subject (see, for example, Marden [7], Rahman and Schmeisser [11], or Milovanović et al. [8]).

If one applies inequality (1.2) to the polynomial $f(z) = z^n f(\frac{1}{z})$ and uses maximum modulus principle, one easily get

$$\max_{|z|=\rho} |f(z)| \geq \rho^n \max_{|z|=1} |f(z)|, \quad 0 < \rho \leq 1. \tag{1.3}$$

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It is noted that equality holds in (1.1), (1.2) and (1.3) if and only if $f(z)$ has all its zeros at the origin. Inequality (1.3) is due to Varga [14], who attributes it to E. H. Zarantonello. It was shown by Govil, Qazi and Rahman [4] that the inequalities (1.1), (1.2) and (1.3) are all equivalent in the sense that any of these inequalities can be derived from the others. In 1960 Rivlin [13] improved inequality (1.3) for a restricted class of polynomials that do not vanish, in the unit disk $|z| < 1$. In fact, he proved that if $f(z) \neq 0$ in $|z| < 1$, then (1.3) can be replaced by

$$|f(\rho z)| \geq \left(\frac{1+\rho}{2}\right)^n |f(z)|, \quad \text{for } 0 < \rho < 1. \tag{1.4}$$

The inequality is best possible and equality holds for $f(z) = \left(\frac{\lambda+\beta z}{2}\right)^n$, where $|\lambda| = |\beta|$. As a generalization of (1.4), Aziz [2] proved that if $f(z) \neq 0$ in $|z| < 1$, then,

$$|f(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n |f(z)|, \quad \text{for } k \geq 1,\ \rho < 1, \tag{1.5}$$

and

$$|f(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n |f(z)|, \quad \text{for } k \leq 1,\ 0 \leq \rho \leq k^2. \tag{1.6}$$

The result is sharp and equality holds for $f(z) = (z + k)^n$.

Although the literature on polynomial inequalities is vast and growing, interested readers can consult the books of Marden [7] and Milovanović et al. [8]. Rather et al. [12] gave a new perspective to the above inequalities (1.5) and (1.6) and extended them to rational functions with prescribed poles. Essentially, in these inequalities, they replaced the polynomial $f(z)$ by a rational function $r(z)$ with poles $a_1, a_2, \ldots, a_n$ all lying in $|z| > 1$ and z^n by a Blaschke product $B(z)$. Before proceeding towards their results, let us introduce the set of rational functions involved.

For $a_j \in \mathbb{C}$ with $j = 1, 2, \ldots, n$, let

$$W(z) := \prod_{j=1}^{n} (z - a_j)$$

and

$$B(z) := \prod_{j=1}^{n} \left(\frac{1 - \overline{a}_j z}{z - a_j}\right), \quad \mathbb{R}_n := \mathbb{R}_n(a_1, a_2, \ldots, a_n) = \left\{\frac{f(z)}{W(z)} : f \in \mathbb{P}_n\right\}.$$

Then, $\mathbb{R}_n$ is the set of rational functions with poles $a_1, a_2, \ldots, a_n$ at most and with finite limit at $+\infty$. Note that $B(z) \in \mathbb{R}_n$ and $|B(z)| = 1$ for $|z| = 1$. Throughout this paper, we shall assume that all the poles $a_1, a_2, \ldots, a_n$ lie in $|z| > k$.

Very recently, Rather et al. [12] obtained certain growth estimates for rational functions $r \in \mathbb{R}_n$ having no zero in $|z| < k$. In this direction, they first present an extension of inequality (1.5) to the rational functions. More precisely, they proved the following result.

Theorem 1.1. *If $r \in \mathbb{R}_n$ having no zeros of r lie in $|z| < k$, $k \geq 1$, then for every $\rho < 1$ and $|z| = 1$,*

$$|r(\rho z)| \geq \left(\frac{\rho + k}{1 + k}\right)^n \prod_{j=1}^{n} \left(\frac{|a_j| - 1}{|a_j| + \rho}\right)|r(z)|. \tag{1.7}$$

Next they proved the following extension of (1.6).

Theorem 1.2. *If $r \in \mathbb{R}_n$ having no zeros in $|z| < k$, $k \leq 1$, then for every $0 \leq \rho \leq k^2$ and $|z| = 1$,*

$$|r(\rho z)| \geq \left(\frac{\rho + k}{1 + k}\right)^n \prod_{j=1}^{n} \left(\frac{|a_j| - 1}{|a_j| + \rho}\right)|r(z)|. \tag{1.8}$$

In this paper, we established some lower bound estimates for the maximal modulus of a certain class of rational functions on a disk with prescribed poles and restricted zeros. The results obtained strengthen some known results for rational functions by using some of the extremal coefficients of the underlying polynomial and in turn produce generalizations of some polynomial inequalities as well.

2. Main Results

Theorem 2.1. *If $r \in \mathbb{R}_n$ having no zeros of r lie in $|z| < k$, $k \geq 1$, then for every $\rho < 1$ and $|z| = 1$,*

$$|r(\rho z)| \geq \left\{\left[\left(\frac{k + \rho}{1 + k}\right)^n + \frac{1}{k^{n-1}}\left(\frac{|a_0| - k^n|a_n|}{|a_0| + |a_n|}\right)\left(\frac{1 - \rho}{1 + k}\right)^n\right] \prod_{j=1}^{n} \frac{|a_j| - 1}{|a_j| + \rho}\right\}|r(z)|. \tag{2.1}$$

Remark 2.1. Since $r(z) = \frac{P(z)}{W(z)}$ has all its zeros in $|z| \geq k$, therefore $P(z)$ has all its zeros in $|z| \geq k$, we always have the situation

$$\frac{|a_0| - k^n|a_n|}{|a_0| + |a_n|} \geq 0. \tag{2.2}$$

Therefore, for all rational functions satisfying the hypothesis of Theorem 2.1 excepting those satisfying $|a_0| = k^n|a_n|$, the inequality (2.1) sharpens the inequality (1.7).

Remark 2.2. If we take $k = 1$ in Theorem 2.1, we get the following extension and refinement of (1.4) to the rational functions.

Corollary 2.1. *If $r \in \mathbb{R}_n$ having no zeros of r lie in $|z| < 1$, then for every $\rho < 1$ and $|z| = 1$,*

$$|r(\rho z)| \geq \left\{\left[\left(\frac{1 + \rho}{2}\right)^n + \left(\frac{|a_0| - |a_n|}{|a_0| + |a_n|}\right)\left(\frac{1 - \rho}{2}\right)^n\right] \prod_{j=1}^{n} \frac{|a_j| - 1}{|a_j| + \rho}\right\}|r(z)|.$$

Remark 2.3. Taking $W(z) = (z-\alpha)^n$, $|\alpha| > 1$, in Theorem 2.1, inequality (2.1) then reduces to the following inequality

$$|f(\rho z)| \geq \left\{ \left[\left(\frac{k+\rho}{1+k}\right)^n + \frac{1}{k^{n-1}} \left(\frac{|a_0| - k^n|a_n|}{|a_0| + |a_n|}\right) \left(\frac{1-\rho}{1+k}\right)^n \right] \left(\frac{|\alpha|-1}{|\alpha|+\rho}\right)^n \left|\frac{\rho z - \alpha}{z-\alpha}\right|^n \right\} |f(z)|. \tag{2.3}$$

Letting $|\alpha| \to +\infty$ in (2.3), we get refinement of (1.5).

Theorem 2.2. *If $r \in \mathbb{R}_n$ having no zeros of r lie in $|z| < k$, $k \leq 1$, then for every $0 \leq \rho \leq k^2$ and $|z| = 1$,*

$$|r(\rho z)| \geq \left\{ \left[\left(\frac{k+\rho}{1+k}\right)^n + \left(\frac{|a_0| - k^n|a_n|}{|a_0| + |a_n|}\right) \left(\frac{\lambda}{1+k}\right)^n \right] \prod_{j=1}^{n} \frac{|a_j|-1}{|a_j|+\rho} \right\} |r(z)|, \tag{2.4}$$

where $\lambda = \min\{1-\rho, k+\rho\}$.

Remark 2.4. As in remark 2.1, inequality (2.4) refines (1.8).

Remark 2.5. Taking $W(z) = (z-\alpha)^n$, $|\alpha| > 1$, in Theorem 2.2, inequality (2.4) then reduces to the following inequality

$$|f(\rho z)| \geq \left\{ \left[\left(\frac{k+\rho}{1+k}\right)^n + \frac{1}{k^{n-1}} \left(\frac{|a_0| - k^n|a_n|}{|a_0| + |a_n|}\right) \left(\frac{\lambda}{1+k}\right)^n \right] \left(\frac{|\alpha|-1}{|\alpha|+\rho}\right)^n \left|\frac{\rho z - \alpha}{z-\alpha}\right|^n \right\} |f(z)|. \tag{2.5}$$

Letting $|\alpha| \to +\infty$ in (2.5), we get refinement of (1.6).

3. Lemmas

For the proofs of our results, we shall make use of the following lemmas.

Lemma 3.1. *For any $0 \leq \rho \leq 1$ and $\rho_j \geq k \geq 1$, $1 \leq j \leq n$, we have*

$$\prod_{j=1}^{n} \left(\frac{\rho+\rho_j}{1+\rho_j}\right) \geq \left(\frac{k+\rho}{k+1}\right)^n + \frac{1}{k^{n-1}} \left[\frac{\rho_1\rho_2\cdots\rho_n - k^n}{\rho_1\rho_2\cdots\rho_n + 1}\right] \left(\frac{1-\rho}{1+k}\right)^n.$$

Lemma 3.2. *For any $0 \leq \rho \leq 1$ and $\rho_j \geq k \geq 1$, $1 \leq j \leq n$, we have*

$$\prod_{j=1}^{n} \left(\frac{\rho+\rho_j}{1+\rho_j}\right) \geq \left(\frac{k+\rho}{k+1}\right)^n + \frac{1}{k^{n-1}} \left[\frac{\rho_1\rho_2\cdots\rho_n - k^n}{\rho_1\rho_2\cdots\rho_n + 1}\right] \left(\frac{\lambda}{1+k}\right)^n,$$

where $\lambda = \min\{1-\rho, k+\rho\}$.

The above two lemmas are due to Kumar and Milovanović [6].

4. Proof of the Theorems

Proof of Theorem 2.1. By hypothesis, $r(z) = \frac{P(z)}{W(z)} \in \mathbb{R}_n$ has all its zeros in $|z| \leq k,\ k \geq 1$. It follows that $P(z)$ has all its zeros in $|z| \leq k,\ k \geq 1$, therefore if $z_j = \rho_j e^{i\theta_j}$, $0 \leq \theta < 2\pi$, $1 \leq j \leq n$, are zeros of $P(z)$, then we write $P(z) = c\prod_{j=1}^{n}(z - \rho_j e^{i\theta_j})$, where $\rho_j \geq k \geq 1,\ j = 1, 2, \ldots, n$. Hence, for $\rho < 1$ and $0 \leq \theta < 2\pi$, we have

$$\left|\frac{r(\rho e^{i\theta})}{r(e^{i\theta})}\right| = \left|\frac{P(\rho e^{i\theta})}{P(e^{i\theta})}\right| \cdot \left|\frac{W(e^{i\theta})}{W(\rho e^{i\theta})}\right| = \prod_{j=1}^{n}\left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| \cdot \prod_{j=1}^{n}\left|\frac{e^{i\theta} - a_j}{\rho e^{i\theta} - a_j}\right|. \tag{4.1}$$

Now,

$$\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| = \prod_{j=1}^{n}\left|\frac{\rho e^{i(\theta-\theta_j)} - \rho_j}{e^{i(\theta-\theta_j)} - \rho_j}\right| = \prod_{j=1}^{n}\left(\frac{\rho^2 + \rho_j^2 - 2\rho\rho_j\cos(\theta - \theta_j)}{1 + \rho_j^2 - 2\rho_j\cos(\theta - \theta_j)}\right)^{\frac{1}{2}}$$

$$\geq \prod_{j=1}^{n}\frac{\rho + \rho_j}{1 + \rho_j} \quad \text{as } \rho < 1. \tag{4.2}$$

Using Lemma 3.1 to the right hand side of (4.2), we get

$$\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| \geq \left(\frac{k + \rho}{1 + k}\right)^n + \frac{1}{k^{n-1}}\left(\frac{\rho_1\rho_2\cdots\rho_n - k^n}{\rho_1\rho_2\cdots\rho_n + 1}\right)\left(\frac{1 - \rho}{1 + k}\right)^n.$$

Using the fact that $\rho_1\rho_2\cdots\rho_n = \frac{|a_0|}{|a_n|}$,

$$\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| \geq \left(\frac{k + \rho}{1 + k}\right)^n + \frac{1}{k^{n-1}}\left(\frac{|a_0| - k^n|a_n|}{|a_0| + |a_n|}\right)\left(\frac{1 - \rho}{1 + k}\right)^n, \tag{4.3}$$

also for $|a_j| > 1,\ j = 1, 2, \ldots, n$, we have

$$\prod_{j=1}^{n}\left|\frac{e^{i\theta} - a_j}{\rho e^{i\theta} - a_j}\right| \geq \prod_{j=1}^{n}\frac{|a_j| - 1}{|a_j| + \rho}. \tag{4.4}$$

Using inequalities (4.3) and (4.4) in (4.1), we get

$$\left|\frac{r(\rho e^{i\theta})}{r(e^{i\theta})}\right| \geq \left[\left(\frac{k + \rho}{1 + k}\right)^n + \frac{1}{k^{n-1}}\left(\frac{|a_0| - k^n|a_n|}{|a_0| + |a_n|}\right)\left(\frac{1 - \rho}{1 + k}\right)^n\right]\prod_{j=1}^{n}\frac{|a_j| - 1}{|a_j| + \rho}.$$

Thus,

$$|r(\rho z)| \geq \left\{\left[\left(\frac{k + \rho}{1 + k}\right)^n + \frac{1}{k^{n-1}}\left(\frac{|a_0| - k^n|a_n|}{|a_0| + |a_n|}\right)\left(\frac{1 - \rho}{1 + k}\right)^n\right]\prod_{j=1}^{n}\frac{|a_j| - 1}{|a_j| + \rho}\right\}|r(z)|.$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. By hypothesis, $r(z) = \frac{P(z)}{W(z)} \in \mathbb{R}_n$ has all its zeros in $|z| \leq k,\ k \geq 1$. It follows that $P(z)$ has all its zeros in $|z| \leq k,\ k \geq 1$, therefore if $z_j = \rho_j e^{i\theta_j}, 0 \leq \theta < 2\pi,\ 1 \leq j \leq n$, are zeros of $P(z)$, then we write $P(z) =$

$c\prod_{j=1}^{n}(z-\rho_j e^{i\theta_j})$, where $\rho_j \geq k \geq 1,\ j=1,2,\dots,n$. Hence, for $\rho<1$ and $0\leq\theta<2\pi$, we have

$$\left|\frac{r(\rho e^{i\theta})}{r(e^{i\theta})}\right| = \left|\frac{P(\rho e^{i\theta})}{P(e^{i\theta})}\right|\cdot\left|\frac{W(e^{i\theta})}{W(\rho e^{i\theta})}\right| = \prod_{j=1}^{n}\left|\frac{\rho e^{i\theta}-\rho_j e^{i\theta_j}}{e^{i\theta}-\rho_j e^{i\theta_j}}\right|\cdot\prod_{j=1}^{n}\left|\frac{e^{i\theta}-a_j}{\rho e^{i\theta}-a_j}\right|. \tag{4.5}$$

Now,

$$\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta}-\rho_j e^{i\theta_j}}{e^{i\theta}-\rho_j e^{i\theta_j}}\right| = \prod_{j=1}^{n}\left|\frac{\rho e^{i(\theta-\theta_j)}-\rho_j}{e^{i(\theta-\theta_j)}-\rho_j}\right| = \prod_{j=1}^{n}\left(\frac{\rho^2+\rho_j^2-2\rho\rho_j\cos(\theta-\theta_j)}{1+\rho_j^2-2\rho_j\cos(\theta-\theta_j)}\right)^{\frac{1}{2}}$$

$$\geq \prod_{j=1}^{n}\frac{\rho+\rho_j}{1+\rho_j} \quad \text{as } 0\leq\rho\leq k^2. \tag{4.6}$$

Using Lemma 3.2 to the right hand side of (4.6), we get

$$\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta}-\rho_j e^{i\theta_j}}{e^{i\theta}-\rho_j e^{i\theta_j}}\right| \geq \left(\frac{k+\rho}{1+k}\right)^n + \frac{1}{k^{n-1}}\left(\frac{\rho_1\rho_2\cdots\rho_n-k^n}{\rho_1\rho_2\cdots\rho_n+1}\right)\left(\frac{1-\rho}{1+k}\right)^n.$$

Using the fact that $\rho_1\rho_2\cdots\rho_n=\frac{|a_0|}{|a_n|}$,

$$\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta}-\rho_j e^{i\theta_j}}{e^{i\theta}-\rho_j e^{i\theta_j}}\right| \geq \left(\frac{k+\rho}{1+k}\right)^n + \left(\frac{|a_0|-k^n|a_n|}{|a_0|+|a_n|}\right)\left(\frac{\lambda}{1+k}\right)^n, \tag{4.7}$$

also for $|a_j|>1,\ j=1,2,\dots,n$, we have

$$\prod_{j=1}^{n}\left|\frac{e^{i\theta}-a_j}{\rho e^{i\theta}-a_j}\right| \geq \prod_{j=1}^{n}\frac{|a_j|-1}{|a_j|+\rho}. \tag{4.8}$$

Using inequalities (4.7) and (4.8) in (4.5), we get

$$\left|\frac{r(\rho e^{i\theta})}{r(e^{i\theta})}\right| \geq \left[\left(\frac{k+\rho}{1+k}\right)^n + \left(\frac{|a_0|-k^n|a_n|}{|a_0|+|a_n|}\right)\left(\frac{\lambda}{1+k}\right)^n\right]\prod_{j=1}^{n}\frac{|a_j|-1}{|a_j|+\rho}.$$

Thus,

$$|r(\rho z)| \geq \left\{\left[\left(\frac{k+\rho}{1+k}\right)^n + \left(\frac{|a_0|-k^n|a_n|}{|a_0|+|a_n|}\right)\left(\frac{\lambda}{1+k}\right)^n\right]\prod_{j=1}^{n}\frac{|a_j|-1}{|a_j|+\rho}\right\}|r(z)|.$$

This completes the proof of Theorem 2.2. □

5. Conclusions

A sequence of publications (see, for example [5, 9] and [10]) on inequalities for rational functions has been published in recent years, and significant progress has been made. Both mathematics and practical fields are interested in inequalities of these types. In this work, we continue our investigation of inequalities of this nature by taking into consideration the location of all the zeros and extremal coefficients of the underlying polynomial.

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References

[1] N. C. Ankeny and T. J. Rivlin, *On a theorem of S. Bernstein,* Pacific J. Math. **5** (1955), 849–852.

[2] A. Aziz, *Growth of polynomials whose zeros are within or outside a circle,* Bull. Austral. Math. Soc. **35** (1987), 247–256.

[3] S. Bernstein, *Sur l'ordre de la meilleure approximation des functions continues par des polynomes de degré donné*, Mem. Acad. R. Belg. **4** (1912), 1–103.

[4] N. K. Govil, M. A. Qazi and Q. I. Rahman, *Inequalities describing the growth of polynomials not vanishing in a disk of prescribed radius*, Math. Inequal. Appl. **6** (2003), 453–467.

[5] A. Hussain and A. Wani, *Turán-type inequalities for certain class of meromorphic functions*, Int. J. Non linear. Anal. **16** (2025), 17–26. https://doi.org/10.22075/ijnaa.2023.29759.4249

[6] P. Kumar and G. V. Milovanović, *On sharpening and generalization of Rivlin's inequality,* Turk. J. Math. **46** (2022), 1436–1445. https://doi.org/10.3906/mat-2203-82

[7] M. Marden, *Geometry of Polynomials,* 2nd Ed. Math. Surveys, No. 3, Amer. Math. Soc., Providence, RI, 1966.

[8] G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias., *Topics in Polynomials, Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore, 1994.

[9] A. Mir and S. Hans, *Inequalities concerning rational functions in the complex domain,* Sib. Math. J. **63** (2022), 1012–1022. https://doi.org/10.1134/s0037446622050196

[10] A. Mir and T. Fayaz, *Inequalities for a rational function with prescribed poles,* Sib. Math. J. **65** (2024), 899–908. https://doi.org/10.1134/S0037446624040153

[11] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials,* Oxford University Press, 2002.

[12] N. A. Rather, M. Shafi and I. Dar, *Growth estimate for rational functions with prescribed poles and restricted zeros,* Kragujevac J. Math. **49** (2025), 305–311. https://doi.org/10.46793/KgJMat2502.305R

[13] T. J. Rivlin, *On the maximum modulus of polynomials*, Amer. Math. Monthly **67** (1960), 251–253.

[14] R. S. Varga, *A comparision of the successive overrelaxation method and semi-iterative methods using Chebyshev polynomials*, J. Soc. Indust. Appl. Math. **5** (1957), 39–46.

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