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ON THE DISCRETE HARDY INEQUALITY WITH VARIABLE EXPONENT

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Abstract. We obtain the celebrated Hardy's inequality in the context of variable exponent sequence spaces.

1. Introduction

The Hardy inequality is a classical result in analysis and the mathematical inequalities, named after mathematician Godfrey Harold Hardy. It has a long and very rich history, starting in 1915 when G.H. Hardy [1] needed an estimate for arithmetic means, more precisely, an inequality of the form

$$\sum_{n=1}^{+\infty}\left|\frac{1}{n}A_n\right|^2 \leq C\sum_{n=1}^{+\infty}|a_n|^2,$$

with $A_n=\sum_{k=1}^{n} a_k$ and $\{a_k\}_{k\in\mathbb{N}}$ a sequence. Later, this inequality was extended to

$$\sum_{n=1}^{+\infty}\left|\frac{1}{n}A_n\right|^p \leq C_p\sum_{n=1}^{+\infty}|a_n|^p, \quad 1<p<+\infty.$$

In his famous 1925 paper [2], Hardy proved the discrete and the integral version of what is known today as Hardy's inequalities. The discrete version of Hardy's inequality asserts that

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{+\infty} a_n^p, \tag{1.1}$$

where $p>1$ and $\{a_k\}_{k=1}^{+\infty}$ is a sequence of nonnegative real numbers.

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Since then, this type of inequality has been intensively studied. There are a lot of research articles on the topic, as well as surveys and books. For example, the survey [4] contains several proofs and historical aspects are given. In [3], authors proved improved classical Hardy inequality for sequences of non-negative real numbers. In [6], Lefévre obtained a short direct proof of the discrete Hardy inequality provided that constant p' is optimal. For more on Hardy's inequality, we refer the reader to [5] and [7].

In this note, we define the (variable) Lebesgue sequence spaces, equipped with Luxemburg norm. We will prove Hölder's inequality in this setting. Finally, our main goal is to obtain the Hardy inequality (1.1) for these spaces.

2. $\ell_{p(\cdot)}$ spaces and Hölder's Inequality

We give here some definitions and results that will be useful in the coming section.

Let H be a measurable subset of $\mathbb{R}^n$ and let $p : H \to [1, +\infty)$ be a measurable function. We suppose that

$$1 \leq p_-(H) \leq p(x) \leq p_+(H) < +\infty, \tag{2.1}$$

where $p_- := \operatorname{ess\,inf}_{x\in H} p(x)$, $p_+ := \operatorname{ess\,sup}_{x\in H} p(x)$.

Let $p(\cdot) > 1$. The conjugate exponent of $p(\cdot)$ is denoted by $p'(\cdot)$ and is defined as

$$p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}.$$

Definition 2.1. $\ell_{p(\cdot)}$ denotes the variable exponent Lebesgue sequence space (sometimes denoted by $\ell_{p(\cdot)}(\mathbb{N})$), which is the set of all sequences $a = \{a_n\}_{n\in\mathbb{N}}$ such that

$$I_{p(\cdot)}(a) = \sum_{n=1}^{+\infty} |a_n|^{p(x)} < +\infty.$$

We equip $\ell_{p(\cdot)}$ with the Luxemburg norm

$$\|a\|_{p(\cdot)} = \inf\left\{\lambda : I_{p(\cdot)}\left(\frac{a}{\lambda}\right) \leq 1\right\},$$

where we use the convention that $\inf \emptyset = +\infty$.

Note that

$$I_{p(\cdot)}\left(\frac{a}{\|a\|_{p(\cdot)}}\right) \leq 1,$$

where $\|a\|_{p(\cdot)} \neq 0$.

We state and prove Hölder's inequality for $\ell_{p(\cdot)}$.

Theorem 2.1. *Let $a \in I_{p(\cdot)}$ and $b \in I_{q(\cdot)}$ with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. Then,*

$$\sum_{n=1}^{+\infty} |a_n b_n| \leq k_{p(\cdot)} \|a\|_{p(\cdot)} \|b\|_{q(\cdot)}.$$

Proof. If $\|a\|_{p(\cdot)} = 0$ and $\|b\|_{q(\cdot)} = 0$, then $ab = 0$, so there is nothing to prove. Therefore we may assume that $\|a\|_{p(\cdot)} > 0$ and $\|b\|_{q(\cdot)} > 0$, moreover by homogeneity we may assume that $\|a\|_{p(\cdot)} = \|b\|_{q(\cdot)} = 1$.

By Young's inequality for variable exponent we have

$$\begin{aligned}\sum_{n=1}^{+\infty} |a_n b_n| \leq \sum_{n=1}^{+\infty} \frac{1}{p(x)}|a_n|^{p(x)} + \sum_{n=1}^{+\infty} \frac{1}{q(x)}|a_n|^{q(x)} \\ \leq &\frac{1}{p_-} I_{p(\cdot)}(a) + \frac{1}{q_-} I_{q(\cdot)}(b) \\ \leq &\frac{1}{p_-} I_{p(\cdot)}\left(\frac{a}{\|a\|_{p(\cdot)}}\right) + \frac{1}{q_-} I_{q(\cdot)}\left(\frac{b}{\|b\|_{q(\cdot)}}\right) \\ \leq &\frac{1}{p_-} + \frac{1}{q_-} \leq \frac{1}{p_-} + 1 - \frac{1}{p_+}.\end{aligned}$$

Finally

$$\sum_{n=1}^{+\infty} |a_n b_n| \leq \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \|a\|_{p(\cdot)} \|b\|_{q(\cdot)} = k_{p(\cdot)} \|a\|_{p(\cdot)} \|b\|_{q(\cdot)},$$

where $k_{p(\cdot)} = 1 + \frac{1}{p_-} - \frac{1}{p_+}$. So, the proof is now complete. □

Remark 2.1. For the sake of completeness and the convenience of the reader, we would like to present another proof of Theorem 2.1. In order to do so, let us first check that $q_- = q_+$. Since $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, then $\frac{1}{p(x)} = \frac{q(x)-1}{q(x)}$ so $q(x) = p(x)(q(x) - 1)$. Now,

$$\begin{aligned}q_- &= \operatorname{ess\,inf} q(x) = \operatorname{ess\,inf} (p(x))\,(q(x) - 1) = \operatorname{ess\,inf} (-p(x))\,(1 - q(x)) \\ &= -\operatorname{ess\,sup} (p(x))\,(1 - q(x)) = -\operatorname{ess\,sup} (-p(x))\,(q(x) - 1) = -\operatorname{ess\,sup} (-q(x)) \\ &= -\operatorname{ess\,sup} (-q_+) \\ &= q_+.\end{aligned}$$

Thus, $\frac{1}{p_-} + \frac{1}{q_-} = \frac{1}{p_-} + \frac{1}{q_+} = \frac{1}{p_-} + 1 - \frac{1}{p_+}$.

Alternative proof of Theorem 2.1. Young's inequality can also be viewed as

$$|a_n b_n| \leq \frac{1}{p(x)}|a_n|^{p(x)} + \frac{1}{q(x)}|a_n|^{q(x)} \leq \frac{1}{p_-}|a_n|^{p_+} + \frac{1}{q_-}|a_n|^{q_+}.$$

Then,

$$\begin{aligned}\sum_{n=1}^{+\infty} \left|\frac{a_n b_n}{\|a\|_{p(\cdot)}\|b\|_{q(\cdot)}}\right| &\leq \frac{1}{p_-} \sum_{n=1}^{+\infty} \left|\frac{a_n}{\|a\|_{p(\cdot)}}\right|^{p_+} + \frac{1}{q_-} \sum_{n=1}^{+\infty} \left|\frac{b_n}{\|b\|_{q(\cdot)}}\right|^{q_+} \\ &\leq \frac{1}{p_-} \frac{\sum_{n=1}^{+\infty} |a_n|^{p_+}}{\|a\|_{p(\cdot)}^{p_+}} + \frac{1}{q_-} \frac{\sum_{n=1}^{+\infty} |b_n|^{q_+}}{\|b\|_{q(\cdot)}^{q_+}} \\ &= \frac{1}{p_-} + \frac{1}{q_-} = \frac{1}{p_-} + 1 - \frac{1}{p_+} = 1 + \frac{1}{p_-} - \frac{1}{p_+}.\end{aligned}$$

So,

$$\sum_{n=1}^{+\infty} |a_n b_n| \leq \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \|a\|_{p(\cdot)} \|b\|_{q(\cdot)} = k_{p(\cdot)} \|a\|_{p(\cdot)} \|b\|_{q(\cdot)},$$

where $k_{p(\cdot)} = 1 + \frac{1}{p_-} - \frac{1}{p_+}$. □

3. Hardy's Inequality on $\ell_{p(\cdot)}$ Spaces

We state and prove our main result, which is the $\ell_{p(\cdot)}$ version of (1.1).

Theorem 3.1. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that the series $\sum_{n=1}^{+\infty} a_n^{p_+}$ converges, then*

$$\sum_{n=1}^{+\infty} \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{p_+} \leq C(p_-, p_+) \sum_{n=1}^{+\infty} a_n^{p_+},$$

where $C(p_-, p_+) = \left[\frac{p_-}{p_- - 1}\left(\frac{1}{p_-} - \frac{1}{p_+} + 1\right)\right]^{p_+}$.

Proof. Let $\alpha_n = \frac{A_n}{n}$ where $A_n = a_1 + a_2 + \cdots + a_n$. Then, $A_n = n\alpha_n$, so $a_1 + a_2 + \cdots + a_n = n\alpha_n$, from which we get that

$$a_n = n\alpha_n - (n-1)\alpha_{n-1}.$$

Now, let us consider

$$\begin{aligned}\alpha_n^{p_+} - q_-\alpha_n^{p_+-1}a_n =& \alpha_n^{p_+} - q_-[n\alpha_n - (n-1)\alpha_{n-1}]\alpha_n^{p_+-1} \\ =& \alpha_n^{p_+} - q_- n\alpha_n^{p_+} + q_-(n-1)\alpha_{n-1}\alpha_n^{p_+-1}.\end{aligned}$$

By Young's inequality

$$\begin{aligned}q_-(n-1)\alpha_{n-1}\alpha_n^{p_+-1} \leq& q_-(n-1)\left[\frac{\alpha_{n-1}^{p_+}}{p_-} + \frac{\alpha_n^{q_+(p_+-1)}}{q_-}\right] \\ =& q_-(n-1)\left[\frac{\alpha_{n-1}^{p_+}}{p_-}\right] + (n-1)\alpha_n^{p_+} \\ =& \frac{(n-1)\alpha_{n-1}^{p_+}}{p_- - 1} + (n-1)\alpha_n^{p_+}.\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha_n^{p_+} - q_-\alpha_n^{p_+-1}a_n \leq& \alpha_n^{p_+} - q_- n\alpha_n^{p_+} + \frac{(n-1)}{p_- - 1}\alpha_{n-1}^{p_+} + (n-1)\alpha_n^{p_+} \\ =& \alpha_n^{p_+} - \frac{p_-}{p_- - 1} n\alpha_n^{p_+} + \frac{(n-1)}{p_- - 1}\alpha_{n-1}^{p_+} + (n-1)\alpha_n^{p_+} \\ =& \frac{1}{p_- - 1}\left[(n-1)\alpha_{n-1}^{p_+} - n\alpha_n^{p_+}\right].\end{aligned}$$

From which

$$\begin{aligned}\sum_{n=1}^{N}\alpha_n^{p_+}-\frac{p_-}{p_--1}\sum_{n=1}^{N}\alpha_n^{p_+-1}a_n \leq&\frac{1}{p_--1}\sum_{n=1}^{N}\Big[(n-1)\alpha_{n-1}^{p_+}-n\alpha_n^{p_+}\Big]\\ =&\frac{1}{p_--1}\Big[-\alpha_1^{p_+}+\alpha_1^{p_+}-2\alpha_2^{p_+}+\cdots-N\alpha_N^{p_+}\Big]\\ =&-\frac{N\alpha_N^{p_+}}{p_--1}\\ \leq&0.\end{aligned}$$

Then,

$$\sum_{n=1}^{N}\alpha_n^{p_+}\leq\frac{p_-}{p_--1}\sum_{n=1}^{N}\alpha_n^{p_+-1}a_n.$$

By Hölder's inequality we have

$$\begin{aligned}\sum_{n=1}^{+\infty}\alpha_n^{p_+}\leq&\frac{p_-}{p_--1}\left(\frac{1}{p_-}-\frac{1}{p_+}+1\right)\left(\sum_{n=1}^{+\infty}a_n^{p_+}\right)^{\frac{1}{p_+}}\left(\sum_{n=1}^{+\infty}\alpha_n^{q_+(p_+-1)}\right)^{\frac{1}{q_+}}\\ =&\frac{p_-}{p_--1}\left(\frac{1}{p_-}-\frac{1}{p_+}+1\right)\left(\sum_{n=1}^{+\infty}a_n^{p_+}\right)^{\frac{1}{p_+}}\left(\sum_{n=1}^{+\infty}\alpha_n^{p_+}\right)^{\frac{1}{q_+}}.\end{aligned}$$

Then,

$$\left(\sum_{n=1}^{+\infty}\alpha_n^{p_+}\right)^{1-\frac{1}{q_+}}\leq\frac{p_-}{p_--1}\left(\frac{1}{p_-}-\frac{1}{p_+}+1\right)\left(\sum_{n=1}^{+\infty}a_n^{p_+}\right)^{\frac{1}{p_+}}.$$

This implies

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{p_+}\leq C(p_-,p_+)\sum_{n=1}^{+\infty}a_n^{p_+},$$

where $C(p_-,p_+)=\left[\frac{p_-}{p_--1}\left(\frac{1}{p_-}-\frac{1}{p_+}+1\right)\right]^{p_+}$, which completes the proof. □

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