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APOSTOL TYPE (p,q)-FROBENIUS-EULER POLYNOMIALS AND NUMBERS

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Abstract. In the present paper, we introduce (p,q)-extension of Apostol type Frobenius-Euler polynomials and numbers and investigate some basic identities and properties for these polynomials and numbers, including addition theorems, difference equations, derivative properties, recurrence relations and so on. Then, we provide integral representations, explicit formulas and relations for these polynomials and numbers. Moreover, we discover (p,q)-extensions of Carlitz's result [L. Carlitz, Mat. Mag. **32** (1959), 247-260] and Srivastava and Pintér addition theorems in [H. M. Srivastava, A. Pinter, Appl. Math. Lett. **17** (2004), 375-380].

1. Introduction

Throughout this paper, we use the standart notions: $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{N}$ denotes the set of the natural numbers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The (p,q)-numbers are defined as

$$[n]_{p,q} := p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \frac{p^n - q^n}{p-q}.$$

We can write easily that $[n]_{p,q} = p^{n-1}[n]_{q/p}$, where $[n]_{q/p}$ is the q-number in q-calculus given by $[n]_{q/p} = \frac{(q/p)^n - 1}{(q/p) - 1}$. Thereby, this implies that (p,q)-numbers and q-numbers are different, that is, we can not obtain (p,q)-numbers just by substituting q by q/p in the definition of q-numbers. In the case of $p = 1$, (p,q)-numbers reduce to q-numbers, see [9, 10].

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The (p,q)-derivative of a function f with respect to x is defined by

$$D_{p,q;x}f(x) := D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x} \qquad (x \neq 0) \tag{1.1}$$

and $(D_{p,q}f)(0) = f'(0)$, provided that f is differentiable at 0. The linear (p,q)-derivative operator holds the following properties

$$D_{p,q}(f(x)g(x)) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x) \tag{1.2}$$

and

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}. \tag{1.3}$$

The (p,q)-analogue of $(x+a)^n$ is given by

$$\begin{aligned}(x+a)^n_{p,q} &= \begin{cases} (x+a)(px+aq)\cdots(p^{n-2}x+aq^{n-2})(p^{n-1}x+aq^{n-1}), & \text{if } n \geq 1 \\ 1, & \text{if } n = 0\end{cases} \\ &= \sum_{k=0}^{n}\binom{n}{k}_{p,q} p^{\binom{k}{2}}q^{\binom{n-k}{2}}x^k a^{n-k} \quad ((p,q)\text{-Gauss Binomial Formula}),\end{aligned}$$

where the (p,q)-Gauss Binomial coefficients $\binom{n}{k}_{p,q}$ and (p,q)-factorial $[n]_{p,q}!$ are defined by

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!} \qquad (n \geq k) \text{ and } [n]_{p,q}! = [n]_{p,q}\cdots[2]_{p,q}[1]_{p,q} \qquad (n \in \mathbb{N}).$$

The (p,q)-exponential functions,

$$e_{p,q}(x) = \sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}x^n}{[n]_{p,q}!} \text{ and } E_{p,q}(x) = \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}x^n}{[n]_{p,q}!}$$

hold the identities

$$e_{p,q}(x)E_{p,q}(-x) = 1 \text{ and } e_{p^{-1},q^{-1}}(x) = E_{p,q}(x), \tag{1.4}$$

and have the (p,q)-derivatives

$$D_{p,q}e_{p,q}(x) = e_{p,q}(px) \text{ and } D_{p,q}E_{p,q}(x) = E_{p,q}(qx). \tag{1.5}$$

The definite (p,q)-integral is defined by

$$\int_0^a f(x)\,d_{p,q}x = (p-q)\,a\sum_{k=0}^{\infty}\frac{p^k}{q^{k+1}}f\left(\frac{p^k}{q^{k+1}}a\right),$$

in conjunction with

$$\int_a^b f(x)\,d_{p,q}x = \int_0^b f(x)\,d_{p,q}x - \int_0^a f(x)\,d_{p,q}x \quad \text{(see [22])}. \tag{1.6}$$

A more detailed statement of above is found in [2, 7, 9, 10, 20, 22].

The classical Bernoulli polynomials and numbers, $B_n(x)$ and B_n, classical Euler polynomials and numbers, $E_n(x)$ and E_n, and classical Genocchi polynomials and numbers, $G_n(x)$ and G_n, are defined by the following generating functions

$$\sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{z}{e^z - 1} e^{xz} \text{ and } \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1} \quad (|z| < 2\pi),$$
$$\sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \frac{2}{e^z + 1} e^{xz} \text{ and } \sum_{n=0}^{\infty} E_n \frac{z^n}{n!} = \frac{2}{e^z + 1} \quad (|z| < \pi),$$
$$\sum_{n=0}^{\infty} G_n(x) \frac{z^n}{n!} = \frac{2z}{e^z + 1} e^{xz} \text{ and } \sum_{n=0}^{\infty} G_n \frac{z^n}{n!} = \frac{2z}{e^z + 1} \quad (|z| < \pi),$$

for detailed information about these numbers and polynomials, see [5, 6, 25, 26].

Apostol-type polynomials and numbers were firstly introduced by Apostol [1] and also Srivastava [26]. Then, Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials and numbers have been studied and investigated by many mathematicians, for instance, Apostol, Araci, Acikgoz, He, Jia, Mahmudov, Luo, Ozarslan, Srivastava, Wang, Wang and Yasar, in [1, 3, 4, 11, 15–18, 26–28]. q-generalizations of mentioned Apostol type polynomials and numbers have been considered and discussed by Kurt, Mahmudov, Keleshteri, Simsek, Srivastava in [14, 19, 23, 24]. Recently, unification of Apostol type polynomials and numbers have been worked by El-Desouky, Gomaa, Kurt, Ozarslan in [8, 13, 21].

Duran et al. [7] defined Apostol type (p, q)-Bernoulli polynomials $B_n^{(\alpha)}(x, y; \lambda : p, q)$ of order α, the Apostol type (p, q)-Euler polynomials $E_n^{(\alpha)}(x, y; \lambda : p, q)$ of order α and the Apostol type (p, q)-Genocchi polynomials $G_n^{(\alpha)}(x, y; \lambda : p, q)$ of order α by the following generating functions:

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x, y; \lambda : p, q) \frac{z^n}{[n]_{p,q}!} = \left(\frac{z}{\lambda e_{p,q}(z) - 1} \right)^{\alpha} e_{p,q}(xz) E_{p,q}(yz)$$
$$(|z + \log \lambda| < 2\pi, 1^{\alpha} = 1),$$
$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x, y; \lambda : p, q) \frac{z^n}{[n]_{p,q}!} = \left(\frac{2}{\lambda e_{p,q}(z) + 1} \right)^{\alpha} e_{p,q}(xz) E_{p,q}(yz)$$
$$(|z + \log \lambda| < \pi, 1^{\alpha} = 1),$$
$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x, y; \lambda : p, q) \frac{z^n}{[n]_{p,q}!} = \left(\frac{2z}{\lambda e_{p,q}(z) + 1} \right)^{\alpha} e_{p,q}(xz) E_{p,q}(yz)$$
$$(|z + \log \lambda| < \pi, 1^{\alpha} = 1),$$

where λ and α are suitable (real or complex) parameters and $p, q, \in \mathbb{C}$ with $0 < |q| < |p| \leq 1$.

Letting $x = 0$ and $y = 0$ above, we then have $B_n^{(\alpha)}(0, 0; \lambda : p, q) := B_n^{(\alpha)}(\lambda : p, q)$, $E_n^{(\alpha)}(0, 0; \lambda : p, q) := E_n^{(\alpha)}(\lambda : p, q)$ and $G_n^{(\alpha)}(0, 0; \lambda : p, q) := G_n^{(\alpha)}(\lambda : p, q)$ called, n-th Apostol type (p, q)-Bernoulli number of order α, n-th Apostol type (p, q)-Euler number of order α and n-th Apostol type (p, q)-Genocchi number of order α, respectively. In the case $\alpha = 1$, we have

$$\mathcal{B}_n^{(1)}(x, y; \lambda : p, q) := \mathcal{B}_n(x, y; \lambda : p, q), \qquad \mathcal{E}_n^{(1)}(x, y; \lambda : p, q) := \mathcal{E}_n(x, y; \lambda : p, q)$$

and

$$\mathcal{G}_n^{(1)}(x, y; \lambda : p, q) := \mathcal{G}_n(x, y; \lambda : p, q)$$

named as n-th Apostol type (p, q)-Bernoulli polynomial, n-th Apostol type (p, q)-Euler polynomial and n-th Apostol type (p, q)-Genocchi polynomial.

Classical Frobenius-Euler polynomials $H_n^{(\alpha)}(x; u)$ of order α are defined by the following Taylor series expansion at $z = 0$:

$$\sum_{n=0}^{\infty} \mathcal{H}_n^{(\alpha)}(x; u) \frac{z^n}{n!} = \left(\frac{1-u}{e^z - u}\right)^{\alpha} e^{zt},$$

where α is suitable (real or complex) parameter and u is an algebraic number, see [4, 5, 12, 28].

Apostol type Frobenius-Euler polynomials $H_n^{(\alpha)}(x; u; \lambda)$ of order α are defined as follows:

$$\sum_{n=0}^{\infty} \mathcal{H}_n^{(\alpha)}(x; u; \lambda) \frac{z^n}{n!} = \left(\frac{1-u}{\lambda e^z - u}\right)^{\alpha} e^{zt},$$

where λ and α are suitable (real or complex) parameters and u is an algebraic number, see [3, 23].

The following definition is new and plays an important role in deriving the main results of this paper. Now we are ready to state the following Definition 1.1.

Definition 1.1. Apostol type (p, q)-Frobenius-Euler polynomials $H_n^{(\alpha)}(x, y; u; \lambda : p, q)$ of order α are defined by means of the following Taylor series expansion about $z = 0$:

$$\sum_{n=0}^{\infty} \mathcal{H}_n^{(\alpha)}(x, y; u; \lambda : p, q) \frac{z^n}{[n]_{p,q}!} = \left(\frac{1-u}{\lambda e_{p,q}(z) - u}\right)^{\alpha} e_{p,q}(xz) E_{p,q}(yz),$$

where λ and α are suitable (real or complex) parameters, $p, q, \in \mathbb{C}$ with $0 < |q| < |p| \leq 1$ and u is an algebraic number.

Upon setting $x = 0$ and $y = 0$ in Definition 1.1, we have $H_n^{(\alpha)}(0, 0; u; \lambda : p, q) := H_n^{(\alpha)}(u; \lambda : p, q)$ called n-th Apostol type (p, q)-Frobenius-Euler number of order α. We remark that

$$\begin{aligned}
\mathcal{H}_n^{(1)}(x, y; u; \lambda : p, q) &:= \mathcal{H}_n(x, y; u; \lambda : p, q), \\
\mathcal{H}_n^{(\alpha)}(x, y; u; \lambda : p, q)\Big|_{p=1} &:= \mathcal{H}_{n,q}^{(\alpha)}(x, y; u; \lambda) \qquad \text{(see [14] and [23])}, \\
\lim_{\substack{q \to 1^- \\ p=1}} \mathcal{H}_n^{(\alpha)}(x, y; u; \lambda : p, q) &:= \mathcal{H}_n^{(\alpha)}(x + y; u; \lambda) \qquad \text{(see [3])}.
\end{aligned}$$

This paper is organized as follows. The second section provides some basic identities and properties for Apostol type (p, q)-Frobenius-Euler polynomials and numbers of order α, including addition theorems, difference equations, derivative properties, recurrence relations and so on. The third section not only includes integral representations, explicit formulas and relations for $H_n^{(\alpha)}(x,y;u;\lambda:p,q)$, but also provides (p,q)-extensions of Carlitz's result Eq. (2.19) in [5] and Srivastava and Pintér's addition theorems between Apostol type (p,q)-Frobenius-Euler polynomials of order α and Apostol type (p,q)-Bernoulli polynomials. The last section deals with some special cases for the results obtained this paper.

2. Preliminaries and Lemmas

In this section, we provide some basic formulas and identites for Apostol type (p,q)-Frobenius-Euler polynomials of order α so that we derive the main outcomes of this paper in the next section. We now begin with the following addition theorems for $H_n^{(\alpha)}(x,y;u;\lambda:p,q)$ as Lemma 2.1.

Lemma 2.1. (Additions theorems) *The following relationships hold true:*

$$\begin{aligned}\mathcal{H}_n^{(\alpha)}(x,y;u;\lambda:p,q) &= \sum_{k=0}^{n}\binom{n}{k}_{p,q}\mathcal{H}_k^{(\alpha)}(u;\lambda:p,q)\,(x+y)_{p,q}^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}_{p,q}q^{\binom{n-k}{2}}\mathcal{H}_k^{(\alpha)}(x,0;u;\lambda:p,q)\,y^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}_{p,q}p^{\binom{n-k}{2}}\mathcal{H}_k^{(\alpha)}(0,y;u;\lambda:p,q)\,x^{n-k}.\end{aligned}$$

Some special cases of Lemma 2.1 are investigated in Corollary 2.1 and 2.2.

Corollary 2.1. *In the case $x=0$ (or $y=0$) in Lemma 2.1, we get the following formulas*

$$\begin{aligned}\mathcal{H}_n^{(\alpha)}(x,0;u;\lambda:p,q) &= \sum_{k=0}^{n}\binom{n}{k}_{p,q}p^{\binom{n-k}{2}}\mathcal{H}_k^{(\alpha)}(u;\lambda:p,q)\,x^{n-k},\\ \mathcal{H}_n^{(\alpha)}(0,y;u;\lambda:p,q) &= \sum_{k=0}^{n}\binom{n}{k}_{p,q}q^{\binom{n-k}{2}}\mathcal{H}_k^{(\alpha)}(u;\lambda:p,q)\,y^{n-k}.\end{aligned}$$

Corollary 2.2. *In the case $x=1$ (or $y=1$) in Lemma 2.1, we have the following formulas*

$$\mathcal{H}_n^{(\alpha)}(x,1;u;\lambda:p,q) = \sum_{k=0}^{n}\binom{n}{k}_{p,q}q^{\binom{n-k}{2}}\mathcal{H}_k^{(\alpha)}(x,0;u;\lambda:p,q)\,, \tag{2.1}$$

$$\mathcal{H}_n^{(\alpha)}(1,y;u;\lambda:p,q) = \sum_{k=0}^{n}\binom{n}{k}_{p,q}p^{\binom{n-k}{2}}\mathcal{H}_k^{(\alpha)}(0,y;u;\lambda:p,q)\,. \tag{2.2}$$

We remark that equations (2.1) and (2.2) are (p,q)-generalizations of the following familiar formula:

$$\mathcal{H}_n^{(\alpha)}(x+1;u;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k^{(\alpha)}(x;u;\lambda)\,.$$

Note that

$$\mathcal{H}_n^{(0)}(x,y;u;\lambda:p,q)=\sum_{k=0}^{n}\binom{n}{k}_{p,q}p^{\binom{k}{2}}q^{\binom{n-k}{2}}x^ky^{n-k}=(x+y)_{p,q}^n. \tag{2.3}$$

Lemma 2.2. (Difference equations) *We have*

$$\lambda\mathcal{H}_n^{(\alpha)}(1,y;u;\lambda:p,q)-u\mathcal{H}_n^{(\alpha)}(0,y;u;\lambda:p,q)=(1-u)\mathcal{H}_n^{(\alpha-1)}(0,y;u;\lambda:p,q),$$
$$\lambda\mathcal{H}_n^{(\alpha)}(x,0;u;\lambda:p,q)-u\mathcal{H}_n^{(\alpha)}(x,-1;u;\lambda:p,q)=(1-u)\mathcal{H}_n^{(\alpha-1)}(x,-1;u;\lambda:p,q).$$

Lemma 2.3. (Derivative properties) *We have*

$$D_{p,q;x}\mathcal{H}_n^{(\alpha)}(x,y;u;\lambda:p,q)=[n]_{p,q}\mathcal{H}_{n-1}^{(\alpha)}(px,y;u;\lambda:p,q),$$
$$D_{p,q;y}\mathcal{H}_n^{(\alpha)}(x,y;u;\lambda:p,q)=[n]_{p,q}\mathcal{H}_{n-1}^{(\alpha)}(x,qy;u;\lambda:p,q).$$

Apostol type (p,q)-Frobenius-Euler polynomials $H_n(x,y;u;\lambda:p,q)$ are related to the $B_n(x,y;\lambda:p,q)$, $E_n(x,y;\lambda:p,q)$ and $G_n(x,y;\lambda:p,q)$ as in below.

Lemma 2.4. *We have*

$$\begin{aligned}
\mathcal{B}_n(\lambda:p,q)&=\frac{[n]_{p,q}}{\lambda-1}\mathcal{H}_{n-1}\left(\lambda^{-1};1:p,q\right),\\
\mathcal{B}_n(x,y;\lambda:p,q)&=\frac{[n]_{p,q}}{\lambda-1}\mathcal{H}_{n-1}\left(x,y;\lambda^{-1};1:p,q\right),\\
\mathcal{E}_n(\lambda:p,q)&=\frac{2}{\lambda+1}\mathcal{H}_n\left(-\lambda^{-1};1:p,q\right),\\
\mathcal{E}_n(x,y;\lambda:p,q)&=\frac{2}{\lambda+1}\mathcal{H}_n\left(x,y;-\lambda^{-1};1:p,q\right),\\
\mathcal{G}_n(\lambda:p,q)&=\frac{2[n]_{p,q}}{\lambda+1}\mathcal{H}_n\left(-\lambda^{-1};1:p,q\right),\\
\mathcal{G}_n(x,y;\lambda:p,q)&=\frac{2[n]_{p,q}}{\lambda+1}\mathcal{H}_n\left(x,y;-\lambda^{-1};1:p,q\right).
\end{aligned}$$

Lemma 2.5. *For $\alpha,\beta\in\mathbb{N}$, Apostol type (p,q)-Frobenius-Euler polynomials satisfy the following relations*

$$\begin{aligned}
\mathcal{H}_n^{(\alpha+\beta)}(x,y;u;\lambda:p,q)&=\sum_{k=0}^{n}\binom{n}{k}_{p,q}\mathcal{H}_k^{(\alpha)}(x,0;u;\lambda:p,q)\mathcal{H}_{n-k}^{(\beta)}(x,0;u;\lambda:p,q),\\
\frac{u\mathcal{H}_n(x,y;u;\lambda:p,q)}{(u-1)}&=\frac{\lambda}{(u-1)}\sum_{k=0}^{n}\binom{n}{k}_{p,q}p^{\binom{n-k}{2}}\mathcal{H}_k(x,y;u;\lambda:p,q)+(x+y)_{p,q}^n,\\
\mathcal{H}_n^{(\alpha-\beta)}(x,y;u;\lambda:p,q)&=\sum_{k=0}^{n}\binom{n}{k}_{p,q}\mathcal{H}_k^{(\alpha)}(x,0;u;\lambda:p,q)\mathcal{H}_{n-k}^{(-\beta)}(0,y;u;\lambda:p,q).
\end{aligned}$$

Lemma 2.6. (Recurrence relationship) $H_n^{(\alpha)}(x, y; u; \lambda : p, q)$ *holds the following equality:*

$$\begin{aligned}&\lambda \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{\binom{n-k}{2}} m^k \mathcal{H}_k^{(\alpha)}(x, 0; u; \lambda : p, q)\\&- u \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{\binom{n-k}{2}} m^k \mathcal{H}_k^{(\alpha)}(x, -1; u; \lambda : p, q)\\=&(1-u) \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{\binom{n-k}{2}} m^k \mathcal{H}_k^{(\alpha-1)}(x, -1; u; \lambda : p, q).\end{aligned}$$

3. Main Results

This section includes integral representations, some identities and explicit formulas for $H_n^{(\alpha)}(x, y; u; \lambda : p, q)$. Also, we present new theorems and some (p, q)-extensions of known results in Carlitz [5], Kurt [14], Simsek [23], Srivastava and Pintér [25] and so on. We start with the following explicit formula for Apostol type (p, q)-Frobenius-Euler polynomials of order α by the following theorem.

Theorem 3.1. *Apostol type (p, q)-Frobenius-Euler polynomials of order α hold the following relation:*

$$\begin{aligned}\mathcal{H}_n^{(\alpha)}(x, y; u; \lambda : p, q) =& \frac{1}{1-u} \sum_{k=0}^{n} \binom{n}{k}_{p,q} [\lambda \mathcal{H}_k(1, y; u; \lambda : p, q)\\&- u\mathcal{H}_k(0, y; u; \lambda : p, q)] \mathcal{H}_{n-k}^{(\alpha)}(x, 0; u; \lambda : p, q).\end{aligned}$$

Proof. Indeed,

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{H}_n^{(\alpha)}(x, y; u; \lambda : p, q) \frac{z^n}{[n]_{p,q}!} &= \left(\frac{1-u}{\lambda e_{p,q}(z) - u}\right)^{\alpha} e_{p,q}(xz) E_{p,q}(yz)\\&= \left(\sum_{n=0}^{\infty} q^{\binom{n}{2}} y^n \frac{z^n}{[n]_{p,q}!}\right) \left(\sum_{n=0}^{\infty} \mathcal{H}_n^{(\alpha)}(x, 0; u; \lambda : p, q) \frac{z^n}{[n]_{p,q}!}\right)\\&= \sum_{n=0}^{\infty} q^{\binom{k}{2}} y^k \mathcal{H}_{n-k}^{(\alpha)}(x, 0; u; \lambda : p, q) \frac{z^n}{[n]_{p,q}!}.\end{aligned}$$

It remains to use Lemma 2.2 and Eq. (2.3). □

The (p, q)-integral representations of $H_n^{(\alpha)}(x, y; u; \lambda : p, q)$ are given by the following theorem.

Theorem 3.2. *We have*

$$\int_a^b \mathcal{H}_n^{(\alpha)}(x, y; u; \lambda : p, q)\, d_{p,q}x = p \frac{\mathcal{H}_{n+1}^{(\alpha)}\left(\frac{b}{p}, y; u; \lambda : p, q\right) - \mathcal{H}_{n+1}^{(\alpha)}\left(\frac{a}{p}, y; u; \lambda : p, q\right)}{[n+1]_{p,q}}, \tag{3.1}$$

(3.2)
$$\int_a^b \mathcal{H}_n^{(\alpha)}\left(x,y;u;\lambda:p,q\right)d_{p,q}y = p\frac{\mathcal{H}_{n+1}^{(\alpha)}\left(x,\frac{b}{q};u;\lambda:p,q\right)-\mathcal{H}_{n+1}^{(\alpha)}\left(x,\frac{a}{q};u;\lambda:p,q\right)}{[n+1]_{p,q}}.$$

Proof. Since
$$\int_a^b D_{p,q}f\left(x\right)d_{p,q}x = f\left(b\right)-f\left(a\right) \quad \text{(see [22])}$$
in terms of Lemma 2.3 and equations. (1.5) and (1.6), we arrive at the asserted result
$$\begin{aligned}\int_a^b \mathcal{H}_n^{(\alpha)}\left(x,y;u;\lambda:p,q\right)d_{p,q}x =& \frac{p}{[n+1]_{p,q}}\int_a^b D_{p,q}\mathcal{H}_{n+1}^{(\alpha)}\left(\frac{x}{p},y;u;\lambda:p,q\right)d_{p,q}x \\ =& p\frac{\mathcal{H}_{n+1}^{(\alpha)}\left(\frac{b}{p},y;u;\lambda:p,q\right)-\mathcal{H}_{n+1}^{(\alpha)}\left(\frac{a}{p},y;u;\lambda:p,q\right)}{[n+1]_{p,q}}.\end{aligned}$$
The other can be shown using similar method. Therefore, we complete the proof of this theorem. □

The integral identities (3.1) and (3.2) are (p,q)-generalizations of the formula
$$\int_a^b \mathcal{H}_n^{(\alpha)}\left(x;u;\lambda\right)dx = \frac{\mathcal{H}_{n+1}^{(\alpha)}\left(b;u;\lambda\right)-\mathcal{H}_{n+1}^{(\alpha)}\left(a;u;\lambda\right)}{n+1}.$$

Here is a recurrence relation of Apostol type (p,q)-Frobenius-Euler polynomials by the following theorem.

Theorem 3.3. *We have*
$$\begin{aligned}\mathcal{H}_{n+1}\left(x,y;u;\lambda:p,q\right) =& yp^n\mathcal{H}_n\left(\frac{q}{p}x,\frac{q}{p}y;u;\lambda:p,q\right)+xp^n\mathcal{H}_n\left(x,y;u;\lambda:p,q\right) \\ & -\lambda\sum_{k=0}^{n}\binom{n}{k}_{p,q}q^kp^{n-k}\mathcal{H}_k\left(x,y;u;\lambda:p,q\right)\mathcal{H}_{n-k}\left(1,0;u;\lambda:p,q\right).\end{aligned}$$

Proof. For $\alpha=1$, applying the (p,q)-derivative to $H_n\left(x,y;u;\lambda:p,q\right)$ with respect to z yields to
$$\sum_{n=0}^{\infty}D_{p,q;z}\mathcal{H}_n\left(x,y;u;\lambda:p,q\right)\frac{z^n}{[n]_{p,q}!} = (1-u)\,D_{p,q;z}\left\{\frac{e_{p,q}\left(xz\right)E_{p,q}\left(yz\right)}{\lambda e_{p,q}\left(z\right)-u}\right\},$$
using equations (1.2) and (1.3), it becomes
$$\begin{aligned}&(1-u)\frac{\left(\lambda e_{p,q}\left(qz\right)-u\right)D_{p,q;z}\left[e_{p,q}\left(xz\right)E_{p,q}\left(yz\right)\right]}{\left(\lambda e_{p,q}\left(pz\right)-u\right)\left(\lambda e_{p,q}\left(qz\right)-u\right)} \\ &+(u-1)\frac{e_{p,q}\left(pxz\right)E_{p,q}\left(pyz\right)D_{p,q;z}\left(\lambda e_{p,q}\left(z\right)-u\right)}{\left(\lambda e_{p,q}\left(pz\right)-u\right)\left(\lambda e_{p,q}\left(qz\right)-u\right)} \\ =& y\sum_{n=0}^{\infty}\mathcal{H}_n\left(\frac{q}{p}x,\frac{q}{p}y;u;\lambda:p,q\right)p^n\frac{z^n}{[n]_{p,q}!}+x\sum_{n=0}^{\infty}\mathcal{H}_n\left(x,y;u;\lambda:p,q\right)p^n\frac{z^n}{[n]_{p,q}!}\end{aligned}$$

$$-\lambda\sum_{n=0}^{\infty}\mathcal{H}_n\left(x,y;u;\lambda:p,q\right)q^n\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}\mathcal{H}_n\left(1,0;u;\lambda:p,q\right)p^n\frac{z^n}{[n]_{p,q}!}.$$

By making use of Cauchy product, comparing the coefficients of $\frac{z^n}{[n]_{p,q}}$, then we have desired result. □

We now present the symmetric identity for $H_n\left(x,y;u;\lambda:p,q\right)$ as given below.

Theorem 3.4. *We have*

$$\begin{aligned}(2u-1)\sum_{k=0}^{n}\binom{n}{k}_{p,q}&\mathcal{H}_k(u;\lambda:p,q)\mathcal{H}_{n-k}(x,y;1-u;\lambda:p,q)\\=&u\mathcal{H}_n(x,y;u;\lambda:p,q)-(1-u)\mathcal{H}_n(x,y;1-u;\lambda:p,q).\end{aligned}\tag{3.3}$$

Proof. If we consider the identity

$$\frac{2u-1}{\left(\lambda e_{p,q}\left(z\right)-u\right)\left(\lambda e_{p,q}\left(z\right)-\left(1-u\right)\right)}=\frac{1}{\lambda e_{p,q}\left(z\right)-u}-\frac{1}{\lambda e_{p,q}\left(z\right)-\left(1-u\right)},$$

then

$$\begin{aligned}&(2u-1)\frac{\left(1-u\right)e_{p,q}\left(xz\right)\left(1-\left(1-u\right)\right)E_{p,q}\left(yz\right)}{\left(\lambda e_{p,q}\left(z\right)-u\right)\left(\lambda e_{p,q}\left(z\right)-\left(1-u\right)\right)}\\=&u\frac{\left(1-u\right)e_{p,q}\left(xz\right)E_{p,q}\left(yz\right)}{\lambda e_{p,q}\left(z\right)-u}-\frac{\left(1-u\right)e_{p,q}\left(xz\right)\left(1-\left(1-u\right)\right)E_{p,q}\left(yz\right)}{\lambda e_{p,q}\left(z\right)-\left(1-u\right)}.\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}&(2u-1)\sum_{n=0}^{\infty}\mathcal{H}_n\left(0,0;u;\lambda:p,q\right)\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}\mathcal{H}_n\left(x,y;1-u;\lambda:p,q\right)\frac{z^n}{[n]_{p,q}!}\\=&u\sum_{n=0}^{\infty}\mathcal{H}_n\left(x,y;u;\lambda:p,q\right)\frac{z^n}{[n]_{p,q}!}-(1-u)\sum_{n=0}^{\infty}\mathcal{H}_n\left(x,y;1-u;\lambda:p,q\right)\frac{z^n}{[n]_{p,q}!}.\end{aligned}$$

Checking against the coefficients of $\frac{z^n}{[n]_{p,q}}$, then we have desired result in (3.3). □

The relation (3.3) is a (p, q)-generalization of Carlitz's result Eq. (2.19) in [5].

Theorem 3.5. *The following relation is valid for Apostol type (p, q)-Frobenius-Euler polynomials:*

$$u\mathcal{H}_n\left(x,y;u;\lambda:p,q\right)=\lambda\sum_{k=0}^{n}\binom{n}{k}_{p,q}p^{\binom{n-k}{2}}\mathcal{H}_k\left(x,y;u;\lambda:p,q\right)-(1-u)\left(x+y\right)_{p,q}^n.$$

Proof. From the relation (1.4) and the identity

$$\frac{u}{\lambda\left(\lambda e_{p,q}\left(z\right)-u\right)e_{p,q}\left(z\right)}=\frac{1}{\lambda e_{p,q}\left(z\right)-u}-\frac{1}{\lambda e_{p,q}\left(z\right)},$$

we write

$$\frac{u\left(1-u\right)e_{p,q}\left(xz\right)E_{p,q}\left(yz\right)}{\left(\lambda e_{p,q}\left(z\right)-u\right)\lambda e_{p,q}\left(z\right)}$$

$$=\frac{(1-u)\,e_{p,q}\,(xz)\,E_{p,q}\,(yz)}{\lambda e_{p,q}\,(z)-u}-\frac{(1-u)\,e_{p,q}\,(xz)\,E_{p,q}\,(yz)}{\lambda e_{p,q}\,(z)},$$

which gives

$$\frac{u}{\lambda}\sum_{n=0}^{\infty}\mathcal{H}_n\,(x,y;u;\lambda:p,q)\,\frac{z^n}{[n]_{p,q}!}$$
$$=\sum_{n=0}^{\infty}\mathcal{H}_n\,(x,y;u;\lambda:p,q)\,\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{z^n}{[n]_{p,q}!}-\frac{1-u}{\lambda}\sum_{n=0}^{\infty}(x+y)_{p,q}^{n}\,\frac{z^n}{[n]_{p,q}!}.$$

Equating the coefficients of $\frac{z^n}{[n]_{p,q}}$, we derive asserted result. □

We are in a position to provide some relationships for Apostol type (p,q)-Frobenius-Euler polynomials of order α related to Apostol type (p,q)-Bernoulli polynomials, Apostol type (p,q)-Euler polynomials and Apostol type (p,q)-Genocchi polynomials in Theorems 3.6, 3.7 and 3.8.

Theorem 3.6. *The following recurrence relations are valid:*

$$\mathcal{H}_n^{(\alpha)}\,(x,y;u;\lambda:p,q)=\sum_{s=0}^{n+1}\binom{n+1}{s}_{p,q}\frac{\mathcal{H}_{n+1-s}^{(\alpha)}\,(0,y;u;\lambda:p,q)}{[n+1]_{p,q}}$$
$$\times\left\{\lambda\sum_{k=0}^{s}\binom{s}{k}_{p,q}\mathcal{B}_{s-k}\,(x,0;\lambda:p,q)\,p^{\binom{k}{2}}-\mathcal{B}_s\,(x,0;\lambda:p,q)\right\},$$

$$\mathcal{H}_n^{(\alpha)}\,(x,y;u;\lambda:p,q)=\sum_{s=0}^{n+1}\binom{n+1}{s}_{p,q}\frac{\mathcal{H}_{n+1-s}^{(\alpha)}\,(x,0;u;\lambda:p,q)}{[n+1]_{p,q}}$$
$$\times\left\{\lambda\sum_{k=0}^{s}\binom{s}{k}_{p,q}\mathcal{B}_{s-k}\,(0,y;\lambda:p,q)\,p^{\binom{k}{2}}-\mathcal{B}_s\,(0,y;\lambda:p,q)\right\}.$$

Proof. Indeed,

$$\left(\frac{1-u}{\lambda e_{p,q}(z)-u}\right)^{\alpha}e_{p,q}(xz)E_{p,q}(yz)$$
$$=\left(\frac{1-u}{\lambda e_{p,q}(z)-u}\right)^{\alpha}E_{p,q}(yz)\frac{z}{\lambda e_{p,q}(z)-1}\frac{\lambda e_{p,q}(z)-1}{z}e_{p,q}(xz)$$
$$=\frac{1}{z}\left[\lambda\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_{p,q}\mathcal{B}_{n-k}(x,0;\lambda:p,q)p^{\binom{k}{2}}\frac{z^n}{[n]_{p,q}!}-\sum_{n=0}^{\infty}\mathcal{B}_n(x,0;\lambda:p,q)\frac{z^n}{[n]_{p,q}!}\right]$$
$$\times\sum_{n=0}^{\infty}\mathcal{H}_n^{(\alpha)}(0,y;u;\lambda:p,q)\frac{z^n}{[n]_{p,q}!}$$
$$=\sum_{n=0}^{\infty}\left[\lambda\sum_{s=0}^{n}\binom{n}{s}_{p,q}\sum_{k=0}^{s}\binom{s}{k}_{p,q}\mathcal{B}_{s-k}(x,0;\lambda:p,q)p^{\binom{k}{2}}-\sum_{s=0}^{n}\binom{n}{s}_{p,q}\mathcal{B}_s(x,0;\lambda:p,q)\right]$$

$$\times \mathcal{H}_{n-s}^{(\alpha)}(0,y;u;\lambda:p,q)\frac{z^{n-1}}{[n]_{p,q}!}.$$

By using Cauchy product and comparing the coefficients of $\frac{z^n}{[n]_{p,q}}$, the proof is completed. ☐

Theorem 3.7. *We have*

$$\mathcal{H}_n^{(\alpha)}(x,y;u;\lambda:p,q)=\sum_{s=0}^{n}\binom{n}{s}_{p,q}\frac{\mathcal{H}_{n-s}^{(\alpha)}(0,y;u;\lambda:p,q)}{2}$$
$$\times\left\{\lambda\sum_{k=0}^{s}\binom{s}{k}_{p,q}\mathcal{E}_k(x,0;\lambda:p,q)\,p^{\binom{s-k}{2}}+\mathcal{E}_s(x,0;\lambda:p,q)\right\},$$
$$\mathcal{H}_n^{(\alpha)}(x,y;u;\lambda:p,q)=\sum_{s=0}^{n}\binom{n}{s}_{p,q}\frac{\mathcal{H}_{n-s}^{(\alpha)}(x,0;u;\lambda:p,q)}{2}$$
$$\times\left\{\lambda\sum_{k=0}^{s}\binom{s}{k}_{p,q}\mathcal{E}_k(0,y;\lambda:p,q)\,p^{\binom{s-k}{2}}+\mathcal{E}_s(0,y;\lambda:p,q)\right\}.$$

Proof. The proof is based on the following equalities

$$\left(\frac{1-u}{\lambda e_{p,q}(z)-u}\right)^{\alpha}e_{p,q}(xz)\,E_{p,q}(yz)=\left(\frac{1-u}{\lambda e_{p,q}(z)-u}\right)^{\alpha}e_{p,q}(xz)$$
$$\times\frac{2}{\lambda e_{p,q}(z)+1}\frac{\lambda e_{p,q}(z)+1}{2}E_{p,q}(yz),$$
$$\left(\frac{1-u}{\lambda e_{p,q}(z)-u}\right)^{\alpha}e_{p,q}(xz)\,E_{p,q}(yz)=\left(\frac{1-u}{\lambda e_{p,q}(z)-u}\right)^{\alpha}E_{p,q}(yz)$$
$$\times\frac{2}{\lambda e_{p,q}(z)+1}\frac{\lambda e_{p,q}(z)+1}{2}e_{p,q}(xz)$$

and is similar to that of Theorem 3.6. ☐

Theorem 3.8. *Each of the following recurrence relations holds true:*

$$\mathcal{H}_n^{(\alpha)}(x,y;u;\lambda:p,q)=\sum_{s=0}^{n+1}\binom{n+1}{s}_{p,q}\frac{\mathcal{H}_{n+1-s}^{(\alpha)}(0,y;u;\lambda:p,q)}{2\,[n+1]_{p,q}}$$
$$\times\left\{\lambda\sum_{k=0}^{s}\binom{s}{k}_{p,q}\mathcal{G}_{s-k}(x,0;\lambda:p,q)\,p^{\binom{k}{2}}+\mathcal{G}_s(x,0;\lambda:p,q)\right\},$$
$$\mathcal{H}_n^{(\alpha)}(x,y;u;\lambda:p,q)=\sum_{s=0}^{n+1}\binom{n+1}{s}_{p,q}\frac{\mathcal{H}_{n+1-s}^{(\alpha)}(x,0;u;\lambda:p,q)}{2\,[n+1]_{p,q}}$$
$$\times\left\{\lambda\sum_{k=0}^{s}\binom{s}{k}_{p,q}\mathcal{G}_{s-k}(0,y;\lambda:p,q)\,p^{\binom{k}{2}}+\mathcal{G}_s(0,y;\lambda:p,q)\right\}.$$

Proof. By making use of the following equalities

$$\left(\frac{1-u}{\lambda e_{p,q}(z)-u}\right)^{\alpha} e_{p,q}(xz)\, E_{p,q}(yz) = \left(\frac{1-u}{\lambda e_{p,q}(z)-u}\right)^{\alpha} e_{p,q}(xz) \times \frac{2z}{\lambda e_{p,q}(z)+1} \frac{\lambda e_{p,q}(z)+1}{2z} E_{p,q}(yz),$$

$$\left(\frac{1-u}{\lambda e_{p,q}(z)-u}\right)^{\alpha} e_{p,q}(xz)\, E_{p,q}(yz) = \left(\frac{1-u}{\lambda e_{p,q}(z)-u}\right)^{\alpha} E_{p,q}(yz) \times \frac{2z}{\lambda e_{p,q}(z)+1} \frac{\lambda e_{p,q}(z)+1}{2z} e_{p,q}(xz),$$

the proof of these theorem is completed similar to that of Theorem 3.6. □

4. Conclusion

In the present paper, we have introduced (p, q)-extension of Apostol type Frobenius-Euler polynomials and numbers and investigated some basic identities and properties for these polynomials and numbers, including addition theorems, difference equations, derivative properties, recurrence relations and so on. Thereafter, we have given integral representations, explicit formulas and relations for mentioned newly defined polynomials and numbers. Henceforth, we have obtained (p, q)-extensions of Carlitz's result [5] and Srivastava and Pintér addition theorems in [25]. The results obtained here reduce to known properties of q-polynomials mentioned in this paper when $p = 1$. Also, in the event of $q \to p = 1$, our results reduce to ordinary results for Apostol type Frobenius-Euler polynomials and numbers.

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