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$\overline{q}$-LAPLACE TRANSFORM ON QUANTUM INTEGRAL

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Abstract. In this paper, we present $\overline{q}$-Laplace transform by $\overline{q}$-integral definition on quantum analogue. We present some properties and obtain formulaes of $\overline{q}$-Laplace transform with its aplications.

1. Introduction

Quantum calculus is the modern name for the investigation of calculus without limits. The quantum calculus or q-calculus began with FH Jackson in the early twentieth century, but this kind of calculus had already been worked out by Euler and Jacobi. Recently it arose interest due to high demand of mathematics that models quantum computing. q-calculus appeared as a connection between mathematics and physics. It has a lot of applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions and other sciences quantum theory, mechanics and the theory of relativity.

There are many of the fundamental aspects of quantum calculus. It has been shown that quantum calculus is a subfield of the more general mathematical field of time scales calculus. Time scales provide a unified framework for studying dynamic equations on both discrete and continuous domains.

In 2017, Alp and Sarikaya [1] gave a new defination of q-integral which is showed $\overline{q}$-integral.

The aim of this paper present Laplace transform on $\overline{q}$-integral. In second section we give notations and preliminaries for q-analogue. In third section we give definition of Laplace transform on $\overline{q}$-integral and obtain some auxiliary results. In fourth section

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we calculate $\overline{q}$-Laplace transforms of functions and some properties of $\overline{q}$-Laplace transform.

Now remember following Laplace transform on classical analysis.

For $t > 0$ Laplace transform of $f(t)$ is defined as

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)dt. \tag{1.1}$$

We say that transform converges if the limit exists, and diverges if not.

2. Notations and Preliminaries

In this section, first we give definition and notations of q-analogue with q-derivates then definition and properties of $\overline{q}$-integral. For $0 < q < 1$ here and further we use the following notations [3, 4]:

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}, \tag{2.1}$$

(2.2)

$$(x-a)_q^n = \prod_{i=0}^{n-1} \left(x - q^i a\right) = (x-a)(x-qa)\left(x-q^2a\right)\cdots\left(x-q^{n-1}a\right), \quad n \in \mathbb{Z}^+,$$

$$(a:q)_0 = 1,$$

$$(1-a)_q^n = (a:q)_n = \prod_{i=0}^{n} \left(1-q^i a\right),$$

(2.3)

$$(1-a)_q^\infty = (a:q)_\infty = \prod_{i=0}^{\infty} \left(1-q^i a\right),$$

(2.4)

$$(1-a)_q^n = \frac{(1-a)_q^\infty}{(1-q^n a)_q^\infty} = \frac{(a:q)_\infty}{(q^n a:q)_\infty}, \quad n \in \mathbb{C}.$$

Notice that, under our assumptions on q, the infinite product (2.3) is convergent. Moreover, the definitions (2.2) and (2.4) are consistent.

Definition 2.1. In [2], for f has $D_q^n f(a)$, Jackson introduced the following q-counterpart of Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q;q)_n} D_q^n f(a)(x-a)_q^n = \sum_{n=0}^{\infty} \frac{D_q^n f(a)(x-a)_q^n}{[n]_q!}, \tag{2.5}$$

D_q is the q-difference operator.

Here E_q^x and e_q^x are two q-analogues of the exponential functions and their q-Taylor series ([4]):

$$E_q^x = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!} = (1 + (1-q)\,x)_q^{\infty} = ((q-1)\,x : q)_{\infty}\,, \tag{2.6}$$

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{(1-(1-q)\,x)_q^{\infty}} = \frac{1}{((1-q)\,x : q)_{\infty}}. \tag{2.7}$$

Lemma 2.1 ([4])**.** *The q-exponential functions satisfy the following properties:*

$$e_q^x E_q^{-x} = E_q^x e_q^{-x} = 1, \quad E_q^x = e_{1/q}^x.$$

For $E_q^{-x} = \frac{1}{e_q^x}$ we have

$$\lim_{x\to\infty} E_q^{-x} = \lim_{x\to\infty} \frac{1}{e_q^x} = 0.$$

Let $J := [a,b] \subset \mathbb{R}$, $J^{\circ} := (a,b)$ be interval and $0 < q < 1$ be a constant. Definiton of q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a,b]$ as follows.

Definition 2.2 ([5])**.** Assume $f : J \to \mathbb{R}$ is a continuous function and let $x \in J$. Then the expression

$$\begin{aligned} {}_aD_q f(x) &= \frac{f(x) - f(qx + (1-q)\,a)}{(1-q)(x-a)}, \quad x \neq a, \\ {}_aD_q f(a) &= \lim_{x\to a} {}_aD_q f(x)\,, \end{aligned} \tag{2.8}$$

is called the q-derivative on J of function f at x.

We say that f is q-differentiable on J provided ${}_aD_q f(x)$ exists for all $x \in J$. Note that if $a = 0$ in (2.8), then ${}_aD_q f = D_q f$, where D_q is the well-known q-derivative of the function $f(x)$ defined by

$${}_aD_q f(x) = \frac{f(x) - f(qx)}{(1-q)\,x}.$$

For more details, see [4].

Lemma 2.2 ([5])**.** *Let $\alpha \in \mathbb{R}$, then we have*

$${}_aD_q (x-a)^{\alpha} = [\alpha]_q (x-a)^{\alpha-1}\,. \tag{2.9}$$

The following definitions and theorems with respect to $\overline{q}$-integral were referred in [1, page 148].

Definition 2.3. Let $f : J \to \mathbb{R}$ is continuous function. For $0 < q < 1$

$$\int_a^b f(s)\, {}_ad_{\overline{q}}s = \frac{(1-q)(b-a)}{2q} \left[(1+q) \sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n)\,a) - f(b) \right], \tag{2.10}$$

which second sense quantum integral definition that call $\overline{q}$-integral for $x \in J$.

Moreover, if $c \in (a,x)$ then the definite $\overline{q}$-integral on J is defined by

$$\begin{aligned}\int_c^x f(s)\,{}_a d_{\overline{q}}s &= \int_a^x f(s)\,{}_a d_{\overline{q}}s - \int_a^c f(s)\,{}_a d_{\overline{q}}s \\ &= \frac{(1-q)(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - f(x)\right] \\ &\quad - \frac{(1-q)(c-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a) - f(c)\right].\end{aligned} \tag{2.11}$$

Theorem 2.1 ([1])**.** *Let $f : J \to \mathbb{R}$ be a continuous function. Then we have the following properties of $\overline{q}$-integral*

i)

$${}_aD_q \int_a^x f(s)\,{}_a d_{\overline{q}}s = \frac{f(x) + f(qx + (1-q)a)}{2};$$

ii)

$$\int_0^1 f(sb + (1-s)a)\,{}_0 d_{\overline{q}}s = \frac{1}{b-a}\int_a^b f(t)\,{}_a d_{\overline{q}}t;$$

iii)

$$\begin{aligned}&\int_c^x {}_aD_q f(s)\,{}_a d_{\overline{q}}s \\ &= \frac{qf(x) + f(qx + (1-q)a) - qf(c) - f(qc + (1-q)a)}{2q}, \quad \text{for } c \in (a,x);\end{aligned}$$

iv)

$$\int_a^x [f(s) + g(s)]\,{}_a d_{\overline{q}}s = \int_a^x f(s)\,{}_a d_{\overline{q}}s + \int_a^x g(s)\,{}_a d_{\overline{q}}s;$$

v)

$$\int_a^x (\alpha f)(s)\,{}_a d_{\overline{q}}s = \alpha \int_a^x f(s)\,{}_a d_{\overline{q}}s, \quad \alpha \in \mathbb{R};$$

vi) partial integration property:

$$\begin{aligned}&\int_c^x f(s)\,{}_aD_q g(s)\,{}_a d_{\overline{q}}s \\ &= \left.\frac{qf(s)g(s) + f(qs + (1-q)a)\,g(qs + (1-q)a)}{2q}\right|_c^x\end{aligned} \tag{2.12}$$

$$-\int_c^x g\left(qs+(1-q)a\right){}_aD_qf\left(s\right){}_ad_{\overline{q}}s;$$

vii)

$$\int_a^x (s-a)^{\alpha}\,{}_ad_{\overline{q}}s=\frac{1}{[\alpha+1]_q}\left(\frac{1+q^{\alpha}}{2}\right)(x-a)^{\alpha+1}.$$

3. Auxiliary Results

For using in further theorems lets give an example on q-derivative.

Example 3.1. For $s>0,\ t\in\mathbb{R}$, we have

$$\begin{aligned} D_qE_q^{-st}&=-\,sE_q^{-qst},\\ D_qE_q^{-qst}&=-\,qsE_q^{-q^2st}. \end{aligned}\tag{3.1}$$

Proof. By using q-derivative and (2.6), we obtain that:

$$\begin{aligned} D_qE_q^{-st}=&D_q\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!}(st)^n\right)\\ =&\sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}s^n}{[n]_q!}D_qt^n\\ =&\sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}s^n}{[n-1]_q!}t^{n-1}\\ =&\sum_{n=0}^{\infty}\frac{(-1)^{n+1} q^{\frac{n(n+1)}{2}}s^{n+1}}{[n]_q!}t^{n}\\ =&-\,s\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!}(qst)^n\\ =&-\,sE_q^{-qst} \end{aligned}$$

and in the same way we have

$$D_qE_q^{-qst}=-qsE_q^{-q^2st}$$

and the proof is completed. □

Now we present $\overline{q}$-Laplace transform on $\overline{q}$-integral below.

Definition 3.1. Let $s>0$ and $f:[0,\infty)\to\mathbb{R}$ be a function. Then the $\overline{q}$-Laplace transform is defined by

$$L_{\overline{q}}\left\{f(t)\right\}=F(s)=\int_0^{\infty}f\left(t\right)E_q^{-qst}d_{\overline{q}}t.\tag{3.2}$$

Assume f, g are two functions and $\alpha, \beta \in \mathbb{C}$ by using (3.2) linearity property of $\overline{q}$-Laplace transform is written as follow:

$$L_{\overline{q}}\left\{\alpha f(t)+\beta g(t)\right\}=\alpha L_{\overline{q}}\left\{f(t)\right\}+\beta L_{\overline{q}}\left\{g(t)\right\}.$$

4. $\overline{q}$-Laplace Transform of Functions

In this section, we proved $\overline{q}$-Laplace transform of functions and n degrees of quantum derivative function. Let's first calculate the $\overline{q}$-Laplace transformation of the constant function as below.

Theorem 4.1. *The $\overline{q}$-Laplace transform of function $f(t)=1$ is*

$$L_{\overline{q}}\left\{1\right\}=F(s)=\frac{1+q}{2q}\cdot\frac{1}{s}.$$

Proof. From definition of $\overline{q}$-Laplace transform, it follows that

$$F(s)=L_{\overline{q}}\left\{1\right\}=\lim_{\alpha\to\infty}\int_0^{\alpha}E_q^{-qst}d_{\overline{q}}t=\int_0^{\infty}E_q^{-qst}d_{\overline{q}}t.$$

Then calculate above integral by using the $\overline{q}$-integral, we have

$$\begin{aligned}
&\int_0^{\alpha}E_q^{-qst}d_{\overline{q}}t\\
&=\int_0^{\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!}(qst)^n\,d_{\overline{q}}t\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}(qs)^n}{[n]_q!}\int_0^{\alpha}t^n d_{\overline{q}}t\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}(qs)^n}{[n]_q!}\cdot\frac{1+q^n}{2\left[n+1\right]_q}\alpha^{n+1}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}(qs)^n}{2\left[n+1\right]_q!}\alpha^{n+1}+\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}(qs)^n q^n}{2\left[n+1\right]_q!}\alpha^{n+1}\\
&=-\frac{1}{2s}\sum_{n=0}^{\infty}\frac{(-1)^{n+1} q^{\frac{n(n+1)}{2}}}{[n+1]_q!}(s\alpha)^{n+1}-\frac{1}{2qs}\sum_{n=0}^{\infty}\frac{(-1)^{n+1} q^{\frac{n(n+1)}{2}}}{[n+1]_q!}(qs\alpha)^{n+1}\\
&=-\frac{1}{2s}\sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!}(s\alpha)^n-\frac{1}{2qs}\sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!}(qs\alpha)^n\\
&=-\frac{1}{2s}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!}(s\alpha)^n+\frac{1}{2s}-\frac{1}{2qs}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!}(qs\alpha)^n+\frac{1}{2qs}\\
&=-\frac{1}{2s}E_q^{-s\alpha}+\frac{1}{2s}-\frac{1}{2qs}E_q^{-qs\alpha}+\frac{1}{2qs}
\end{aligned}$$

and by taking the limit the proof is obtained as follows

$$L_{\overline{q}}\{1\} = F(s) = \lim_{\alpha\to\infty}\left(-\frac{1}{2s}E_q^{-s\alpha} + \frac{1}{2s} - \frac{1}{2qs}E_q^{-qs\alpha} + \frac{1}{2qs}\right) = \frac{1+q}{2q}\cdot\frac{1}{s},$$

where

$$\lim_{\alpha\to\infty} E_q^{-qs\alpha} = \lim_{\alpha\to\infty} E_q^{-s\alpha} = 0.$$

□

Theorem 4.2. *For $n \in \mathbb{R}$ with $n > -1$, the $\overline{q}$-Laplace transform of function $f(t) = t^n$ is*

$$L_{\overline{q}}\{t^n\} = \frac{[n]_q}{s} L_{\overline{q}}\left\{t^{n-1}\right\}. \tag{4.1}$$

Proof. From definition of $\overline{q}$-Laplace transform, it follows that

$$L_{\overline{q}}\{t^n\} = F(s) = \lim_{\alpha\to\infty}\int_0^\alpha t^n E_q^{-qst} d_{\overline{q}}t = \int_0^\infty t^n E_q^{-qst} d_{\overline{q}}t.$$

Then, calculate above integral by using (2.12) and (3.1) with the $\overline{q}$-integral, we have

$$\begin{aligned}
\int_0^\alpha t^n E_q^{-qst} d_{\overline{q}}t &= -\frac{1}{s}\int_0^\alpha t^n D_q E_q^{-st} d_{\overline{q}}t \\
&= -\frac{1}{s}\left[\frac{qt^n E_q^{-st} + (qt)^n E_q^{-qst}}{2q}\Bigg|_0^\alpha - [n]_q \int_0^\alpha t^{n-1} E_q^{-qst} d_{\overline{q}}t\right] \\
&= \frac{[n]_q}{s}\int_0^\alpha t^{n-1} E_q^{-qst} d_{\overline{q}}t - \frac{q\alpha^n E_q^{-s\alpha} + (q\alpha)^n E_q^{-qs\alpha}}{2qs} \\
&= \frac{[n]_q}{s}\int_0^\alpha t^{n-1} E_q^{-qst} d_{\overline{q}}t - \frac{q\alpha^n E_q^{-s\alpha} + (q\alpha)^n E_q^{-qs\alpha}}{2qs}
\end{aligned}$$

and by taking the limit

$$\begin{aligned}
L_{\overline{q}}\{t^n\} = & F(s) = \lim_{\alpha\to\infty}\int_0^\alpha t^n E_q(-qst) d_{\overline{q}}t \\
&= \lim_{\alpha\to\infty}\left[\frac{[n]_q}{s}\int_0^\alpha t^{n-1} E_q^{-qst} d_{\overline{q}}t - \frac{q\alpha^n E_q^{-s\alpha} + (q\alpha)^n E_q^{-qs\alpha}}{2qs}\right] \\
&= \frac{[n]_q}{s}\int_0^\infty t^{n-1} E_q^{-qst} d_{\overline{q}}t = \frac{[n]_q}{s} L_{\overline{q}}\left\{t^{n-1}\right\}
\end{aligned}$$

and the proof is completed. □

Theorem 4.3. *Let $n \in \mathbb{N}$, then the $\overline{q}$-Laplace transform of function $f(t) = t^n$ is*

$$L_{\overline{q}}\{t^n\} = \frac{1+q}{2q}\cdot\frac{[n]_q!}{s^{n+1}}.$$

Proof. By using (4.1), it follows that

$$\begin{aligned}
L_{\overline{q}}\left\{t^{n}\right\} &= \frac{[n]_q}{s} L_{\overline{q}}\left\{t^{n-1}\right\} \\
&= \frac{[n]_q}{s} \cdot \frac{[n-1]_q}{s} L_{\overline{q}}\left\{t^{n-2}\right\} \\
&\vdots \\
&= \frac{[n]_q}{s} \cdot \frac{[n-1]_q}{s} \cdots L_{\overline{q}}\left\{1\right\} \\
&= \frac{[n]_q}{s} \cdot \frac{[n-1]_q}{s} \cdots \frac{1+q}{2q} \cdot \frac{1}{s} \\
&= \frac{1+q}{2q} \cdot \frac{[n]_q!}{s^{n+1}}.
\end{aligned}$$

□

Theorem 4.4. *The $\overline{q}$-Laplace transform of function $f(t) = e_q^{at}$ is*

$$L_{\overline{q}}\left\{e_q^{at}\right\} = \frac{1+q}{2q} \cdot \frac{1}{s-a}, \quad s > a.$$

Proof. From definition of $\overline{q}$-Laplace transform, it follows that

$$\begin{aligned}
L_{\overline{q}}\left\{e_q^{at}\right\} &= \lim_{\alpha\to\infty} \int_0^{\alpha} e_q^{at} E_q^{-qst} d_{\overline{q}}t = \lim_{\alpha\to\infty} \sum_{n=0}^{\infty} \frac{a^n}{[n]_q!} \int_0^{\alpha} t^n E_q^{-qst} d_{\overline{q}}t \\
&= \sum_{n=0}^{\infty} \frac{a^n}{[n]_q!} \lim_{\alpha\to\infty} \int_0^{\alpha} t^n E_q^{-qst} d_{\overline{q}}t = \sum_{n=0}^{\infty} \frac{a^n}{[n]_q!} L_{\overline{q}}\left\{t^n\right\} \\
&= \sum_{n=0}^{\infty} \frac{a^n}{[n]_q!} \cdot \frac{1+q}{2q} \cdot \frac{[n]_q!}{s^{n+1}} = \frac{1+q}{2qs} \sum_{n=0}^{\infty} \left(\frac{a}{s}\right)^n = \frac{1+q}{2q} \cdot \frac{1}{s-a},
\end{aligned}$$

and the proof is completed. □

Theorem 4.5. *The $\overline{q}$-Laplace transform of function $f(t) = E_q^{at}$ is*

$$L_{\overline{q}}\left\{E_q^{at}\right\} = \frac{1+q}{2qs} \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \left(\frac{a}{s}\right)^n, \quad s > 0.$$

Proof. From definition of $\overline{q}$-Laplace transform, it follows that

$$\begin{aligned}
L_{\overline{q}}\left\{E_q^{at}\right\} &= \int_0^{\infty} E_q^{at} E_q^{-qst} d_{\overline{q}}t \\
&= \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{a^n}{[n]_q!} \int_0^{\infty} t^n E_q^{-qst} d_{\overline{q}}t
\end{aligned}$$

$$= \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{a^n}{[n]_q!} \cdot \frac{1+q}{2q} \cdot \frac{[n]_q!}{s^{n+1}}$$
$$= \frac{1+q}{2qs} \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \left(\frac{a}{s}\right)^n . \qquad \square$$

Theorem 4.6. *The $\overline{q}$-Laplace transform of q-cosine, q-sine, q-Cosine, q-Sine functions are that*

$$L_{\overline{q}}\{\cos_q at\} = \frac{1+q}{2q} \cdot \frac{s}{s^2+a^2},$$
$$L_{\overline{q}}\{\sin_q at\} = \frac{1+q}{2q} \cdot \frac{a}{s^2+a^2},$$
$$L_{\overline{q}}\{\mathrm{Cos}_q at\} = \frac{1+q}{2qs} \sum_{n=0}^{\infty} (-1)^n q^{n(2n-1)} \left(\frac{a}{s}\right)^{2n},$$
$$L_{\overline{q}}\{\mathrm{Sin}_q at\} = \frac{1+q}{2qs} \sum_{n=0}^{\infty} (-1)^n q^{n(2n+1)} \left(\frac{a}{s}\right)^{2n+1}.$$

Proof. Consider the following definition of q-cosine, q-sine, q-Cosine and q-Sine functions:

$$\cos_q at = \frac{e_q^{iat} + e_q^{-iat}}{2} \quad \text{and} \quad \sin_q at = \frac{e_q^{iat} - e_q^{-iat}}{2i},$$
$$\mathrm{Cos}_q at = \frac{E_q^{iat} + E_q^{-iat}}{2} \quad \text{and} \quad \mathrm{Sin}_q at = \frac{E_q^{iat} - E_q^{-iat}}{2i}.$$

Then, by using linearity of $\overline{q}$-Laplace transform,

$$L_{\overline{q}}\{\cos_q at\} = L_{\overline{q}}\left\{\frac{e_q^{iat} + e_q^{-iat}}{2}\right\} = \frac{1}{2}\left(L_{\overline{q}}\left\{e_q^{iat}\right\} + L_{\overline{q}}\left\{e_q^{-iat}\right\}\right)$$
$$= \frac{1}{2}\left(\frac{1+q}{2q} \cdot \frac{1}{s-ia} + \frac{1+q}{2q} \cdot \frac{1}{s+ia}\right)$$
$$= \frac{1+q}{2q} \cdot \frac{s}{s^2+a^2}$$

and in the same way we have

$$L_{\overline{q}}\{\sin_q at\} = \frac{1+q}{2q} \cdot \frac{a}{s^2+a^2}.$$

Now, we obtain $\overline{q}$-Laplace transform of q-Cosine and q-Sine functions

$$L_{\overline{q}}\{\mathrm{Cos}_q at\} = L_{\overline{q}}\left\{\frac{E_q^{iat} + E_q^{-iat}}{2}\right\}$$
$$= \frac{1}{2}\left(L_{\overline{q}}\left\{E_q^{iat}\right\} + L_{\overline{q}}\left\{E_q^{-iat}\right\}\right)$$
$$= \frac{1+q}{4qs}\left(\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \left(\frac{ia}{s}\right)^n + \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \left(\frac{-ia}{s}\right)^n\right)$$

$$
\begin{aligned}
&= \frac{1+q}{4qs} \sum_{n=0}^{\infty} \left[1+(-1)^n\right] q^{\frac{n(n-1)}{2}} \left(\frac{ia}{s}\right)^n \\
&= \frac{1+q}{2qs} \sum_{n=0}^{\infty} (-1)^n q^{n(2n-1)} \left(\frac{a}{s}\right)^{2n}
\end{aligned}
$$

and

$$
\begin{aligned}
L_{\overline{q}} \{\mathrm{Sin}_q\, at\} &= L_{\overline{q}} \left\{ \frac{E_q^{iat} - E_q^{-iat}}{2i} \right\} \\
&= \frac{1}{2i} \left(L_{\overline{q}} \left\{ E_q^{iat} \right\} - L_{\overline{q}} \left\{ E_q^{-iat} \right\} \right) \\
&= \frac{1+q}{4qsi} \left(\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \left(\frac{ia}{s}\right)^n - \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \left(\frac{-ia}{s}\right)^n \right) \\
&= \frac{1+q}{4qsi} \sum_{n=0}^{\infty} \left[1-(-1)^n\right] q^{\frac{n(n-1)}{2}} \left(\frac{ia}{s}\right)^n \\
&= \frac{1+q}{2qsi} \sum_{n=0}^{\infty} q^{n(2n+1)} \left(\frac{ia}{s}\right)^{2n+1} \\
&= \frac{1+q}{2qs} \sum_{n=0}^{\infty} (-1)^n q^{n(2n+1)} \left(\frac{a}{s}\right)^{2n+1}.
\end{aligned}
$$

So, the proof is completed. □

Theorem 4.7. *The $\overline{q}$-Laplace transform of hyperbolic q-cosine, hyperbolic q-sine functions are*

$$
\begin{aligned}
L_{\overline{q}} \{\cosh_q at\} &= \frac{1+q}{2q} \cdot \frac{s}{s^2 - a^2}, \\
L_{\overline{q}} \{\sinh_q at\} &= \frac{1+q}{2q} \cdot \frac{a}{s^2 - a^2}.
\end{aligned}
$$

Proof. Hyperbolic q-cosine, hyperbolic q-sine are defined by

$$
\cosh_q at = \frac{e_q^{at} + e_q^{-at}}{2} \quad \text{and} \quad \sinh_q at = \frac{e_q^{at} - e_q^{-at}}{2}.
$$

Then, by using linearity of $\overline{q}$-Laplace transform,

$$
\begin{aligned}
L_{\overline{q}} \{\cosh_q at\} &= L_{\overline{q}} \left\{ \frac{e_q^{at} + e_q^{-at}}{2} \right\} = \frac{1}{2} \left(L_{\overline{q}} \left\{ e_q^{at} \right\} + L_{\overline{q}} \left\{ e_q^{-at} \right\} \right) \\
&= \frac{1}{2} \left(\frac{1+q}{2q} \cdot \frac{1}{s-a} + \frac{1+q}{2q} \cdot \frac{1}{s+a} \right) \\
&= \frac{1+q}{2q} \cdot \frac{s}{s^2 - a^2}
\end{aligned}
$$

and in the same way we have

$$
L_{\overline{q}} \{\sinh_q at\} = \frac{1+q}{2q} \cdot \frac{a}{s^2 - a^2}.
$$

□

If $f(t)$ is piecewise continuous on the interval $(0,\infty)$ and of exponential order c, then $L_{\overline{q}}\{f(t)\}$ exists for $s>c$. Therefore, we obtain the following theorem.

Theorem 4.8. *If $f, D_q f, D_q^2 f, \ldots, D_q^{n-1} f$ are continuous and $D_q^n f$ is piecewise continuous on $(0,\infty)$ and are of exponential order then we have*

$$L_{\overline{q}}\left\{D_q^n f(t)\right\} = s^n L_{\overline{q}}\{f(t)\} - \frac{(1+q)}{2q}\sum_{i=0}^{n-1} s^{n-1-i} D_q^i f(0).$$

Proof. A function f is said to be of exponential order c if there exist c, $K>0$ and $T>0$ such that

$$|f(t)| \leq Ke^{ct}, \quad \text{for all } t<T.$$

Therefore, we have

$$\lim_{t\to\infty} E_q^{-qst} f(t) = 0. \tag{4.2}$$

Then, by using (4.2) we write

$$\begin{aligned} L_{\overline{q}}\{D_q f(t)\} &= \int_0^\infty E_q^{-qst} D_q f(t)\, d_{\overline{q}}t \\ &= \left.\frac{qE_q^{-qst} f(t) + E_q^{-q^2st} f(qt)}{2q}\right|_0^\infty - \int_0^\infty f(qt)\, D_q E_q^{-qst} d_{\overline{q}}t \\ &= -\frac{(1+q)}{2q} f(0) + qs\int_0^\infty f(qt)\, E_q^{-q^2st} d_{\overline{q}}t \\ &= -\frac{(1+q)}{2q} f(0) + s\int_0^\infty f(u)\, E_q^{-qsu} d_{\overline{q}}u \\ &= sL_{\overline{q}}\{f(t)\} - \frac{(1+q)}{2q} f(0). \end{aligned}$$

If we replace $f(t)$ by $D_q f(t)$ we have

$$\begin{aligned} L_{\overline{q}}\left\{D_q^2 f(t)\right\} &= \int_0^\infty E_q^{-qst} D_q^2 f(t)\, d_{\overline{q}}t \\ &= \left.\frac{qE_q^{-qst} D_q f(t) + E_q^{-q^2st} D_q f(qt)}{2q}\right|_0^\infty - \int_0^\infty D_q f(qt)\, D_q E_q^{-qst} d_{\overline{q}}t \\ &= -\frac{(1+q)}{2q} D_q f(0) + qs\int_0^\infty f(qt)\, E_q^{-q^2st} d_{\overline{q}}t \\ &= -\frac{(1+q)}{2q} D_q f(0) + s\int_0^\infty D_q f(t)\, E_q^{-qst} d_{\overline{q}}t \end{aligned}$$

$$
\begin{aligned}
&= sL_{\overline{q}}\left\{D_q f(t)\right\} - \frac{(1+q)}{2q} D_q f(0) \\
&= s\left[sL_{\overline{q}}\left\{f(t)\right\} - \frac{(1+q)}{2q} f(0)\right] - \frac{(1+q)}{2q} D_q f(0) \\
&= s^2 L_{\overline{q}}\left\{f(t)\right\} - \frac{(1+q)}{2q}\left(D_q f(0) + s f(0)\right).
\end{aligned}
$$

If we continue with this process, we get

$$
L_{\overline{q}}\left\{D_q^n f(t)\right\} = s^n L_{\overline{q}}\left\{f(t)\right\} - \frac{(1+q)}{2q}\sum_{i=0}^{n-1} s^{n-1-i} D_q^i f(0),
$$

and the proof is completed. □

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