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QUOTIENT HOOPS INDUCED BY QUASI-VALUATION MAPS

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Abstract. In this paper, our aim was making a metric space on hoop algebras, because of that, we introduced the notion of valuation maps from F-quasi-valuation map based on hoops and related properties of them are investigated. By using these notions, we introduced a quasi-metric space. The continuity of operations of a hoop is studied with topology induced by a quasi-valuation. Also, we studied hoop homomorphism and investigated that under which condition this homomorphism is an F-quasi-valuation map. Moreover, we wanted to find a congruence relation on hoops in a new way and study about the quotient structure that is made by it. So, we defined a congruence relation by F-quasi-valuation map and proved that the quotient is a hoop.

1. Introduction

Non-classical logic has become a considerable formal tool for computer science and artificial intelligence to deal with fuzzy information and uncertainty information. Many-valued logic, a great extension and development of classical logic, has always been a crucial direction in non-classical logic. In order to research the many-valued logical system whose propositional value is given in a lattice, Bosbach in [14, 15], proposed the concept of hoops, and discussed their some properties. Hoops are naturally ordered commutative residuated integral monoids. In the last years, hoops theory and related structues was enriched with deep structure theorems [1, 3–10, 12, 13, 16–18, 22, 24, 27]. Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops one obtains an elegant short proof of the completeness theorem for propositional basic logic, introduced by Hájek in [21]. The algebraic structures corresponding to Hájek's propositional

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(fuzzy) basic logic, BL-algebras, are particular cases of hoops. The main example of BL-algebras in interval $[0, 1]$ endowed with the structure induced by a t-norm. MV-algebras, product algebras and Gödel algebras are the most known classes of BL-algebras. Recent investigations are concerned with non-commutative generalizations for these structures. During these years, many researchers study on hoops in different way, and got some results on hoops [11, 20, 23, 26]. Algebra and topology, the two fundamental domains of mathematics, play complementary roles. Topology studies continuity and convergence and provides a general framework to study the concept of a limit. Algebra studies all kinds of operations and provides a basis for algorithms and calculations. Many of the most important objects of mathematics represent a blend of algebraic and of topological structures. Topological function spaces and linear topological spaces in general, topological groups and topological fields and topological lattices are objects of this kind. Very often an algebraic structure and a topology come naturally together. The rules that describe the relationship between a topology and algebraic operation are almost always transparent and natural the operation has to be continuous, jointly continuous, jointly or separately. In the 20th century many topologists and algebraists have contributed to topological algebra. Song, Roh and Jun, in [25] introduced the notion of quasi-valuation maps based on a subalgebra and an ideal in BCK/BCI-algebras, and then they investigated several properties. They provided relations between a quasi-valuation map based on a subalgebra and a quasi-valuation map based on an ideal. In a BCI-algebra, they gave a condition for a quasi-valuation map based on an ideal to be a quasi-valuation map based on a subalgebra, and found conditions for a real-valued function on a BCK/BCI-algebra to be a quasi-valuation map based on an ideal. Using the notion of a quasi-valuation map based on an ideal, they constructed (pseudo) metric spaces, and shew that the binary operation $\star$ in BCK-algebras is uniformly continuous. In [2], Aaly and Rezaei, introduced the notion of quasi-valuation maps such as ($S_{\odot}$, $S_{\rightarrow}$) S-quasi-valuation maps and F-quasi-valuation map based on subalgebras and filters and related properties of them are investigated. Also, they studied the relation between them and proved that every F-quasi-valuation map is an S-quasi-valuation map. Finally, by using the notion F-quasi-valuation map, they introduced a metric space and proved that if λ is an F-quasi-valuation map of hoop H then all operation of H are continuous.

In this paper, our aim was making a metric space on hoop algebras, because of that, we introduced the notion of valuation maps from F-quasi-valuation map based on hoops and related properties of them are investigated. By using these notions, we introduced a quasi-metric space. The continuity of operations of a hoop is studied with topology induced by a quasi-valuation. Also, we studied hoop homomorphism and investigated that under which condition these homomorphism is an F-quasi-valuation map. Moreover, we wanted to find a congruence relation on hoops in a new way and study about the quotient structure that is made by it. So, we defined a congruence relation by F-quasi-valuation map and proved that the quotient is a hoop.

2. Preliminaries

By a *hoop* we mean an algebra $(H, \odot, \to, 1)$ in which $(H, \odot, 1)$ is a commutative monoid and for all $x, y, z \in H$ the following assertions are valid:

(H1) $x \to x = 1$;
(H2) $x \odot (x \to y) = y \odot (y \to x)$;
(H3) $x \to (y \to z) = (x \odot y) \to z$.

We define a relation "$\leq$" on a hoop H by

$$(\forall x, y \in H)(x \leq y \Leftrightarrow x \to y = 1).$$

It is easy to see that $(H, \leq)$ is a poset. A hoop H is *bounded* if there is an element $0 \in H$ such that, for all $x \in H$, $0 \leq x$. Let $x^0 = 1$ and $x^n = x^{n-1} \odot x$, for any $n \in \mathbb{N}$. If H is a bounded hoop, then we define a negation " $'$ " on H such that, for all $x \in H$, $x' = x \to 0$. A nonempty subset S of H is called a *subhoop* of H if it satisfies:

$$(\forall x, y \in S)(x \odot y \in S,\ x \to y \in S).$$

Note that every subhoop contains the element 1.

Proposition 2.1 ([19])**.** *Let $(H, \odot, \to, 1)$ be a hoop. For any $x, y, z \in H$, the following conditions hold:*

($a1$) *$(H, \leq)$ is a meet-semilattice with $x \wedge y = x \odot (x \to y)$;*
($a2$) *$x \odot y \leq z$ if and anly if $x \leq y \to z$;*
($a3$) *$x \odot y \leq x, y$ and $x^n \leq x$ for any $n \in \mathbb{N}$;*
($a4$) *$x \leq y \to x$;*
($a5$) *$1 \to x = x$ and $x \to 1 = 1$;*
($a6$) *$x \odot (x \to y) \leq y$ and $x \odot y \leq x \wedge y \leq x \to y$;*
($a7$) *$x \to y \leq (y \to z) \to (x \to z)$;*
($a8$) *$x \leq y$ implies $x \odot z \leq y \odot z$, $z \to x \leq z \to y$ and $y \to z \leq x \to z$;*
($a9$) *$x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$.*

A nonempty subset F of a hoop H is called a *filter* of H (see [19]) if the following assertions are valid:

$$(\forall x, y \in H)(x, y \in F \Rightarrow x \odot y \in F), \tag{2.1}$$

$$(\forall x, y \in H)(x \in F,\ x \leq y \Rightarrow y \in F). \tag{2.2}$$

Note that the conditions (2.1) and (2.2) mean that F is closed under the operation $\odot$ and F is upward closed, respectively.

Note that a subset F of a hoop H is a filter of H if and only if the following assertions are valid (see [19]):

$$1 \in F,$$
$$(\forall x, y \in H)\,(x \to y \in F,\ x \in F \Rightarrow y \in F)\,.$$

Definition 2.1 ([2])**.** A real valued function λ of H is called

- an $S_{\odot}$-*quasi-valuation map* of H if

$$(\forall x, y \in H)(\lambda(x \odot y) \geq \lambda(x) + \lambda(y));$$

- an $S_{\rightarrow}$-*quasi-valuation map* of H if

$$(\forall x, y \in H)(\lambda(x \rightarrow y) \geq \lambda(x) + \lambda(y));$$

- an S-*quasi-valuation map* of H if it is an $S_{\odot}$-quasi-valuation map and an $S_{\rightarrow}$-quasi-valuation map of H.

Definition 2.2 ([2])**.** A real valued function λ of H is called an F-*quasi-valuation map of* H if

$$\begin{aligned} &\lambda(1) = 0, \\ &(\forall x, y \in H)(\lambda(y) \geq \lambda(x) + \lambda(x \rightarrow y)). \end{aligned}$$

Proposition 2.2 ([2])**.** *Let λ be an F-quasi-valuation map on H. Then the following statements hold:*
(*i*) *λ is an S-quasi-valuation map on H;*
(*ii*) *λ is an order preserving map;*
(*iii*) *for any $x \in H$, $\lambda(x) \leq 0$.*

Theorem 2.1 ([2])**.** *If an F-quasi-valuation map λ of H satisfies the following condition*

$$(\forall x \in H)(\lambda(x) = 0 \Rightarrow x = 1),$$

then (H, d_{λ}) is a metric space.

Note. In what follows, let H denote a hoop unless otherwise specified.

3. Quasi-Valuation Maps on Hoops

In this section, we introduce the notion of valuation maps from F-quasi-valuation map based on hoops and related properties of them are investigated. By using these notions, we introduce a quasi-metric space. The continuity of operations of a hoop was studied with topology induced by a quasi-valuation.

If a F-quasi-valuation map λ of H satisfies:

$$(\forall x \in H)\,(x \neq 1 \Rightarrow \lambda(x) \neq 0)\,,$$

then we say that λ is an F-*valuation map* of H.

Example 3.1. Let $H = \{0, a, b, 1\}$ be a set with Cayley tables (Table 1 and 2). Then $(H, \odot, \rightarrow, 1)$ is a bounded hoop. Define a map λ on H as follows:

$$\lambda : H \rightarrow \mathbb{R}, \quad x \mapsto \begin{cases} -30, & \text{if } x = 0, \\ -25, & \text{if } x = a, \\ -20, & \text{if } x = b, \\ 0, & \text{if } x = 1. \end{cases}$$

Table 1. Cayley table for the binary operation "$\odot$"

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Table 2. Cayley table for the binary operation "$\rightarrow$"

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

It is routine to verify that λ is an F-valuation map of H.

For any non-empty subset F of H and a negative real number k, define a real valued function λ_F on H as follows:

$$\lambda_F : H \rightarrow \mathbb{R}, \quad x \mapsto \begin{cases} 0, & \text{if } x \in F, \\ k, & \text{otherwise.} \end{cases} \tag{3.1}$$

Lemma 3.1 ([2])**.** *If F is a filter of H, then the function λ_F in* (3.1) *is an F-quasi-valuation map of H and $F_{\lambda_F} = F$.*

Theorem 3.1. *The function λ_F in* (3.1) *is an F-valuation map of H if and only if F is the trivial filter of H, that is $F = \{1\}$.*

Proof. Straightforward. □

In the following, we introduce quasi-metric space by using the notion of valuation maps from F-quasi-valuation map based on hoops. The continuity of operations of a hoop will study with topology induced by a quasi-valuation.

Definition 3.1. A function $d : H \times H \rightarrow \mathbb{R}$ is called a *quasi-metric* on H if it satisfies:

$$(\forall x, y \in H)\,(d(x, y) \leq 0,\ d(x, x) = 0)\,,$$
$$(\forall x, y \in H)\,(d(x, y) = d(y, x))\,,$$
$$(\forall x, y, z \in H)\,(d(x, z) \geq d(x, y) + d(y, z))\,.$$

We say that the pair (H, d) is a *quasi-metric space.*

Theorem 3.2. *If λ is an F-quasi-valuation map of H, then (H, d_λ) is a quasi-metric space which is called the quasi-metric space induced by λ, where*

$$d_\lambda : H \times H \rightarrow \mathbb{R}, \quad (x, y) \mapsto \lambda(x \rightarrow y) + \lambda(y \rightarrow x).$$

Proof. Since λ is an F-quasi-valuation map of H, by Proposition 2.2, λ is order preserving and for any $x \in H$, $\lambda(x) \leq 0$. Thus, according to definition of d_λ, it is clear that, for any $x, y \in H$, $d_\lambda(x, y) \leq 0$. Let $x \in H$. Then $d_\lambda(x, x) = \lambda(x \to x) = \lambda(1) = 0$. Also, for any $x, y \in H$,

$$d_\lambda(x, y) = \lambda(x \to y) + \lambda(y \to x) = \lambda(y \to x) + \lambda(x \to y) = d_\lambda(y, x).$$

Moreover, by Proposition 2.1 ($a7$), for any $x, y, z \in H$, $(x \to y) \odot (y \to z) \leq x \to z$. Since λ is an F-quasi-valuation map of H, by Proposition 2.2, λ is order preserving and S-quasi-valuation map of H. Then

$$\lambda(x \to y) + \lambda(y \to z) \leq \lambda((x \to y) \odot (y \to z)) \leq \lambda(x \to z).$$

By the similar way, $\lambda(z \to y) + \lambda(y \to x) \leq \lambda((z \to y) \odot (y \to x)) \leq \lambda(z \to x)$. Hence,

$$\begin{aligned} d_\lambda(x, y) + d_\lambda(y, z) &= \lambda(x \to y) + \lambda(y \to x) + \lambda(y \to z) + \lambda(z \to y) \\ &\leq \lambda(x \to z) + \lambda(z \to x) = d_\lambda(x, z). \end{aligned}$$

Therefore, (H, d_λ) is a quasi-metric space which is called the quasi-metric space induced by λ. □

Proposition 3.1. *Every quasi-metric space (H, d_λ) induced by an F-quasi-valuation map λ of H satisfies:*

$$\begin{aligned} d_\lambda(x, y) &\leq \min\{d_\lambda(x \to a, y \to a), d_\lambda(a \to x, a \to y)\}, \\ d_\lambda(x \to y, a \to b) &\geq d_\lambda(x \to y, a \to y) + d_\lambda(a \to y, a \to b), \\ d_\lambda(x \odot y, a \odot b) &\geq d_\lambda(x \odot y, a \odot y) + d_\lambda(a \odot y, a \odot b), \end{aligned}$$

for all $a, b, x, y \in H$.

Proof. Let (H, d_λ) be a quasi-metric space. By Proposition 2.1 ($a7$) for any $x, y, z \in H$, $x \to y \leq (y \to a) \to (x \to a)$ and $y \to x \leq (x \to a) \to (y \to a)$. Since λ is an F-quasi-valuation map of H, by Proposition 2.2, λ is order preserving and S-quasi-valuation map of H. Then

$$\begin{aligned} d_\lambda(x, y) \leq \lambda(x \to y) + \lambda(y \to x) &\leq \lambda((y \to a) \to (x \to a)) + \lambda((x \to a) \to (y \to a)) \\ &= d_\lambda(x \to a, y \to a). \end{aligned}$$

By the similar way, $d_\lambda(x, y) \leq d_\lambda(a \to x, a \to y)$. Hence,

$$d_\lambda(x, y) \leq \min\{d_\lambda(x \to a, y \to a), d_\lambda(a \to x, a \to y)\}.$$

Now, let $x, y, a \in H$. Then by Proposition 2.1 ($a7$), we have

$$((x \to y) \to (y \to a)) \odot ((y \to a) \to (a \to b)) \leq (x \to y) \to (a \to b).$$

By the similar way,

$$((a \to b) \to (a \to y)) \odot ((a \to y) \to (x \to y)) \leq (a \to b) \to (x \to y).$$

Since λ is an F-quasi-valuation map of H, by Proposition 2.2, λ is order preserving and S-quasi-valuation map of H. Then it is clear that

$$d_\lambda(x \to y, a \to y) + d_\lambda(a \to y, a \to b) \leq d_\lambda(x \to y, a \to b).$$

Also, since $y \odot a \leq y \odot a$, by Proposition 2.1 ($a2$) and ($a8$), we have $y \leq a \to (y \odot a)$, and so $x \to y \leq (x \odot a) \to (y \odot a)$. Then, by Proposition 2.2, λ is order preserving, thus, $\lambda(x \to y) \leq \lambda((x \odot a) \to (y \odot a))$. By the similar way, $\lambda(y \to x) \leq \lambda((y \odot a) \to (x \odot a))$. Hence,

$$\begin{aligned} d_\lambda(x, y) =& \lambda(x \to y) + \lambda(y \to x) \leq \lambda((x \odot a) \to (y \odot a)) + \lambda((y \odot a) \to (x \odot a)) \\ =& d_\lambda(x \odot a, y \odot a). \end{aligned}$$

Then, for any $x, y, a, b \in H$, since (H, d_λ) is a quasi-metric space, we have,

$$d_\lambda(x \odot y, a \odot y) + d_\lambda(a \odot y, a \odot b) \leq d_\lambda(x \odot y, a \odot b). \qquad \square$$

Theorem 3.3. *If λ is an F-valuation map of H, then the quasi-metric space induced by λ satisfies the following assertion,*

$$(\forall x, y \in H)\,(d_\lambda(x, y) = 0 \Rightarrow x = y)\,. \tag{3.2}$$

Proof. Let λ be an F-valuation map of H. Then λ is an F-quasi-valuation map of H. Thus, by Theorem 3.2, $d_\lambda(x, y)$ is quasi-metric. Now, for any $x, y \in H$, if $d_\lambda(x, y) = 0$, then $\lambda(x \to y) + \lambda(y \to x) = 0$. Since λ is an F-quasi-valuation map of H, by Proposition 2.2, for any $x \in H$, $\lambda(x) \leq 0$. So, by routine calculations, it is clear that $\lambda(x \to y) = \lambda(y \to x) = 0$, and so by Theorem 2.1, $x \to y = 1$ and $y \to x = 1$. Therefore, $x = y$. $\square$

We consider conditions for an F-quasi-valuation map to be an F-valuation map.

Theorem 3.4. *If the quasi-metric space (H, d_λ) induced by an F-quasi-valuation map λ of H satisfies the condition* (3.2)*, then λ is an F-valuation map of H.*

Proof. Let λ be an F-quasi-valuation map of H and there exists $1 \neq x \in H$ such that $\lambda(x) = 0$. Since λ is an F-quasi-valuation map of H that satisfying the condition (3.2), we have

$$d_\lambda(1, x) = \lambda(1 \to x) + \lambda(x \to 1) = \lambda(x) + \lambda(1) = 0.$$

Then $d_\lambda(1, x) = 0$. Since (H, d_λ) is a quasi metric, we have $x = 1$, which is a contradiction. Hence, for any $1 \neq x \in H$, $\lambda(x) \neq 0$. Therefore, λ is an F-valuation map of H. $\square$

Note. If (H, d_λ) is a quasi-metric space, then for any $x \in H$ and $\varepsilon < 0$ the set $B_\varepsilon(x) = \{y \in H \mid d_\lambda(x, y) > \varepsilon\}$ is called a *ball of radius* $|\varepsilon|$ *with center at* x. The set $U \subseteq H$ is open in (H, d_λ) if, for any $x \in U$, there is an $\varepsilon < 0$ such that $B_\varepsilon(x) \subseteq U$. The topology $\mathcal{T}_{d_\lambda}$ induced by d_λ is the collection of all open sets in (H, d_λ).

Theorem 3.5. *If $\mathcal{T}_\lambda$ is an induced topology by d_λ, then $(H, \odot, \to, \mathcal{T}_\lambda)$ is a topological hoop.*

Proof. By Theorem 3.3, (H, d_λ) is a quasi-metric space. Let $x, y \in H$ such that $x \to y \in B_\varepsilon(x \to y)$ for any $\varepsilon < 0$. We claim that $B_\varepsilon(x) \to B_\varepsilon(y) \subseteq B_\varepsilon(x \to y)$. For this, suppose $z \in B_\varepsilon(x) \to B_\varepsilon(y)$. Then there exist $p \in B_\varepsilon(x)$ and $q \in B_\varepsilon(y)$ such that $z = p \to q$. Thus, $d_\lambda(x, p) \geq \frac{\varepsilon}{2}$ and $d_\lambda(y, q) \geq \frac{\varepsilon}{2}$. By Proposition 3.1, it is clear that $d_\lambda(x \to y, p \to y) \geq d_\lambda(x, p)$ and $d_\lambda(p \to y, p \to q) \geq d_\lambda(y, q)$. Thus,

$$d_\lambda(x \to y, p \to q) \geq d_\lambda(x \to y, p \to y) + d_\lambda(p \to y, p \to q) \geq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, $d_\lambda(x \to y, p \to q) \geq \varepsilon$. Hence, $z \in B_\varepsilon(x \to y)$ and so, $(H, \to, \mathcal{T}_\lambda)$ is a topological hoop. By the similar way, we can prove that $(H, \odot, \mathcal{T}_\lambda)$ is a topological hoop. Therefore, $(H, \odot, \to, \mathcal{T}_\lambda)$ is a topological hoop. $\square$

Theorem 3.6. *For any F-quasi-valuation map λ of H, if we define a relation R_λ on H as follows:*

$$(\forall x, y)\left((x, y) \in R_\lambda \Leftrightarrow d_\lambda(x, y) = 0\right),$$

then R_λ is a congruence relation on H.

We say that R_λ is a congruence relation on H induced by λ.

Proof. Let $x, y, z \in H$. For proving that R_λ is a congruence relation on H, first of all we have to prove that R_λ is an equivalence relation on H. It is clear that R_λ is reflexive and symetric relation on H. Suppose $(x, y) \in R_\lambda$ and $(y, z) \in R_\lambda$. Then $d_\lambda(x, y) = 0$ and $d_\lambda(y, z) = 0$. By Proposition 2.1 $(a7)$, for any $x, y, z \in H$, we have $(x \to y) \odot (y \to z) \leq x \to z$. Since λ is an F-quasi-valuation map of H, by Proposition 2.2, λ is order preserving and S-quasi-valuation map of H, then $\lambda(x \to y) + \lambda(y \to z) \leq \lambda(x \to z)$. By the similar way, $\lambda(z \to y) + \lambda(y \to x) \leq \lambda(z \to x)$. Hence,

$$0 = \lambda(x \to y) + \lambda(y \to x) + \lambda(y \to z) + \lambda(z \to y) \leq \lambda(x \to z) + \lambda(z \to x) = d_\lambda(x, z).$$

Since λ is an F-quasi-valuation map of H, by Proposition 2.2, for any $x \in H$, $\lambda(x) \leq 0$. Then $d_\lambda(x, z) = 0$ and so $(x, z) \in R_\lambda$. Therefore, R_λ is a transitive relation on H. Now, we prove that R_λ is a congruence relation on H. For any $x, y, z \in H$ such that $(x, y) \in R_\lambda$. Since $y \leq z \to (y \odot z)$, by Proposition 2.1 $(a8)$,

$$x \to y \leq x \to (z \to (y \odot z)) = (x \odot z) \to (y \odot z).$$

Since λ is an F-quasi-valuation map of H, by Proposition 2.2, λ is order preserving, then $\lambda(x \to y) \leq \lambda((x \odot z) \to (y \odot z))$. By the similar way, it is clear that

$$\lambda(y \to x) \leq \lambda((y \odot z) \to (x \odot z)).$$

Hence,

$$0 = d_\lambda(x, y) \leq \lambda((x \odot z) \to (y \odot z)) + \lambda((y \odot z) \to (x \odot z)) = d_\lambda(x \odot z, y \odot z).$$

Since λ is an F-quasi-valuation map of H, by Proposition 2.2, for any $x \in H$, $\lambda(x) \leq 0$. Then $d_\lambda(x \odot z, y \odot z) = 0$ and so $(x \odot z, y \odot z) \in R_\lambda$. Moreover, if $(x, y) \in R_\lambda$, then by Proposition 3.1, it is clear that $(x \to z, y \to z) \in R_\lambda$ and $(z \to x, z \to y) \in R_\lambda$. Therefore, R_λ is a congruence relation on H induced by λ. $\square$

For any congruence relation R_λ induced by F-quasi-valuation map λ of H, let H_λ denote the set of all equivalence classes, that is,

$$H_\lambda := \{x_\lambda \mid x \in H\},$$

where $x_\lambda := \{y \in H \mid (x, y) \in R_\lambda\}$.

Theorem 3.7. *If λ is an F-quasi-valuation map of H, then $(H_\lambda, \boxdot, \twoheadrightarrow, 1_\lambda)$ is a hoop, where*

$$(\forall x_\lambda, y_\lambda \in H_\lambda)\,(x_\lambda \boxdot y_\lambda = (x \odot y)_\lambda,\ x_\lambda \twoheadrightarrow y_\lambda = (x \to y)_\lambda)\,.$$

Proof. Let $x \in H$. Then it is clear that $x_\lambda \twoheadrightarrow x_\lambda = (x \to x) = 1_\lambda$ and $(H_\lambda, \boxdot, 1_\lambda)$ is a commutative monoid. Suppose $x_\lambda, y_\lambda, z_\lambda \in H_\lambda$. Then

$$\begin{aligned}(x_\lambda \boxdot y_\lambda) \twoheadrightarrow z_\lambda =&(x \odot y)_\lambda \twoheadrightarrow z_\lambda = ((x \odot y) \to z)_\lambda = (x \to (y \to z))_\lambda\\ =&x_\lambda \twoheadrightarrow (y_\lambda \twoheadrightarrow z_\lambda).\end{aligned}$$

Moreover, by routine calculations, we have

$$x_\lambda \boxdot (x_\lambda \twoheadrightarrow y_\lambda) = (x \odot (x \to y))_\lambda = (y \odot (y \to x))_\lambda = y_\lambda \boxdot (y_\lambda \twoheadrightarrow x_\lambda).$$

Therefore, $(H_\lambda, \boxdot, \twoheadrightarrow, 1_\lambda)$ is a hoop. □

Theorem 3.7 is illustrated by the following example.

Example 3.2. According to Example 3.1, $H_\lambda = \{1_\lambda, x_\lambda, y_\lambda, 0_\lambda\}$.

Lemma 3.2 ([2]). *If $\lambda : H \to \mathbb{R}$ is an F-quasi-valuation map of H, then the set*

$$F_\lambda := \{x \in H \mid \lambda(x) = 0\}$$

is a filter of H.

Proposition 3.2. *If λ is an F-quasi-valuation map of H, then $F_\lambda = 1_\lambda$.*

Proof. Let λ be an F-quasi-valuation map of H. Then, by Lemma 3.2, we have

$$\begin{aligned}F_\lambda = \{x \in H \mid \lambda(x) = 0\} &= \{x \in H \mid \lambda(1 \to x) + \lambda(x \to 1) = 0\}\\ &= \{x \in H \mid d_\lambda(1, x) = 0\}\\ &= \{x \in H \mid (x, 1) \in R_\lambda\}\\ &= 1_\lambda.\end{aligned}$$

□

For any filter F of H, let η_F be a relation on H defined by

$$(\forall x, y \in H)\,((x, y) \in \eta_F \Leftrightarrow x \to y \in F,\ y \to x \in F)\,.$$

Then η_F is a congruence relation on H (induced by F). Denote by H/F the set of all equivalence classes, that is,

$$H/F := \{[x] \mid x \in H\},$$

where $[x] = \{y \in H \mid (x, y) \in \eta_F\}$.

Theorem 3.8. *If λ is an F-quasi-valuation map of H, then $\eta_{F_\lambda} = R_\lambda$.*

Proof. Let $x, y \in H$. Then

$$\begin{aligned}(x,y) \in \eta_{F_\lambda} &\Leftrightarrow x \to y \in F_\lambda \text{ and } y \to x \in F_\lambda \\ &\Leftrightarrow \lambda(x \to y) = \lambda(y \to x) = 0 \\ &\Leftrightarrow \lambda(x \to y) + \lambda(y \to x) = 0 \\ &\Leftrightarrow d_\lambda(x,y) = 0 \\ &\Leftrightarrow (x,y) \in R_\lambda.\end{aligned}$$ □

Theorem 3.9. *Let λ and g be F-quasi-valuation maps of H with $\lambda \neq g$. If $1_\lambda = 1_g$, then R_λ and R_g coincide and so $H_\lambda = H_g$.*

Proof. By routine calculations, we can see that $1_\lambda = \{x \in H \mid \lambda(x) = 0\}$. Suppose $x, y \in H$ such that $(x,y) \in R_\lambda$. Then $d_\lambda(x,y) = 0$ and so $\lambda(x \to y) + \lambda(y \to x) = 0$. Thus, $\lambda(x \to y) \geq -\lambda(y \to x)$. Since λ is an F-quasi-valuation map of H, by Proposition 2.2, we get that $\lambda(x \to y) = \lambda(y \to x) = 0$. Thus, $x \to y$, $y \to x \in 1_\lambda$. By assumption, $1_\lambda = 1_g$ we get that $x \to y$, $y \to x \in 1_g$, and so $g(x \to y) = g(y \to x) = 0$. Hence, $g(x \to y) + g(y \to x) = 0$, and so $d_g(x,y) = 0$. So $(x,y) \in R_g$. The proof of converse is similar. Therefore, R_λ and R_g coincide and so $H_\lambda = H_g$. □

Theorem 3.10. *For any filter F and any F-quasi-valuation map λ of H such that $1_\lambda \subseteq F$ consider the set*

$$\overline{F_\lambda} := \{x_\lambda \mid x \in F\}.$$

Then the following assertions are valid:

(1) $(\forall x \in H)\left(x \in F \Leftrightarrow x_\lambda \in \overline{F_\lambda}\right)$;

(2) F_λ *is a filter of* H_λ.

Proof. (1) It is clear that if $x \in F$, then $x_\lambda \in \overline{F_\lambda}$. Suppose $x_\lambda \in \overline{F_\lambda}$. Then there exists $y \in F$ such that $x_\lambda = y_\lambda$. Thus, $(x,y) \in R_\lambda$. Since R_λ is a congruence relation on H, we have $(y \to x, 1) = (y \to x, y \to y) \in R_\lambda$. Hence, $y \to x \in 1_\lambda$. Since $1_\lambda \subseteq F$, we have $y \to x \in F$. Moreover, $y \in F$ and F is a filter of H, then $x \in F$.

(2) Since F is a filter of H, $1 \in F$, and so $1_\lambda \in \overline{F_\lambda}$. Suppose $x_\lambda, x_\lambda \twoheadrightarrow y_\lambda \in \overline{F_\lambda}$. Then by (1), $x \in F$ and $x \to y \in F$. Since F is a filter of H, $y \in F$. Thus, by (1), $y_\lambda \in \overline{F_\lambda}$. Therefore, $\overline{F_\lambda}$ is a filter of H_λ. □

Proposition 3.3. *For any F-quasi-valuation map λ of H, let F^* be a filter of F_λ. Then the set*

$$F := \{x \in H \mid x_\lambda \in F^*\}$$

is a filter of H and $1_\lambda \subseteq F$.

Proof. Since F^* is a filter of $\overline{F_\lambda}$, $1_\lambda \in F^*$ and so $1 \in F$. Now, suppose $x, x \to y \in F$. Then $x_\lambda, (x \to y)_\lambda \in F^*$. Since F^* is a filter of $\overline{F_\lambda}$, we have $y_\lambda \in F^*$ and so $y \in F$. Hence, F is a filter of H. □

Let $\mathcal{F}(H_\lambda)$ denote the set of all filters of F_λ and let $\mathcal{F}(H,\lambda)$ denote the set of all filters of H containing 1_λ. Then there exists a bijection between $\mathcal{F}(H_\lambda)$ and $\mathcal{F}(H,\lambda)$, that is,

$$f:\mathcal{F}(H_\lambda)\to\mathcal{F}(H,\lambda),\quad F\mapsto F_\lambda$$

is a bijection.

Theorem 3.11. *Let $g:H\to G$ be a homomorphism of hoops. Then the following hold.*

(1) *If λ is an F-quasi-valuation map of G, then the composition $\lambda\circ g$ of λ and g is an F-quasi-valuation map of H.*

(2) *If g is an isomorphism and if λ is an F-quasi-valuation map of G, then $H_{\lambda\circ g}$ and G_λ are isomorphic.*

Proof. (1) Since g is a homomorphism of hoops, we have $(\lambda\circ g)(1)=\lambda(g(1))=\lambda(1)$. Since λ is an F-quasi-valuation map of H, we have $\lambda(1)=0$ and so $(\lambda\circ g)(1)=0$. Now, suppose $x,y\in H$. Since λ is an F-quasi-valuation map of H, we have

$$\begin{aligned}(\lambda\circ g)(x\to y)+(\lambda\circ g)(x)&=\lambda(g(x\to y))+\lambda(g(x))\\&=\lambda(g(x)\to g(y))+\lambda(g(x))\\&\le\lambda(g(y))=(\lambda\circ g)(y).\end{aligned}$$

(2) Let define the map $\phi:H_{\lambda\circ g}\to G_\lambda$ such that, for any $x_{\lambda\circ g}\in H_{\lambda\circ g}$, $\phi(x_{\lambda\circ g})=(g(x))_\lambda$. Now, we prove that ϕ is an isomorphism. For this, let $x_{\lambda\circ g},y_{\lambda\circ g}\in H_{\lambda\circ g}$. Then

$$\begin{aligned}\phi(x_{\lambda\circ g}\boxdot y_{\lambda\circ g})&=\phi((x\odot y)_{\lambda\circ g})\\&=(g(x\odot y))_\lambda=(g(x)\odot g(y))_\lambda\\&=(g(x))_\lambda\boxdot(g(y))_\lambda=\phi(x_{\lambda\circ g})\boxdot\phi(y_{\lambda\circ g})\end{aligned}$$

and

$$\begin{aligned}\phi(x_{\lambda\circ g}\twoheadrightarrow y_{\lambda\circ g})&=\phi((x\to y)_{\lambda\circ g})\\&=(g(x\to y))_\lambda=(g(x)\to g(y))_\lambda\\&=(g(x))_\lambda\twoheadrightarrow(g(y))_\lambda=\phi(x_{\lambda\circ g})\twoheadrightarrow\phi(y_{\lambda\circ g}),\end{aligned}$$

Hence, ϕ is a homomorphism of hoop.

Let $x,y\in H$ such that $\phi(x)=\phi(y)$. Then $\lambda\circ g(x)=\lambda\circ g(y)$. Thus, $g(x)_\lambda=g(y)_\lambda$, and so $(g(x),g(y))\in R_\lambda$. Hence, $d_\lambda(g(x),g(y))=0$. Since d_λ is a quasi-metric and g is an isomorphism, we have $g(x)=g(y)$ and so $x=y$. Hence, ϕ is a one to one homomorphism.

Let $x_\lambda\in G_\lambda$. Since g is unto, there exists $y\in G$, such that $g(y)_\lambda=x_\lambda$. Then $(\lambda\circ g)(y)=x_\lambda$, thus, $\phi(y)=x_\lambda$. Hence, ϕ is an isomorphism and so $H_{\lambda\circ g}$ and G_λ are isomorphic. □

Theorem 3.12. *For any F-quasi-valuation map λ of H, we have the following assertions.*

(1) *The map* $\pi : H \to H_\lambda$, $x \mapsto x_\lambda$, *is an onto homomorphism.*
(2) *For any* F*-quasi-valuation map* ϕ^* *of* H_λ, *there exists an* F*-quasi-valuation map* ϕ *of* H *such that* $\phi = \phi^* \circ \pi$.
(3) *The map* $\lambda^* : H_\lambda \to \mathbb{R}$, $x_\lambda \mapsto \lambda(x)$, *is an* F*-quasi-valuation map of* H_λ.

Proof. (1) By definition of H_λ, the proof is clear.

(2) Let define $\phi = \phi^*(x_\lambda)$. We show that ϕ is an F-quasi-valuation map of H. For this, since $1 \in H$, we have $\phi(1) = \phi^*(1_\lambda)$. Moreover, ϕ^* is an F-quasi-valuation map of H, $\phi(1) = 0$. Suppose $x, y \in H$ such that

$$\phi(x) + \phi(x \to y) = \phi^*(x_\lambda) + \phi^*(x_\lambda \twoheadrightarrow y_\lambda) \leq \phi^*(y_\lambda).$$

Since ϕ^* is an F-quasi-valuation map of H, we have $\phi(x) + \phi(x \to y) \leq \phi(y)$.

(3) Let $x_\lambda \in H_\lambda$. Since λ is an F-quasi-valuation map of H, we have

$$\lambda^*(x_\lambda) + \lambda^*(x_\lambda \twoheadrightarrow y_\lambda) = \lambda(x) + \lambda(x \to y) \leq \lambda(y) = \lambda^*(y_\lambda).$$ □

Proposition 3.4. *Let* H *and* G *be two hoops and* $\lambda : H \to \mathbb{R}$ *and* $\gamma : G \to \mathbb{R}$ *be quasi-valuations. If* $f : H \to G$ *is a homomorphism, then the following statements are equivalent:*
(i) f *is a quasi-valuation preserving;*
(ii) f *is an isometry.*

Proof. $(i) \Rightarrow (ii)$ Let f be a quasi-valuation preserving. Then, for any $x \in H$, define $\gamma(f(x)) = \lambda(x)$. For any $x, y \in H$, we have

$$\begin{aligned} d_\gamma(f(x), f(y)) &= \gamma(f(x) \to f(y)) + \gamma(f(y) \to f(x)) \\ &= \gamma(f(x \to y)) + \gamma(f(y \to x)) \\ &= \gamma \circ f(x \to y) + \gamma \circ f(y \to x) \\ &= \lambda(x \to y) + \lambda(y \to x) \\ &= d_\lambda(x, y). \end{aligned}$$

Hence, f is an isometry.

$(ii) \Rightarrow (i)$ Let f be an isometry. Then, for any $x \in H$,

$$\begin{aligned} \lambda(x) =& d_\lambda(x, 1) = d_\gamma(f(x), f(1)) = \gamma(f(x) \to f(1)) + \gamma(f(1) \to f(x)) = \gamma(f(x)) \\ =& \gamma \circ f(x). \end{aligned}$$

Hence, f is a quasi-valuation preserving. □

Proposition 3.5. *Let* $f : H \to G$ *be a hoop isomorphism. If* λ *is a quasi-valuation on* H, *then* $\gamma : G \to \mathbb{R}$ *that, for any* $y \in G$, *is defined by* $\gamma(y) = \lambda \circ f^{-1}(y)$ *is a quasi-valuation. Moreover, if* λ *is an* F*-quasi-valuation on* H, *then* γ *is an* F*-quasi-valuation on* G.

Proof. Let $y_1, y_2 \in G$. Since f is an isomorphism, there exist $x_1, x_2 \in H$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then

$$\begin{aligned}\gamma(y_1 \to y_2) &= \lambda \circ f^{-1}(y_1 \to y_2) = \lambda(f^{-1}(y_1 \to y_2)) \\ &= \lambda(f^{-1}(y_1) \to f^{-1}(y_2)) = \lambda(x_1 \to x_2) \\ &\geq \lambda(x_1) + \lambda(x_2) = \lambda(f^{-1}(y_1)) + \lambda(f^{-1}(y_2)) \\ &= \gamma(y_1) + \gamma(y_2).\end{aligned}$$

By the similar way, we can prove that $\gamma(y_1 \odot y_2) \geq \gamma(y_1) + \gamma(y_2)$. Hence, γ is a quasi-valuation.

Since f is a hoop isomorphism, it is clear that $f(1_H) = 1_G$. Since λ is an F-quasi-valuation on H, we have $\gamma(1_G) = \lambda \circ f^{-1}(1_G) = \lambda(f^{-1}(1_G)) = \lambda(1_H) = 0$, and so $\gamma(1_G) = 0$. Let $x, y \in H$ and λ be an F-quasi-valuation on H. Since f is an isomorphism, there exist $a, b \in H$ such that $f(a) = x$ and $f(b) = y$. Then

$$\begin{aligned}\gamma(y) &= \lambda \circ f^{-1}(y) = \lambda(f^{-1}(y)) = \lambda(b) \\ &\geq \lambda(a \to b) + \lambda(a) = \lambda(f^{-1}(x)) + \lambda(f^{-1}(x) \to f^{-1}(y)) \\ &= \lambda(f^{-1}(x)) + \lambda(f^{-1}(x \to y)) = \lambda \circ f^{-1}(x) + \lambda \circ f^{-1}(x \to y) \\ &= \gamma(x) + \gamma(x \to y).\end{aligned}$$

□

4. Conclusions and Future Works

In this paper, our aim was making a metric space on hoop algebras, because of that we introduced the notion of valuation maps from F-quasi-valuation map based on hoops and related properties of them are investigated. By using these notions, we introduce a quasi-metric space. The continuity of operations of a hoop was studied with topology induced by a quasi-valuation. Also, we study hoop homomorphism and investigate that under which condition these homomorphism is an F-quasi-valuation map. Moreover, we wanted to find a congruence relation on hoops in a new way and study about the quotient structure that is made by it. Because of that, we define a congruence relation by F-quasi-valuation map and prove that the quotient is a hoop. In our future work, we want to study about the product of finite number of this quasi-metric space and investigate that the quotient space of hoop has a quasi-metric or not. Finally we study the completion of this pace.

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