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STRUCTURE OF 3-PRIME NEAR RINGS WITH GENERALIZED (σ,τ)-n-DERIVATIONS

ASMA ALI[1], ABDELKARIM BOUA[2], AND INZAMAM UL HUQUE[2]

Abstract. In this paper, we define generalized (σ,τ)-n-derivation for any mappings σ and τ of a near ring N and also investigate the structure of a 3-prime near ring satisfying certain identities with generalized (σ,τ)-n-derivation. Moreover, we characterize the aforementioned mappings.

1. Introduction

A left near ring N is a triplet $(N,+,.)$, where $+$ and $.$ are two binary operations such that $(i)\,(N,+)$ is a group (not necessarily abelian); $(ii)\,(N,.)$ is a semigroup, and $(iii)\,x.(y+z)=x.y+x.z$ for all $x,y,z\in N$. Analogously, if N satisfies the right distributive law, i.e., $(x+y).z=x.z+y.z$ for all $x,y\in N$, then N is said to be a right near ring. The most natural example of a left near ring is the set of all identity preserving mappings acting from right of an additive group G (not necessarily abelian) into itself with pointwise addition and composition of mappings as multiplication. If these mappings act from left on G, then we get a right near ring (Pilz [10, Example 1.4]). Throughout the paper, N denotes a zero-symmetric left near ring with multiplicative centre Z and for any pair of elements $x,y\in N$, $[x,y]=xy-yx$, $x\circ y=xy+yx$ and $(x,y)=x+y-x-y$ stand for the Lie product, Jordan Product and additive commutator respectively. Let σ and τ be mappings on N. For any $x,y\in N$, set the symbol $[x,y]_{\sigma,\tau}$ will denote $x\sigma(y)-\tau(y)x$, while the symbol $(x\circ y)_{\sigma,\tau}$ will denote $x\sigma(y)+\tau(y)x$. The terminology multiplicative mappings on a near ring N is used for the mappings $\sigma,\tau:N\to N$ satisfying $\sigma(xy)=\sigma(x)\sigma(y)$

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and $\tau(xy) = \tau(x)\tau(y)$ for all $x, y \in N$. A near ring N is called zero-symmetric if $0x = 0$, for all $x \in N$ (recall that left distributivity yields that $x0 = 0$). A near ring N is said to be 3-prime if $xNy = \{0\}$ for $x, y \in N$ implies that $x = 0$ or $y = 0$. A near ring N is called 2-torsion free if $(N, +)$ has no element of order 2. A nonempty subset U of N is called a semigroup right (resp. semigroup left) ideal if $UN \subseteq U$ (resp. $NU \subseteq U$) and if U is both a semigroup right ideal and a semigroup left ideal, it is called a semigroup ideal.

Let $n \geq 2$ be a fixed positive integer and $N^n = \underbrace{N \times N \times \cdots \times N}_{n-\text{times}}$. A map $\Delta : N^n \to N$ is said to be permuting (symmetric) on a near ring N if the relation $\Delta(x_1, x_2, \ldots, x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ holds for all $x_i \in N$, $i = 1, 2, \ldots, n$, and for every permutation $\pi \in S_n$, where S_n is the permutation group on $\{1, 2, \ldots, n\}$. An additive mapping $F : N \to N$ is said to be a right (resp. left) generalized derivation with associated derivation d if $F(xy) = F(x)y + xd(y)$ (resp. $F(xy) = d(x)y + xF(y)$), for all $x, y \in N$ and F is said to be a generalized derivation with associated derivation d on N if it is both a right generalized derivation and a left generalized derivation on N with associated derivation d.

Ozturk et al. [9] and Park et al. [6] studied bi-derivations and tri-derivations in near rings. Further, Ceven et al. [4] and Ozturk et al. [8] defined (σ, τ) bi-derivations and (σ, τ) tri-derivations in near rings. Let σ, τ be automorphisms on a near ring N. A symmetric bi-additive (additive in both arguments) mapping $d : N \times N \to N$ is said to be a (σ, τ) bi-derivation if $d(xx', y) = d(x, y)\sigma(x') + \tau(x)d(x', y)$ holds for all $x, x', y \in N$. A symmetric tri-additive (additive in each argument) mapping $d : N \times N \times N \to N$ is said to be a (σ, τ) tri-derivation if $d(xx', y, z) = d(x, y, z)\sigma(x') + \tau(x)d(x', y, z)$ holds for all $x, x', y, z \in N$.

Motivated by these concepts, we define (σ, τ)-n-derivation and generalized (σ, τ)-n-derivation for any arbitrary mappings σ and τ of a near ring N in place of automorphisms.

Definition 1.1 ((σ, τ)-n-derivation)**.** Let $\sigma, \tau : N \to N$ be mappings on N. An n-additive (additive in each argument) mapping $d : \underbrace{N \times N \times \cdots \times N}_{n-\text{times}} \to N$ is called a (σ, τ)-n-derivation of N if the following equations

$$\begin{aligned}
d(x_1x_1', x_2, \ldots, x_n) =& d(x_1, x_2, \ldots, x_n)\sigma(x_1') + \tau(x_1)d(x_1', x_2, \ldots, x_n),\\
d(x_1, x_2x_2', \ldots, x_n) =& d(x_1, x_2, \ldots, x_n)\sigma(x_2') + \tau(x_2)d(x_1, x_2', \ldots, x_n),\\
&\vdots\\
d(x_1, x_2, \ldots, x_nx_n') =& d(x_1, x_2, \ldots, x_n)\sigma(x_n') + \tau(x_n)d(x_1, x_2, \ldots, x_n')
\end{aligned}$$

hold for all $x_1, x_1', x_2, x_2', \ldots, x_n, x_n' \in N$.

Definition 1.2 (Right generalized (σ, τ)-n-derivation)**.** An n-additive (additive in each argument) mapping $F : \underbrace{N \times N \times \cdots \times N}_{n-\text{times}} \to N$ is called a right generalized

(σ,τ)-n-derivation associated with (σ,τ)-n-derivation d on N if the relations

$$\begin{aligned}
F(x_1x_1',x_2,\dots,x_n) =& F(x_1,x_2,...,x_n)\sigma(x_1')+\tau(x_1)d(x_1',x_2,\dots,x_n),\\
F(x_1,x_2x_2',\dots,x_n) =& F(x_1,x_2,\dots,x_n)\sigma(x_2')+\tau(x_2)d(x_1,x_2',\dots,x_n),\\
&\vdots\\
F(x_1,x_2,\dots,x_nx_n') =& F(x_1,x_2,\dots,x_n)\sigma(x_n')+\tau(x_n)d(x_1,x_2,\dots,x_n')
\end{aligned}$$

hold for all $x_1,x_1',x_2,x_2',\dots,x_n,x_n'\in N$.

Definition 1.3 (Left generalized (σ,τ)-n-derivation)**.** An n-additive (additive in each argument) mapping $F:\underbrace{N\times N\times\cdots\times N}_{n-\text{times}}\to N$ is called a left generalized (σ,τ)-n-derivation associated with (σ,τ)-n-derivation d on N if the relations

$$\begin{aligned}
F(x_1x_1',x_2,\dots,x_n) =& d(x_1,x_2,\dots,x_n)\sigma(x_1')+\tau(x_1)F(x_1',x_2,\dots,x_n),\\
F(x_1,x_2x_2',\dots,x_n) =& d(x_1,x_2,\dots,x_n)\sigma(x_2')+\tau(x_2)F(x_1,x_2',\dots,x_n),\\
&\vdots\\
F(x_1,x_2,\dots,x_nx_n') =& d(x_1,x_2,\dots,x_n)\sigma(x_n')+\tau(x_n)F(x_1,x_2,\dots,x_n')
\end{aligned}$$

hold for all $x_1,x_1',x_2,x_2',\dots,x_n,x_n'\in N$.

A mapping $F:\underbrace{N\times N\times\cdots\times N}_{n-\text{times}}\to N$ is called a generalized (σ,τ)-n-derivation associated with (σ,τ)-n-derivation d on N if F is both a right generalized (σ,τ)-n-derivation and a left generalized (σ,τ)-n-derivation associated with (σ,τ)-n-derivation d on N.

Example 1.1. *Let S be a zero-symmetric left near ring and*

$$N=\left\{\begin{pmatrix}0&x&y\\0&0&z\\0&0&0\end{pmatrix}\mid x,y,z,0\in S\right\}.$$

Then N is a zero-symmetric left near ring with respect to matrix addition and matrix multiplication. Define mappings $F,d:\underbrace{N\times N\times\cdots\times N}_{n-times}\to N$ by

$$F\left(\begin{pmatrix}0&x_1&y_1\\0&0&z_1\\0&0&0\end{pmatrix},\begin{pmatrix}0&x_2&y_2\\0&0&z_2\\0&0&0\end{pmatrix},\dots,\begin{pmatrix}0&x_n&y_n\\0&0&z_n\\0&0&0\end{pmatrix}\right)=\begin{pmatrix}0&0&0\\0&0&z_1z_2\dots z_n\\0&0&0\end{pmatrix},$$

$$d\left(\begin{pmatrix}0&x_1&y_1\\0&0&z_1\\0&0&0\end{pmatrix},\begin{pmatrix}0&x_2&y_2\\0&0&z_2\\0&0&0\end{pmatrix},\dots,\begin{pmatrix}0&x_n&y_n\\0&0&z_n\\0&0&0\end{pmatrix}\right)=\begin{pmatrix}0&x_1x_2\dots x_n&0\\0&0&0\\0&0&0\end{pmatrix}.$$

Define $\sigma, \tau : N \to N$ *by*

$$\sigma \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & xy & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that F *is a nonzero right (but not left) generalized* (σ, τ)*-*n*-derivation associated with a nonzero* (σ, τ)*-*n*-derivation* d *of* N*, where* σ *and* τ *are any arbitrary mappings on* N*.*

Example 1.2. *Let* N *be a zero-symmetric left near ring as in Example* 1.1. *Define mappings* $F, d : \underbrace{N \times N \times \cdots \times N}_{n-times} \to N$ *by*

$$F\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & x_1x_2\ldots x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$d\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z_1z_2\ldots z_n \\ 0 & 0 & 0 \end{pmatrix}.$$

Define $\sigma, \tau : N \to N$ *by*

$$\sigma \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x^2 & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & z^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be easily seen that F *is a nonzero left (but not right) generalized* (σ, τ)*-*n*-derivation associated with a nonzero* (σ, τ)*-*n*-derivation* d *of* N *for any arbitrary mappings* σ *and* τ *on* N*.*

Example 1.3. *Let* S *be a zero-symmetric left near ring and*

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid x, y, z, 0 \in S \right\}.$$

It is easy to see that N *is a zero-symmetric left near ring with respect to matrix addition and matrix multiplication. Define mappings* $F, d : \underbrace{N \times N \times \cdots \times N}_{n-times} \to N$ *by*

$$F\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & z_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & z_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & z_n & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & y_1y_2\ldots y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$d\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & z_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & z_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & z_n & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & z_1z_2\ldots z_n & 0 \end{pmatrix}.$$

Define $\sigma, \tau : N \to N$ *by*

$$\sigma\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & x^2 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & yz & 0 \end{pmatrix}.$$

It can be easily verified that F *is a nonzero right as well as left generalized* (σ, τ)*-*n*-derivation associated with a nonzero* (σ, τ)*-*n*-derivation* d *of* N*, where* σ *and* τ *are any arbitrary mappings on* N*.*

Obviously this notion covers the notion of a generalized n-derivation (in case $\sigma = \tau = I$), notion of an n-derivation (in case $F = d$, $\sigma = \tau = I$), notion of a left n-centralizer (in case $d = 0$, $\sigma = I$), notion of a (σ, τ)-n-derivation (in case $F = d$) and the notion of a left σ-n-multiplier (in case $d = 0$). Thus, it is interesting to investigate the properties of this general notion. In [7], Bresar has proved that if R is a 2-torsion free semiprime ring and $F : R \to R$ is an additive map on R such that $F(x)x + xF(x) = 0$ for all $x \in R$, then $F = 0$. Further, Vukman [5] proved that if there exist a derivation $d : R \to R$ and an automorphism $\alpha : R \to R$, where R is 2-torsion free semiprime ring such that $[d(x)x + xd(x), x] = 0$ for all $x \in R$, then d and $\alpha - I$, I denotes the identity mapping on R, map R into its centre. Motivated by the mentioned results we prove that if a 3-prime near ring N with a generalized (σ, τ)-n-derivation F satisfies certain identity, then N is a commutative ring and F is a left σ-n-multiplier on N.

2. Some Preliminaries

Lemma 2.1. ([1, Lemmas 1.2]). *Let* N *be* 3*-prime near ring.*

(i) *If* $z \in Z \setminus \{0\}$*, then* z *is not a zero divisor.*
(ii) *If* $Z \setminus \{0\}$ *and* x *is an element of* N *for which* $xz \in Z$*, then* $x \in Z$*.*

Lemma 2.2. ([1, Lemmas 1.3 and Lemma 1.4]). *Let* N *be* 3*-prime near ring and* U *be a nonzero semigroup ideal of* N*.*

(i) *If* $x, y \in N$ *and* $xUy = \{0\}$*, then* $x = 0$ *or* $y = 0$*.*
(ii) *If* $x \in N$ *and* $xU = \{0\}$ *or* $Ux = \{0\}$*, then* $x = 0$*.*

Lemma 2.3. ([1, Lemma 1.5]). *If* N *is a* 3*-prime near ring and* Z *contains a nonzero semigroup left ideal or a nonzero semigroup right ideal, then* N *is a commutative ring.*

Lemma 2.4. If N is a 3-prime near ring admitting a generalized (σ, τ)-n-derivation F associated with a (σ, τ)-n-derivation d of N such that σ and τ are multiplicative mappings on N, then

$$\begin{aligned} &\{d(x_1, x_2, \ldots, x_n)\sigma(y_1) + \tau(x_1)F(y_1, x_2, \ldots, x_n)\}\sigma(z_1) \\ =&d(x_1, x_2, \ldots, x_n)\sigma(y_1)\sigma(z_1) + \tau(x_1)F(y_1, x_2, \ldots, x_n)\sigma(z_1), \\ &\{d(x_1, x_2, \ldots, x_n)\sigma(y_2) + \tau(x_2)F(x_1, y_2, \ldots, x_n)\}\sigma(z_2) \\ =&d(x_1, x_2, \ldots, x_n)\sigma(y_2)\sigma(z_2) + \tau(x_2)F(x_1, y_2, \ldots, x_n)\sigma(z_2), \end{aligned}$$

$$\begin{aligned}&\vdots\\&\{d(x_1,x_2,\dots,x_n)\sigma(y_n)+\tau(x_n)F(x_1,x_2,\dots,y_n)\}\sigma(z_n)\\=&d(x_1,x_2,\dots,x_n)\sigma(y_n)\sigma(z_n)+\tau(x_n)F(x_1,x_2,\dots,y_n)\sigma(z_n),\end{aligned}$$

for all $x_1,y_1,z_1,x_2,y_2,z_2,\dots,x_n,y_n,z_n\in N$.

Proof. For all $x_1,y_1,z_1,x_2,y_2,z_2,\dots,x_n,y_n,z_n\in N$

$$\begin{aligned}F(x_1y_1z_1,x_2,\dots,x_n)=&F(x_1y_1,x_2,\dots,x_n)\sigma(z_1)+\tau(x_1y_1)d(z_1,x_2,\dots,x_n)\\=&\{d(x_1,x_2,\dots,x_n)\sigma(y_1)+\tau(x_1)F(y_1,x_2,\dots,x_n)\}\sigma(z_1)\\&+\tau(x_1)\tau(y_1)d(z,u_2,\dots,u_n)\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}F(x_1y_1z_1,x_2,\dots,x_n)=&d(x_1,x_2,\dots,x_n)\sigma(y_1z_1)+\tau(x_1)F(y_1z_1,x_2,\dots,x_n)\\=&d(x_1,x_2,\dots,x_n)\sigma(y_1)\sigma(z_1)+\tau(x_1)F(y_1,x_2,\dots,x_n)\sigma(z_1)\\&+\tau(x_1)\tau(y_1)d(z_1,x_2,\dots,x_n).\end{aligned}\tag{2.2}$$

Combining (2.1) and (2.2), we get

$$\begin{aligned}&\{d(x_1,x_2,\dots,x_n)\sigma(y_1)+\tau(x_1)F(y_1,x_2,\dots,x_n)\}\sigma(z_1)\\=&d(x_1,x_2,\dots,x_n)\sigma(y_1)\sigma(z_1)+\tau(x_1)F(y_1,x_2,\dots,x_n)\sigma(z_1).\end{aligned}$$

Similarly, we can prove other relations for $i=2,3,\dots,n$. □

Remark 2.1. If σ is an onto map on N, then Lemma 2.4 becomes

$$\begin{aligned}&\{d(x_1,x_2,\dots,x_n)\sigma(y_1)+\tau(x_1)F(y_1,x_2,\dots,x_n)\}a\\=&d(x_1,x_2,\dots,x_n)\sigma(y_1)a+\tau(x_1)F(y_1,x_2,\dots,x_n)a,\\&\{d(x_1,x_2,\dots,x_n)\sigma(y_2)+\tau(x_2)F(x_1,y_2,\dots,x_n)\}a\\=&d(x_1,x_2,\dots,x_n)\sigma(y_2)a+\tau(x_2)F(x_1,y_2,\dots,x_n)a,\\&\vdots\\&\{d(x_1,x_2,\dots,x_n)\sigma(y_n)+\tau(x_n)F(x_1,x_2,\dots,y_n)\}a\\=&d(x_1,x_2,\dots,x_n)\sigma(y_n)a+\tau(x_n)F(x_1,x_2,\dots,y_n)a,\end{aligned}$$

for all $x_1,y_1,x_2,y_2,\dots,x_n,y_n,a\in N$.

Lemma 2.5. Let N be a 3-prime near ring and $U_1,U_2,\dots,U_n$ be nonzero semigroup ideals of N. Let σ and τ be mappings on N such that $U_i\subseteq\tau(U_i)$ for $i=1,2,\dots,n$. If d is a nonzero (σ,τ)-n-derivation on N, then $d(U_1,U_2,\dots,U_n)\neq\{0\}$.

Proof. Assume that

$$d(u_1,u_2,\dots,u_n)=0,\quad\text{for all } u_1\in U_1,u_2\in U_2,\dots,u_n\in U_n.\tag{2.3}$$

Replacing u_1 by u_1r_1, where $r_1\in N$ in (2.3) and using (2.3), we get

$$\tau(u_1)d(r_1,u_2,\dots,u_n)=0.$$

Since $U_i \subseteq \tau(U_i)$ for $i = 1, 2, \ldots, n$, we have $U_1 d(r_1, u_2, \ldots, u_n) = \{0\}$. Applying Lemma 2.2 (ii), we obtain $d(r_1, u_2, \ldots, u_n) = 0$ for all $u_2 \in U_2, \ldots, u_n \in U_n$ and $r_1 \in N$. Replacing u_2 by $u_2 r_2$, where $r_2 \in N$ in the last expression and another application of Lemma 2.2(ii) yields that $d(r_1, r_2, \ldots, u_n) = 0$. Proceeding inductively, we conclude that $d(r_1, r_2, \ldots, r_n) = 0$ for all $r_1, r_2, \ldots, r_n \in N$, a contradiction which completes the proof. □

Lemma 2.6. Let N be a 3-prime near-ring and $U_1, U_2, \ldots, U_n$ be nonzero semigroup ideals of N. Let σ, τ be multiplicative mappings on U_i such that $U_1 \subseteq \sigma(U_1)$. If d is a nonzero (σ, τ)-n-derivation on N such that $d(U_1, U_2, \ldots U_n)\sigma(a) = \{0\}$ or $\sigma(a) d(U_1, U_2, \ldots U_n) = \{0\}$ for all $a \in N$, then $\sigma(a) = 0$.

Proof. Suppose that $d(U_1, U_2, \ldots, U_n)\sigma(a) = \{0\}$. Then

$$d(u_1, u_2, \ldots, u_n)\sigma(a) = 0, \quad \text{for all } u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{2.4}$$

Replacing u_1 by $u_1 u_1'$ in (2.4) and using it again yields that

$$d(u_1, u_2, \ldots, u_n)\sigma(u_1')\sigma(a) = 0, \quad \text{for all } u_1, u_1' \in U_1, u_2 \in U_2, \ldots, u_n \in U_n.$$

Equivalently,

$$d(u_1, u_2, \ldots, u_n)\sigma(U_1)\sigma(a) = \{0\}, \quad \text{for all } u_1, \in U_1, u_2 \in U_2, \ldots, u_n \in U_n.$$

Since $U_1 \subseteq \sigma(U_1)$, we obtain

$$d(u_1, u_2, \ldots, u_n) U_1 \sigma(a) = \{0\}, \quad \text{for all } u_1, \in U_1, u_2 \in U_2, \ldots, u_n \in U_n.$$

Applying Lemma 2.2 (i) and Lemma 2.5, we obtain $\sigma(a) = 0$. Similarly, we can prove the result for later case. □

Lemma 2.7. Let N be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ be nonzero semigroup ideals of N. Let σ be a onto map on N such that $U_1 \subseteq \sigma(U_1)$ and $U_1 \cap Z \neq \emptyset$. If d is a (σ, σ)-n-derivation on N, then $d(Z, U_2, U_3, \ldots, U_n) \subseteq Z$.

Proof. Suppose that $z \in U_1 \cap Z$. Then

$$d(zx_1, x_2, \ldots, x_n) = d(x_1 z, x_2, \ldots, x_n), \quad \text{for all } x_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n,$$

and

$$\begin{aligned} &d(z, x_2, \ldots, x_n)\sigma(x_1) + \sigma(z) d(x_1, x_2, \ldots, x_n) \\ =&\sigma(x_1) d(z, x_2, \ldots, x_n) + d(x_1, x_2, \ldots, x_n)\sigma(z). \end{aligned}$$

Substituting $x_1' \in U_1$ and $z' \in U_1 \cap Z$ for $\sigma(x_1)$ and $\sigma(z)$ respectively, we get

$$d(z, x_2, \ldots, x_n) x_1' = x_1' d(z, x_2, \ldots, x_n), \quad \text{for all } x_1' \in U_1, x_2 \in U_2, \ldots, x_n \in U_n.$$

Replacing x_1' by $x_1' r$ for $r \in N$ in above expression and using it again, we find that $x_1'[d(z, x_2, \ldots, x_n), r] = 0$. Hence, $d(Z, U_2, U_3, \ldots, U_n) \subseteq Z$ by Lemma 2.2 (ii). □

Lemma 2.8. Let N be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ be nonzero semigroup ideals of N. Let σ, τ be mappings on N such that $U_i \subseteq \sigma(U_i)$ and $U_i \subseteq \tau(U_i)$ for $i = 1, 2, \ldots, n$. If F is a nonzero right generalized (σ, τ)-n-derivation associated with a (σ, τ)-n-derivation d on N, then $F(U_1, U_2, \ldots, U_n) \neq \{0\}$.

Proof. Let

$$F(u_1, u_2, \ldots, u_n) = 0, \quad \text{for all } u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{2.5}$$

Replacing u_1 by $u_1 r_1$, where $r_1 \in N$ in (2.5) and using (2.5), we get

$$\tau(u_1)d(r_1, u_2, \ldots, u_n) = \{0\}.$$

Since $U_1 \subseteq \tau(U_1)$, we have

$$U_1 d(r_1, u_2, \ldots, u_n) = \{0\}, \quad \text{for all } u_2 \in U_2, \ldots, u_n \in U_n \text{ and } r_1 \in N.$$

Applying Lemma 2.2(ii), we find

$$d(r_1, u_2, \ldots, u_n) = 0, \quad \text{for all } u_2 \in U_2, \ldots, u_n \in U_n \text{ and } r_1 \in N. \tag{2.6}$$

Now replacing u_2 by $u_2 r_2$ in (2.6) for $r_2 \in N$ and another application of Lemma 2.2 (ii) yields that $d(r_1, r_2, u_3, \ldots, u_n) = 0$ for all $u_3 \in U_3, \ldots, u_n \in U_n$ and $r_1, r_2 \in N$. Proceeding inductively, we get $d(r_1, r_2, \ldots, r_n) = 0$ for all $r_1, r_2, \ldots, r_n \in N$, i.e., $d = 0$. Therefore, our hypothesis reduces to

$$F(r_1 u_1, u_2, \ldots, u_n) = F(r_1, u_2, \ldots, u_n)\sigma(u_1) = 0,$$

for all $u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$ and $r_1 \in N$ which implies that

$$F(r_1, u_2, \ldots, u_n) = 0, \quad \text{for all } u_2 \in U_2, \ldots, u_n \in U_n \text{ and } r_1 \in N. \tag{2.7}$$

Replacing u_2 by $r_2 u_2$ in (2.7), we get $F(r_1, r_2, \ldots, u_n)U_2 = \{0\}$ and Lemma 2.2 (ii) gives $F(r_1, r_2, u_3, \ldots, u_n) = 0$ for all $u_3 \in U_3, \ldots, u_n \in U_n$ and $r_1, r_2 \in N$. Proceeding inductively, we obtain $F = 0$ on N, a contradiction. ☐

3. Main Results

Theorem 3.1. *Let N be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of N. Suppose that σ, τ are multiplicative mappings on U_i for $i = 1, 2, \ldots, n$, such that $U_i \subseteq \tau(U_i)$ for $i = 1, 2, \ldots, n$, and σ is onto on N. If N admits a generalized (σ, τ)-n-derivation F associated with a (σ, τ)-n-derivation d such that $F(x_1 x_1', x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)F(x_1', x_2, \ldots, x_n)$ for all $x_1, x_1' \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$, then F is a left σ-n-multiplier on N.*

Proof. By hypothesis

$$\begin{aligned} F(x_1 x_1', x_2, \ldots, x_n) &= d(x_1, x_2, \ldots, x_n)\sigma(x_1') + \tau(x_1)F(x_1', x_2, \ldots, x_n) \\ &= F(x_1, x_2, \ldots, x_n)F(x_1', x_2, \ldots, x_n), \end{aligned}$$

for all $x_1, x_1' \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$. Replacing x_1' by $x_1'z$ for $z \in U_1$ in the above relation, we get

$$\begin{aligned}&\{d(x_1, x_2, \ldots, x_n)\sigma(x_1') + \tau(x_1)F(x_1', x_2, \ldots, x_n)\}F(z, x_2, \ldots, x_n)\\ =&d(x_1, x_2, \ldots, x_n)\sigma(x_1'z) + \tau(x_1)\{d(x_1', x_2, \ldots, x_n)\sigma(z) + \tau(x_1')F(z, x_2, \ldots, x_n)\}.\end{aligned}$$

Applying Lemma 2.4 and using the hypothesis, we obtain

$$\begin{aligned}&d(x_1, x_2, \ldots, x_n)\sigma(x_1')F(z, x_2, \ldots, x_n) + \tau(x_1)d(x_1', x_2, \ldots, x_n)\sigma(z)\\ &+ \tau(x_1)\tau(x_1')F(z, x_2, \ldots, x_n)\\ =&d(x_1, x_2, \ldots, x_n)\sigma(x_1'z) + \tau(x_1)d(x_1', x_2, \ldots, x_n)\sigma(z) + \tau(x_1)\tau(x_1')F(z, x_2, \ldots, x_n),\end{aligned}$$

which reduces to

$$d(x_1, x_2, \ldots, x_n)\sigma(x_1')(F(z, x_2, \ldots, x_n) - \sigma(z)) = 0,$$

for all $x_1, x_1', z \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$. This implies that

$$d(x_1, x_2, \ldots, x_n)U_1(F(z, x_2, \ldots, x_n) - \sigma(z)) = \{0\}.$$

By Lemma 2.2 (i), we obtain $d(x_1, x_2, \ldots, x_n) = 0$ or $F(z, x_2, \ldots, x_n) = \sigma(z)$ for all $z \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$.

If $F(z, x_2, \ldots, x_n) = \sigma(z)$ for all $z \in U_1$, replacing z by zt, we get

$$\tau(z)d(t, x_2, \ldots, x_n) = 0.$$

Putting $u \in U_1$ in place of $\tau(z)$ and using Lemma 2.2 (ii), we obtain $d(t, x_2, \ldots, x_n) = 0$ for all $t \in U_1$. Therefore, in both cases we arrive at $d(U_1, U_2, \ldots, U_n) = \{0\}$. Now arguing in the similar manner as we have done in Lemma 2.5, we can get $d = 0$ on N, which completes the proof. ☐

Theorem 3.2. *Let N be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ be nonzero semigroup ideals of N. Suppose that σ is a multiplicative mapping on U_i for $i = 1, 2, \ldots, n$, such that $U_i \subseteq \sigma(U_i)$ for $i = 1, 2, \ldots, n$. If N admits a nonzero generalized (σ, σ)-n-derivation F associated with a (σ, σ)-n-derivation d such that $F(U_1, U_2, \ldots, U_n) \subseteq Z(N)$, then N is a commutative ring.*

Proof. If $d \neq 0$, then for all $u_1, u_1' \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$

$$F(u_1u_1', u_2, \ldots, u_n) = d(u_1, u_2, \ldots, u_n)\sigma(u_1') + \sigma(u_1)F(u_1', u_2, \ldots, u_n) \in Z(N). \tag{3.1}$$

Now commuting (3.1) with the element $\sigma(u_1)$ and using Lemma 2.4, we get

$$d(u_1, u_2, \ldots, u_n)\sigma(u_1')\sigma(u_1) = \sigma(u_1)d(u_1, u_2, \ldots, u_n)\sigma(u_1').$$

Since σ is an onto map on N, replacing $\sigma(u_1')$ by $r_1 \in N$ in above expression, we find that

$$d(u_1, u_2, \ldots, u_n)r_1\sigma(u_1) = \sigma(u_1)d(u_1, u_2, \ldots, u_n)r_1. \tag{3.2}$$

Substituting r_1r_2 where $r_2 \in N$ in place of r_1 in (3.2) and using it again, we obtain

$$d(u_1, u_2, \ldots, u_n)N[\sigma(u_1), r_2] = \{0\}.$$

By 3-primeness of N, we get $d(u_1, u_2, \dots, u_n) = 0$ or $[\sigma(u_1), r] = 0$ for all $u_1 \in U_1, u_2 \in U_2, \dots, u_n \in U_n$ and $r \in N$.

Case 1. Suppose there exists $x_0 \in U_1$ such that $d(x_0, u_2, \dots, u_n) = 0$ for all $u_2 \in U_2, \dots, u_n \in U_n$. Then

$$F(u_1 x_0, u_2, \dots, u_n) = F(u_1, u_2, \dots, u_n)\sigma(x_0) \in Z(N),$$

for all $u_1 \in U_1, u_2 \in U_2, \dots, u_n \in U_n$. Since $F(u_1, u_2, \dots, u_n) \neq 0$, then $\sigma(x_0) \in Z(N)$ by Lemma 2.1 (ii).

Case 2. Suppose there exists $x_0 \in U_1$ such that $[\sigma(x_0), r] = 0$ for all $r \in N$, then $\sigma(x_0) \in Z(N)$.

In both cases, we obtain $\sigma(U_1) \subseteq Z(N)$ which implies that $U_1 \subseteq Z(N)$. Hence, by Lemma 2.3, we conclude that N is a commutative ring.

Assume that $d = 0$, then another application of Lemma 2.1 (ii) and Lemma 2.8, our hypothesis gives $U_1 \subseteq Z(N)$ and N is a commutative ring by Lemma 2.3. □

The following example shows that the 3-primeness hypothesis in Theorem 3.2 can not be omitted.

Example 3.1. *Let us consider Example* 1.3. *Consider*

$$U = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid x, y, z, 0 \in S \right\}.$$

Then clearly U is a nonzero semigroup ideal of a non 3*-prime zero-symmetric left near ring N. If we choose $U_1 = U_2 = \dots = U_n = U$, then $F(U_1, U_2, \dots, U_n) \subseteq Z(N)$. However, N is not commutative.*

Theorem 3.3. Let N be a 3-prime near-ring and $U_1, U_2, \dots, U_n$ are nonzero semigroup ideals of N. Suppose that σ, τ are multiplicative mappings on U_i for $i = 1, 2, \dots, n$, such that $U_i \subseteq \sigma(U_i)$, $U_i \subseteq \tau(U_i)$ for $i = 1, 2, \dots, n$, and σ is onto on N. If N admits a generalized (σ, τ)-n-derivation F associated with a (σ, τ)-n-derivation d such that $F(x_1 x_1', x_2, \dots, x_n) = F(x_1', x_2, \dots, x_n)F(x_1, x_2, \dots, x_n)$ for all $x_1, x_1' \in U_1, x_2 \in U_2, \dots, x_n \in U_n$, then N is commutative ring.

Proof. By hypothesis,

$$\begin{aligned} F(x_1 x_1', x_2, \dots, x_n) &= d(x_1, x_2, \dots, x_n)\sigma(x_1') + \tau(x_1)F(x_1', x_2, \dots, x_n) \\ &= F(x_1', x_2, \dots, x_n)F(x_1, x_2, \dots, x_n), \end{aligned} \tag{3.3}$$

for all $x_1, x_1' \in U_1, x_2 \in U_2, \dots, x_n \in U_n$. Substituting $x_1 x_1'$ for x_1' in (3.3) and using Remark 2.1, we obtain

$$\begin{aligned} F(x_1(x_1 x_1'), x_2, \dots, x_n) =& F(x_1 x_1', x_2, \dots, x_n)F(x_1, x_2, \dots, x_n) \\ =& d(x_1, x_2, \dots, x_n)\sigma(x_1')F(x_1, x_2, \dots, x_n) \\ &+ \tau(x_1)F(x_1', x_2, \dots, x_n)F(x_1, x_2, \dots, x_n). \end{aligned}$$

Also, using the definition of F, we get

$$\begin{aligned}F(x_1(x_1x_1'), x_2, \ldots, x_n) =& d(x_1, x_2, \ldots, x_n)\sigma(x_1x_1') + \tau(x_1)F(x_1x_1', x_2, \ldots, x_n)\\ =& d(x_1, x_2, \ldots, x_n)\sigma(x_1)\sigma(x_1')\\ &+ \tau(x_1)F(x_1', x_2, \ldots, x_n)F(x_1, x_2, \ldots, x_n).\end{aligned}$$

By comparing the last two equations, we can easily arrive at

$$d(x_1, x_2, \ldots, x_n)\sigma(x_1')F(x_1, x_2, \ldots, x_n) = d(x_1, x_2, \ldots, x_n)\sigma(x_1)\sigma(x_1'). \tag{3.4}$$

Since σ is onto on N, we get

$$d(x_1, x_2, \ldots, x_n)r_1F(x_1, x_2, \ldots, x_n) = d(x_1, x_2, \ldots, x_n)\sigma(x_1)r_1.$$

Now substituting r_1r_2 for r_1 in above expression and using it again, we find that

$$d(x_1, x_2, \ldots, x_n)N[F(x_1, x_2, \ldots, x_n), r_2] = \{0\},$$

for all $x_1, \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$ and $r_2 \in N$. Since N is 3-prime, we have $d(x_1, x_2, \ldots, x_n) = 0$ or $F(x_1, x_2, \ldots, x_n) \in Z(N)$ for all $x_1, \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$. Using the same argument as used in the proof of the Lemma 2.5 and Theorem 3.2, we conclude that N is a commutative ring. □

Theorem 3.4. *Let N be a 3-prime near-ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of N. Let σ be an automorphism and τ be a homomorphism on N such that $U_1 \subseteq \sigma(U_1)$ and $U_i \subseteq \tau(U_i)$ for $i = 1, 2, \ldots, n$. If N admits a left generalized (σ,τ)-n-derivation F associated with a (σ,τ)-n-derivation d such that $F([x, y], u_2, \ldots, u_n) = \pm\tau([x, y])$ for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, then N is a commutative ring.*

Proof. By hypothesis

$$F([x, y], u_2, \ldots, u_n) = \pm\tau([x, y]), \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.5}$$

Replacing y by xy in (3.5) and using $[x, xy] = x[x, y]$, we get

$$d(x, u_2, \ldots, u_n)\sigma([x, y]) + \tau(x)F([x, y], u_2, \ldots, u_n) = \pm(\tau(x)\tau(xy) - \tau(x)\tau(yx)),$$

which reduces to

$$d(x, u_2, \ldots, u_n)\sigma([x, y]) = 0, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.6}$$

This implies that

$$d(x, u_2, \ldots, u_n)\sigma(x)\sigma(y) = d(x, u_2, \ldots, u_n)\sigma(y)\sigma(x).$$

Substituting yz in place of y, where $z \in N$ in the last expression and using it again, we find that

$$d(x, u_2, \ldots, u_n)\sigma(y)[\sigma(x), \sigma(z)] = 0.$$

Since $U_1 \subseteq \sigma(U_1)$, then Lemma 2.2 (i) yields that $d(x, u_2, \ldots, u_n) = 0$ or $\sigma(x) \in Z(N)$ for all $x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. Since σ is an automorphism on N, then $d(x, u_2, \ldots, u_n) = 0$ or $x \in Z(N)$ for all $x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. Using Lemma

2.7, we get $d(U_1, U_2, \ldots, U_n) \in Z(N)$ which forces that N is a commutative ring by Theorem 3.2 which completes the proof. $\square$

Theorem 3.5. *Let N be a 2-torsion free 3-prime near-ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of N. Let σ be an automorphism on N and τ be a homomorphism on N such that $U_1 \subseteq \sigma(U_1)$ and $U_i \subseteq \tau(U_i)$ for $i = 1, 2, \ldots, n$. Then N admits no left generalized (σ,τ)-n-derivation F associated with a nonzero (σ,τ)-n-derivation d satisfying one of the following conditions:*

(i) $F(x \circ y, u_2, \ldots, u_n) = \pm\tau([x, y])$ *for all* $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$*;*
(ii) $F(x \circ y, u_2, \ldots, u_n) = \pm\tau(x \circ y)$ *for all* $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$*;*
(iii) $F(x \circ y, u_2, \ldots, u_n) = 0$ *for all* $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$*.*

Proof. (i) Assume that

$$F(x \circ y, u_2, \ldots, u_n) = \pm\tau([x, y]), \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.7}$$

Replacing y by xy in (3.7), we get

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) + \tau(x)F(x \circ y, u_2, \ldots, u_n) = \pm(\tau(x)\tau(xy) - \tau(x)\tau(yx)),$$

which implies that

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) + \tau(x)F(x \circ y, u_2, \ldots, u_n) = \pm\tau(x)\tau([x, y]).$$

Using the hypothesis, we find that

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) = 0, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n,$$

which implies that

$$d(x, u_2, \ldots, u_n)\sigma(y)\sigma(x) = -d(x, u_2, \ldots, u_n)\sigma(x)\sigma(y). \tag{3.8}$$

Substituting yz for y in (3.8) where $z \in N$, we have

$$\begin{aligned} d(x, u_2, \ldots, u_n)\sigma(y)\sigma(z)\sigma(x) &= -d(x, u_2, \ldots, u_n)\sigma(x)\sigma(y)\sigma(z) \\ &= d(x, u_2, \ldots, u_n)\sigma(x)\sigma(y)(-\sigma(z)) \\ &= (-d(x, u_2, \ldots, u_n)\sigma(y)\sigma(x))(-\sigma(z)) \\ &= d(x, u_2, \ldots, u_n)\sigma(y)(-\sigma(x))(-\sigma(z)) \\ &= d(x, u_2, \ldots, u_n)\sigma(y)\sigma(-x)\sigma(-z), \end{aligned}$$

which implies that

$$\begin{aligned} 0 &= d(x, u_2, \ldots, u_n)\sigma(y)(\sigma(z)\sigma(x) - \sigma(-x)\sigma(-z)) \\ &= d(x, u_2, \ldots, u_n)\sigma(y)(-\sigma(z)\sigma(-x) + \sigma(-x)\sigma(z)). \end{aligned}$$

Since $U_1 \subseteq \sigma(U_1)$, Lemma 2.2 (i) yields that

$$d(x, u_2, \ldots, u_n) = 0 \text{ or } \sigma(-x) \in Z(N), \quad \text{for all } x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.9}$$

Suppose there exists $x_0 \in U_1$ such that $\sigma(-x_0) \in Z(N)$. Since $-U_1$ is a nonzero semigroup left ideal of N, replacing x and y by $-x_0$ in (3.8), we get

$$2d(-x_0, u_2, \ldots, u_n)\sigma(-x_0)\sigma(-x_0) = 0,$$

for all $u_2 \in U_2, \dots, u_n \in U_n$. Using 2-torsion freeness of N, we conclude that $d(-x_0, u_2, \dots, u_n)N\sigma(-x_0)N\sigma(-x_0) = \{0\}$ for all $u_2 \in U_2, \dots, u_n \in U_n$. By 3-primeness of N, we arrive at $d(-x_0, u_2, \dots, u_n) = 0$ or $\sigma(-x_0) = 0$ for all $u_2 \in U_2, \dots, u_n \in U_n$. Since σ is an automorphism of N, by (3.9) we get $d(x, u_2, \dots, u_n) = 0$ for all $x \in U_1, u_2 \in U_2, \dots, u_n \in U_n$, so $d(U_1, U_2, \dots, U_n) = \{0\}$, which contradicts Lemma 2.5.

(ii) Suppose that

$$F(x \circ y, u_2, \dots, u_n) = \pm\tau(x \circ y), \quad \text{for all } x, y \in U_1, u_2 \in U_2, \dots, u_n \in U_n. \tag{3.10}$$

Replacing y by xy in (3.10), we get

$$d(x, u_2, \dots, u_n)\sigma(x \circ y) + \tau(x)F(x \circ y, u_2, \dots, u_n) = \pm\tau(x)\tau(x \circ y),$$

which reduces to

$$d(x, u_2, \dots, u_n)\sigma(y)\sigma(x) = -d(x, u_2, \dots, u_n)\sigma(x)\sigma(y). \tag{3.11}$$

Since (3.11) is same as (3.8), arguing in the similar manner as in (i), we find a contradiction with our hypothesis.

Using the same techniques, we can prove the result for (iii). $\square$

Theorem 3.6. Let N be a 3-prime near ring and $U_1, U_2, \dots, U_n$ are nonzero semigroup ideals of N. Let σ be an homomorphism on N such that $U_i \subseteq \sigma(U_i)$ for $i = 1, 2, \dots, n$. If N admits a left generalized (σ,σ)-n-derivation F associated with a (σ,σ)-n-derivation d such that $F([x, y], u_2, \dots, u_n) = [\sigma(x), y]_{\sigma,\sigma}$ for all $x, y \in U_1, u_2 \in U_2, \dots, u_n \in U_n$, then F is a right σ-n-multiplier on N or N is commutative.

Proof. By hypothesis

$$F([x, y], u_2, \dots, u_n) = [\sigma(x), y]_{\sigma,\sigma}, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \dots, u_n \in U_n. \tag{3.12}$$

Replacing y by xy in (3.12), we get

$$d(x, u_2, \dots, u_n)\sigma([x, y]) + \sigma(x)F([x, y], u_2, \dots, u_n) = \sigma(x)[\sigma(x), y]_{\sigma,\sigma},$$

which reduces to

$$d(x, u_2, \dots, u_n)\sigma([x, y]) = 0, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \dots, u_n \in U_n. \tag{3.13}$$

As (3.13) is same as (3.6), arguing in the similar manner as in Theorem 3.4, we obtain the result. $\square$

Theorem 3.7. *Let N be a 2-torsion free 3-prime near-ring and $U_1, U_2, \dots, U_n$ are nonzero semigroup ideals of N. Let σ be a homomorphism on N such that $U_i \subseteq \sigma(U_i)$ for $i = 1, 2, \dots, n$. Then N admits no left generalized (σ,σ)-n-derivation F associated with a nonzero (σ,σ)-n-derivation d satisfying one of the following conditions:*

(i) *$F(x \circ y, u_2, \dots, u_n) = [\sigma(x), y]_{\sigma,\sigma}$ for all $x, y \in U_1, u_2 \in U_2, \dots, u_n \in U_n$;*
(ii) *$F(x \circ y, u_2, \dots, u_n) = (\sigma(x) \circ y)_{\sigma,\sigma}$ for all $x, y \in U_1, u_2 \in U_2, \dots, u_n \in U_n$.*

Proof. (i) Suppose that

$$F(x \circ y, u_2, \ldots, u_n) = [\sigma(x), y]_{\sigma,\sigma}, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.14}$$

Replacing y by xy in (3.14), we get

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) + \sigma(x)F(x \circ y, u_2, \ldots, u_n) = \sigma(x)[\sigma(x), y]_{\sigma,\sigma},$$

which reduces to

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) = 0, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.15}$$

Since (3.15) is same as (3.8), arguing as in the proof of Theorem 3.5, we find that $d(x, u_2, \ldots, u_n) = 0$ for all $x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$ or N is a commutative ring. If N is a commutative ring, then our hypothesis becomes

$$2F(xy, u_2, \ldots, u_n) = 0,$$

for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. By 2-torsion freeness of N, we have $F(xy, u_2, \ldots, u_n) = 0$ for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. This implies that

$$d(x, u_2, \ldots, u_n)\sigma(y) + \sigma(x)F(y, u_2, \ldots, u_n) = 0.$$

Replacing y by yz in last expression, we obtain $d(x, u_2, \ldots, u_n)\sigma(y)\sigma(z) = 0$ for all $x, y, z \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$ which implies that $d(x, u_2, \ldots, u_n)\sigma(U_1)\sigma(z) = \{0\}$ for all $x, z \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. Since $U_1 \subseteq \sigma(U_1)$, we get

$$d(x, u_2, \ldots, u_n)U_1\sigma(z) = \{0\},$$

for all $x, z \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. Using Lemma 2.2 (i), we have $d(x, u_2, \ldots, u_n) = 0$ for all $x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$ or $\sigma(U_1) = U_1 = \{0\}$. Since $U_1 \neq \{0\}$, we conclude that $d(U_1, U_2, \ldots, U_n) = \{0\}$ which contradicts Lemma 2.5.

(ii) Assume that

$$F(x \circ y, u_2, \ldots, u_n) = (\sigma(x) \circ y)_{\sigma,\sigma}, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.16}$$

Substituting xy for y in (3.16), we have

$$\begin{aligned} F(x(x \circ y), u_2, \ldots, u_n) &= \sigma(x)\sigma(xy) + \sigma(xy)\sigma(x), \\ d(x, u_2, \ldots, u_n)\sigma(x \circ y) + \sigma(x)F(x \circ y, u_2, \ldots, u_n) &= \sigma(x)(\sigma(x) \circ y)_{\sigma,\sigma}, \end{aligned}$$

which implies that

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) = 0, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n.$$

Arguing in the similar manner as we have done above, we obtain $d(x, u_2, \ldots, u_n) = 0$ for all $x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, we again get a contradiction. □

Theorem 3.8. *Let N be a 3-prime near-ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of N. Let σ be an homomorphism on N such that $U_i \subseteq \sigma(U_i)$ for $i = 1, 2, \ldots, n$. If N admits a left generalized (σ, σ)-n-derivation F associated with a nonzero (σ, σ)-n-derivation d such that $F([x, y], u_2, \ldots, u_n) = [d(x, u_2, \ldots, u_n), \sigma(y)]$ for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, then N is a commutative ring.*

Proof. Suppose that for all $x, y \in U_1, u_2 \in U_2, \dots, u_n \in U_n$

$$F([x,y],u_2,\dots,u_n) = [d(x,u_2,\dots,u_n),\sigma(y)]. \tag{3.17}$$

Replacing y by xy in (3.17), we get

$$d(x,u_2,\dots,u_n)\sigma([x,y]) + \sigma(x)F([x,y],u_2,\dots,u_n) = [d(x,u_2,\dots,u_n),\sigma(xy)].$$

In view of our hypothesis, the above expression gives

$$\begin{aligned}&d(x,u_2,\dots,u_n)\sigma(xy) - d(x,u_2,\dots,u_n)\sigma(yx) + \sigma(x)d(x,u_2,\dots,u_n)\sigma(y)\\ &- \sigma(x)\sigma(y)d(x,u_2,\dots,u_n)\\ =&d(x,u_2,\dots,u_n)\sigma(xy) - \sigma(xy)d(x,u_2,\dots,u_n),\end{aligned}$$

which implies that

$$d(x,u_2,\dots,u_n)\sigma(y)\sigma(x) = \sigma(x)d(x,u_2,\dots,u_n)\sigma(y). \tag{3.18}$$

Replacing y by yu in the last equation and using it, we can easily arrive at

$$d(x,u_2,\dots,u_n)\sigma(y)[\sigma(x),\sigma(u)] = 0.$$

Since $U_1 \subseteq \sigma(U_1)$, by Lemma 2.2 (i), we conclude that

$$d(x,u_2,\dots,u_n) = 0 \quad \text{or} \quad \sigma(x) \in Z(U_1), \quad \text{for all } x \in U_1, u_2 \in U_2, \dots, u_n \in U_n. \tag{3.19}$$

Suppose there exists $x_0 \in U$ such that $\sigma(x_0) \in Z(U_1)$. Then $\sigma(x_0)v = v\sigma(x_0)$ for all $v \in U_1$ and replacing v by vn, where $n \in N$ and using it, we conclude that $U[\sigma(x_0), n] = \{0\}$ for all $n \in N$ by Lemma 2.2 (ii), we conclude that $\sigma(x_0) \in Z(N)$. In this case, (3.19) becomes

$$d(x,u_2,\dots,u_n) = 0 \quad \text{or} \quad \sigma(x) \in Z(N) \quad \text{for all } x \in U_1, u_2 \in U_2, \dots, u_n \in U_n. \tag{3.20}$$

In all cases, the equation (3.17) becomes

$$F([x,y],u_2,\dots,u_n) = 0, \quad \text{for all } x, y \in U_1, u_2 \in U_2, \dots, u_n \in U_n. \tag{3.21}$$

This equation is a special case of Theorem 3.4 with $\tau = 0$, which is already treated previously. □

Theorem 3.9. *Let N be a 2-torsion free 3-prime near ring and $U_1, U_2, \dots, U_n$ are nonzero semigroup ideals of N. Let σ be an automorphism on N such that $U_i \subseteq \sigma(U_i)$ for $i = 1, 2, \dots, n$. Then N admits no left generalized (σ,σ)-n-derivation F associated with a nonzero (σ,σ)-n-derivation d satisfying one of the following conditions:*

(i) $F(x \circ y, u_2, \dots, u_n) = d(x,u_2,\dots,u_n) \circ \sigma(y)$;
(ii) $F(x \circ y, u_2, \dots, u_n) = [d(x,u_2,\dots,u_n), \sigma(y)]$,

for all $x, y \in U_1, u_2 \in U_2, \dots, u_n \in U_n$.

Proof. (i) By hypothesis, for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$

$$F(x \circ y, u_2, \ldots, u_n) = d(x, u_2, \ldots, u_n) \circ \sigma(y). \tag{3.22}$$

Substituting xy for y in (3.22) and using $(x \circ xy) = x(x \circ y)$, we obtain

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) + \sigma(x)F(x \circ y, u_2, \ldots, u_n) = d(x, u_2, \ldots, u_n) \circ \sigma(xy).$$

Using the hypothesis, we find that

$$d(x, u_2, \ldots, u_n)\sigma(y)\sigma(x) = -\sigma(x)d(x, u_2, \ldots, u_n)\sigma(y). \tag{3.23}$$

Replacing y by yz where $z \in N$ in the last expression and using the same steps that we introduced previously, we obtain $d(x, u_2, \ldots, u_n)\sigma(y)(-\sigma(z)\sigma(-x) + \sigma(-x)\sigma(z)) = 0$ for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n, z \in N$. Since $\sigma(U_1) = U_1$ and invoking Lemma 2.2 (i) and Lemma 2.3, we conclude that $d(x, u_2, \ldots, u_n) = 0$ or $\sigma(-x) \in Z(N)$.

Suppose there exists $x_0 \in U$ such that $\sigma(-x_0) \in Z(N)$. Since $-U_1$ is a nonzero semigroup left ideal of N, replacing x and y by $-x_0$ in (3.23), we get

$$2d(-x_0, u_2, \ldots, u_n)\sigma(-x_0)\sigma(-x_0) = 0, \quad \text{for all } u_2 \in U_2, \ldots, u_n \in U_n.$$

Using 2-torsion freeness of N, we conclude that

$$d(-x_0, u_2, \ldots, u_n)N\sigma(-x_0)N\sigma(-x_0) = \{0\},$$

for all $u_2 \in U_2, \ldots, u_n \in U_n$. By 3-primeness of N, we arrive at $d(-x_0, u_2, \ldots, u_n) = 0$ or $\sigma(-x_0) = 0$ for all $u_2 \in U_2, \ldots, u_n \in U_n$. Since σ is an automorphism of N, by (3.9) we get $d(x, u_2, \ldots, u_n) = 0$ for all $x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, so $d(U_1, U_2, \ldots, U_n) = \{0\}$, which contradicts Lemma 2.5.

(ii) By hypothesis, we have for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$

$$F(x \circ y, u_2, \ldots, u_n) = [d(x, u_2, \ldots, u_n), \sigma(y)]. \tag{3.24}$$

Substituting xy for y in (3.24) and using $(x \circ xy) = x(x \circ y)$, we obtain

$$d(x, u_2, \ldots, u_n)\sigma(x \circ y) + \sigma(x)F(x \circ y, u_2, \ldots, u_n) = [d(x, u_2, \ldots, u_n), \sigma(xy)],$$

which reduces to

$$d(x, u_2, \ldots, u_n)\sigma(y)\sigma(x) = -\sigma(x)d(x, u_2, \ldots, u_n)\sigma(y). \tag{3.25}$$

(3.25) is same as (3.23), arguing in the similar manner as above, we conclude that $d(U_1, U_2, \ldots, U_n) = \{0\}$, which leads to a contradiction. □

Theorem 3.10. *Let N be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of N. Let σ be an homomorphism on N such that $U_i \subseteq \sigma(U_i)$ for $i = 1, 2, \ldots, n$. If F is a left generalized (σ, σ)-n-derivation associated with a nonzero (σ, σ)-n-derivation d on N such that $d([x, y], u_2, \ldots, u_n) = [F(x, u_2, \ldots, u_n), \sigma(y)]$ for all $x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, then F is a right σ-n-multiplier on N or N is a commutative ring.*

Proof. Assume that

$$d([x,y],u_2,\ldots,u_n)=[F(x,u_2,\ldots,u_n),\sigma(y)], \tag{3.26}$$

for all $x,y\in U_1,u_2\in U_2,\ldots,u_n\in U_n$. Replacing y by xy in (3.26), we get

$$d(x[x,y],u_2,\ldots,u_n)=[F(x,u_2,\ldots,u_n),\sigma(xy)],$$

which implies that

$$d(x,u_2,\ldots,u_n)\sigma([x,y])+\sigma(x)d([x,y],u_2,\ldots,u_n)=[F(x,u_2,\ldots,u_n),\sigma(x)\sigma(y)].$$

Using (3.26), the last equation becomes

$$d(x,u_2,\ldots,u_n)\sigma([x,y])+\sigma(x)F(x,u_2,\ldots,u_n)\sigma(y)=F(x,u_2,\ldots,u_n)\sigma(x)\sigma(y).$$

For $x=y$, (3.26) gives $F(x,u_2,\ldots,u_n)\sigma(x)=\sigma(x)F(x,u_2,\ldots,u_n)$ which implies that $d(x,u_2,\ldots,u_n)\sigma([x,y])=0$. As this equation is same as (3.6), arguing in the similar manner as in Theorem 3.4, we obtain the result. □

Theorem 3.11. *Let N be a 2-torsion free 3-prime near ring and $U_1,U_2,\ldots,U_n$ are nonzero semigroup ideals of N such that U_1 is closed under addition. Let σ be a onto homomorphism on N such that $U_1\subseteq\sigma(U_1)$. Then N admits no generalized (σ,σ)-n-derivation F associated with a (σ,σ)-n-derivation d such that $U_1\cap Z\neq\emptyset$, $d(U_1\cap Z,U_2,U_3,\ldots,U_n)\neq\{0\}$ and $d(x\circ y,u_2,\ldots,u_n)=F(x,u_2,\ldots,u_n)\circ\sigma(y)$ for all $x,y\in U_1,u_2\in U_2,\ldots,u_n\in U_n$.*

Proof. Suppose that

$$d(x\circ y,u_2,\ldots,u_n)=F(x,u_2,\ldots,u_n)\circ\sigma(y), \tag{3.27}$$

for all $x,y\in U_1,u_2\in U_2,\ldots,u_n\in U_n$. Let $z\in U_1\cap Z$ such that $d(z,u_2,u_3,\ldots,u_n)\neq 0$ and replacing y by zy in (3.27), we get

$$d(z,u_2,\ldots,u_n)\sigma(x\circ y)+\sigma(z)d(x\circ y,u_2,\ldots,u_n)=F(x,u_2,\ldots,u_n)\circ\sigma(z)\sigma(y).$$

Substituting arbitrary element $z'\in U_1\cap Z$ for $\sigma(z)$ in above expression and using (3.27), we obtain $d(z,u_2,\ldots,u_n)\sigma(x\circ y)=0$. By Lemma 2.7, it is clear that $d(z,u_2,\ldots,u_n)\in Z\setminus\{0\}$ which means that $d(z,u_2,\ldots,u_n)N\sigma(x\circ y)=\{0\}$. By 3-primeness of N, we conclude that $\sigma(x\circ y)=0$ for all $x,y\in U_1$ which implies that $\sigma(x)\circ\sigma(y)=0$. Now replacing $\sigma(x)$ and $\sigma(y)$ by x' and y' for all $x',y'\in U_1$ respectively, we have $x'\circ y'=0$. In particular $x'^2=0$ for all $x'\in U_1$. Since U_1 is closed under addition, we have $u(u+u')^2=0$ for all $u,u'\in U_1$ this gives $uu'u=0$ for all $u,u'\in U_1$, i.e., $uU_1u=\{0\}$. Thus, $U_1=\{0\}$, which contradicts our hypothesis. □

The following example shows that the 3-primeness hypothesis in Theorems 3.4 to 3.11 can not be omitted.

Example 3.2. Let S be a zero-symmetric left near-ring which is not abelian. Consider

$$N=\left\{\begin{pmatrix}0&x&y\\0&0&0\\0&0&0\end{pmatrix}\mid x,y,0\in S\right\}$$

and

$$U = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, 0 \in S \right\}.$$

Then clearly U is a nonzero semigroup ideal of a non 3-prime zero-symmetric left near ring N. Define mappings $F, d : \underbrace{N \times N \times \cdots \times N}_{n-times} \to N$ by

$$F\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & x_1 x_2 \ldots x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$d\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & y_1 y_2 \ldots y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define $\sigma, \tau : N \to N$ by

$$\tau \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & -y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma = id_N.$$

If we choose $U_1 = U_2 = \cdots = U_n = U$, then it is easy to see that F is a nonzero generalized (σ, σ)-n-derivation associated with a nonzero (σ, σ)-n-derivation d and also a nonzero generalized (σ, τ)-n-derivation associated with a nonzero (σ, τ)-n-derivation d of N satisfying

(i) $F(x \circ y, u_2, \ldots, u_n) = 0$;

(ii) $F([x, y], u_2, \ldots, u_n) = \pm\tau([x, y])$;

(iii) $F(x \circ y, u_2, \ldots, u_n) = \pm\tau([x, y])$;

(iv) $F(x \circ y, u_2, \ldots, u_n) = (\sigma(x) \circ y)_{\sigma,\sigma}$;

(v) $F([x, y], u_2, \ldots, u_n) = [\sigma(x), y]_{\sigma,\sigma}$;

(vi) $F(x \circ y, u_2, \ldots, u_n) = [\sigma(x), y]_{\sigma,\sigma}$;

(vii) $F(x \circ y, u_2, \ldots, u_n) = \pm\tau(x \circ y)$;

(viii) $F([x, y], u_2, \ldots, u_n) = [d(x, u_2, \ldots, u_n), \sigma(y)]$;

(ix) $d([x, y], u_2, \ldots, u_n) = [F(x, u_2, \ldots, u_n), \sigma(y)]$;

(x) $F(x \circ y, u_2, \ldots, u_n) = [d(x, u_2, \ldots, u_n), \sigma(y)]$;

(xi) $F(x \circ y, u_2, \ldots, u_n) = d(x, u_2, \ldots, u_n) \circ \sigma(y)$;

(xii) $d(x \circ y, u_2, \ldots, u_n) = F(x, u_2, \ldots, u_n) \circ \sigma(y)$,

for all $x, y, u_2, \ldots, u_n \in U$. However, N is not a commutative ring.

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[1]Department of Mathematics, Aligarh Muslim University, Aligarh, INDIA
Email address: asma_ali2@rediffmail.com, inzamamulhuque057@gmail.com

[2]Department of Mathematics, Physics and Computer Science,
Polydisciplinary Faculty, LSI, Taza, Sidi Mohammed Ben Abdellah University,
Fez, Morocco
Email address: abedelkarimboua@yahoo.fr

[3]Department of Mathematics, Aligarh Muslim University, Aligarh, INDIA
Email address: asma_ali2@rediffmail.com, inzamamulhuque057@gmail.com