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ON ENTIRE SOLUTIONS OF SYSTEM OF q-SHIFT PARTIAL DIFFERENTIAL EQUATIONS IN $\mathbb{C}^2$

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Abstract. Investigation of entire solutions of system of Fermat-type q-shift partial differential functional equations remain largely unexplored. In this article we have discussed the form of entire solutions of three systems of q-shift partial differential equations in $\mathbb{C}^2$, which are new and important in the literature. The results obtained here could have many applications across various branches of mathematics and mathematical physics. Also we have justified our results by various examples.

1. Introduction

It is well known that Fermat's Last Theorem [23, 24] is one of the most important and longstanding conjectures in mathematics. Nowadays, investigation of Fermat-type functional equations have become an active area of research for the past several decades. Nevanlinna theory [10] is the most powerful tool to study the entire and meromorphic solutions of Fermat type functional equations. The works of Gross [6], Iyer [13] and Montel [18] provided interesting and valuable insights, particularly in establishing solutions when $m = 2$ of the Fermat-type functional equation

$$f^m(z) + g^m(z) = 1 \tag{1.1}$$

and also to determine the non existence of entire solutions of (1.1) for $m > 2$. We recall the famous result of Saleeby [22] regarding meromorphic solutions of (1.1) in $\mathbb{C}^n$ as follows.

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Theorem 1.1 ([22]). *For $m = 2$, the meromorphic solutions of* (1.1) *are characterized as follows.*

(i) *The entire solutions of* (1.1) *are $f(z) = \cos(u(z))$ and $g(z) = \sin(u(z))$, where u is an entire function in $\mathbb{C}^n$.*

(ii) *The meromorphic solutions of* (1.1) *are of the form $f(z) = \frac{2a(z)}{1-a^2(z)}$ and $g(z) = \frac{1-a^2(z)}{1+a^2(z)}$, where a is a meromorphic function in $\mathbb{C}^n$.*

Using difference analogues of Nevanlinna Theory (see [3, 7]), many mathematicians investigated the existence and precise form of entire solutions of some non-linear difference as well as differential difference equations (see [11, 15, 17]). Regarding q-shift partial differential equations of Fermat type, Liu and Cao [16], in 2013, obtained the following result.

Theorem 1.2 ([16]). *Any transcendental entire solution of $f'(z)^2 + f(qz)^2 = 1$ must be of the form $f(z) = \sin(z + b)$ when $q = 1$, whereas $f(z) = \sin(z + k\pi)$ or $f(z) = -\sin(z + k\pi + \pi/2)$ when $q = -1$. There are no transcendental entire solutions of finite order when $q \neq \pm 1$.*

The study of partial differential equations, a generalizations of the well-known eikonal equation in real variable case has a long history. We refer the readers to go through [4, 5, 19] and the references therein. Recently, investigation of entire and meromorphic solutions of partial differential equations of Fermat type in several complex variables has received considerable attention in the literature (see [8, 9, 14, 20, 21, 28, 29]). Hereinafter, we denote by $z+q = (z_1+q_1, z_2+q_2)$, $qz = (q_1z_1, q_1z_2)$, where $z := (z_1, z_2)$, $q := (q_1, q_2) \in \mathbb{C}^2$. The q-shift of f is defined as $f(qz) := f(q_1z_1, q_2z_2)$.

In 2019 Xu-Cao [27] first considered the following partial differential-difference equation

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1, \tag{1.2}$$

and by utilizing the difference Nevanlinna theory in several complex variables, specially the difference version of logarithmic derivative lemma (see [1, 2]), they proved the following result, which opens up a new direction in the field of several complex variable.

Theorem 1.3 ([27]). *Let $c = (c_1, c_2)$ be a constant in $\mathbb{C}^2$. Then, any finite order transcendental entire solution of* (1.2) *has the form $f(z_1, z_2) = \sin(az_1 + bz_2 + H(z_2))$, where a, b are constants in $\mathbb{C}$ satisfying $a^2 = 1$, $ae^{i(ac_1+bc_2)} = 1$ and $H(z_2)$ is a polynomial in z_2 such that $H(z_2 + c_2) = H(z_2)$. In particular, whenever $c_2 \neq 0$, $f(z_1, z_2) = \sin(az_1 + bz_2 + constant)$.*

In 2020, Xu and his coauthors [25, 26] explored the precise form of transcendental entire solutions of the following system of partial differential difference equations

$$\begin{cases} \left(\frac{\partial f_1(z_1,z_2)}{\partial z_1}\right)^2 + f_2(z_1 + c_1, z_2 + c_2)^2 = 1, \\ \left(\frac{\partial f_2(z_1,z_2)}{\partial z_1}\right)^2 + f_1(z_1 + c_1, z_2 + c_2)^2 = 1 \end{cases} \tag{1.3}$$

and

$$\begin{cases} \left(\frac{\partial f_1(z_1,z_2)}{\partial z_1}\right)^2 + 2w\frac{\partial f_1}{\partial z_1} f_2(z_1+c_1, z_2+c_2) + f_2(z_1+c_1, z_2+c_2)^2 = 1, \\ \left(\frac{\partial f_2(z_1,z_2)}{\partial z_1}\right)^2 + 2w\frac{\partial f_2}{\partial z_1} f_1(z_1+c_1, z_2+c_2) + f_1(z_1+c_1, z_2+c_2)^2 = 1, \end{cases} \tag{1.4}$$

where w is a complex constant such that $w^2 \neq 0, 1$. For detail study we refer the readers to take a glance at [25, 26].

Inspired by the above equations and the corresponding results, in this article we investigate the existence and exact form of transcendental entire solutions of the following q-shift analogs of partial differential equations

$$\begin{cases} \left(\frac{\partial f_1(z_1,z_2)}{\partial z_1}\right)^2 + f_2(q_1 z_1, q_2 z_2)^2 = 1, \\ \left(\frac{\partial f_2(z_1,z_2)}{\partial z_1}\right)^2 + f_1(q_1 z_1, q_2 z_2)^2 = 1, \end{cases} \tag{1.5}$$

$$\begin{cases} \left(\frac{\partial f_1(z_1,z_2)}{\partial z_1}\right)^2 + 2w\frac{\partial f_1(z_1,z_2)}{\partial z_1} f_2(q_1 z_1, q_2 z_2) + f_2(q_1 z_1, q_2 z_2)^2 = 1, \\ \left(\frac{\partial f_2(z_1,z_2)}{\partial z_1}\right)^2 + 2w\frac{\partial f_2(z_1,z_2)}{\partial z_1} f_1(q_1 z_1, q_2 z_2) + f_1(q_1 z_1, q_2 z_2)^2 = 1, \end{cases} \tag{1.6}$$

$$\begin{cases} \left(\frac{\partial f_1}{\partial z_1}\right)^2 + [f_2(q_1 z_1, q_2 z_2) - f_1(z_1, z_2)]^2 = 1, \\ \left(\frac{\partial f_2}{\partial z_1}\right)^2 + [f_1(q_1 z_1, q_2 z_2) - f_2(z_1, z_2)]^2 = 1, \end{cases} \tag{1.7}$$

where $q = (q_1, q_2) \in \mathbb{C}^2$ with $q_1 \neq 0$ and $w^2 \neq 0, 1$. Observe that (1.5) and (1.6) are the q-shift versions of (1.3) and (1.4), respectively. We state our main results as follows.

2. Main Results

Theorem 2.1. *Let $q := (q_1, q_2) \in \mathbb{C}^2$ with $q_1 \neq 0$. If $(f_1(z_1, z_2), f_2(z_1, z_2))$ be a pair of finite order transcendental entire solution of* (1.5)*, then one of the following conclusions must hold.*

(a) $q_1 = -1$ *and*

$$f_1(z_1, z_2) = \sin\left(e^{i\xi_1} z_1 + p_2\left(\frac{1}{q_2} z_2\right)\right), \quad f_2(z_1, z_2) = \sin\left(e^{i\xi_2} z_1 + p_1\left(\frac{1}{q_2} z_2\right)\right),$$

where $p_1(z_1, z_2)$, $p_2(z_1, z_2)$ and the value of q_2 are obtained from Lemma 3.1, $\xi_1, \xi_2 \in \mathbb{C}$ such that $\xi_1 = \xi_2 = a_0 + b_0$ and $e^{i\xi_1} = \pm 1$.

(b) $q_1 = 1$ *and the form of solution is same as in* (a)*, where $p_1(z_1, z_2)$, $p_2(z_1, z_2)$ and the value of q_2 are obtained from Lemma* 3.2, $\xi_1, \xi_2 \in \mathbb{C}$ such that $\xi_1 = -\xi_2 = a_0 - b_0$ and $e^{i\xi_1} = \pm 1$.

Theorem 2.2. *If* $(f_1(z_1,z_2), f_2(z_1,z_2))$ *be a pair of finite order transcendental entire solutions of* (1.6), *then* $q_1=-1$ *and*

$$\begin{cases} f_1(z_1,z_2)=\frac{1}{\sqrt{2}}\left(\frac{\cos\left(ie^{i\xi_1}z_1+p_2(z_2/q_2)\right)}{\sqrt{1+w}}-\frac{\sin\left(ie^{i\xi_1}z_1+p_2(z_2/q_2)\right)}{\sqrt{1-w}}\right),\\ f_2(z_1,z_2)=\frac{1}{\sqrt{2}}\left(\frac{\cos\left(ie^{i\xi_2}z_1+p_1(z_2/q_2)\right)}{\sqrt{1+w}}-\frac{\sin\left(ie^{i\xi_2}z_1+p_1(z_2/q_2)\right)}{\sqrt{1-w}}\right), \end{cases} \tag{2.1}$$

where $p_1(z_1,z_2)$, $p_2(z_1,z_2)$, q_2 *are found from Lemma* 3.1, $\xi_1,\xi_2\in\mathbb{C}$ such that $\xi_1=\xi_2=a_0+b_0$ with $e^{2i\xi_1}=-1$ and w is a complex constant such that $w^2\neq 0,1$.

Theorem 2.3. *Let* $(q_1,q_2)\in\mathbb{C}^2$ *be such that* $q_j\neq 0$, $j=1,2$. *If* $(f_1(z_1,z_2), f_2(z_1,z_2))$ *be a pair of transcendental entire solutions with finite order of* (1.7), *then one of the following conclusions holds.*

(A) $q_1=\pm 1$, *and* $f_1(z_1,z_2)=z_1\cos\gamma_1+\phi_1(z_2)$, $f_2(z_1,z_2)=z_1\cos\gamma_2+\phi_2(z_2)$, *where* ϕ_1,ϕ_2 *are entire functions of* z_2, *only satisfying* $\phi_2(q_2z_2)-\phi_1(z_2)=\sin\gamma_1$, $\phi_1(q_2z_2)-\phi_2(z_2)=\sin\gamma_2$, *where* γ_1,γ_2 *are complex constants such that* $\gamma_1\neq(2t+1)\pi/2$ *and* $\gamma_2=2n\pi\pm\gamma_1$, *if* $q_1=1$ *and* $\gamma_2=(2n\pi+1)\pi\pm\gamma_1$, *if* $q_1=-1$, t,n *being two integers.*

(B) $q_1=\pm 1$, *and the form of solutions are given by*

$$f_1(z_1,z_2)=z_1\cos\gamma_1+\phi_1(z_2),\quad f_2(z_1,z_2)=\frac{1}{q_1}\cos\gamma_1 z_1+\phi_1\left(\frac{z_2}{q_2}\right)\pm\sin\gamma_1,$$

where ϕ_1 *is an entire functions of* z_2 *such that*

$$\phi_1(q_2z_2)-\phi_1\left(\frac{z_2}{q_2}\right)=\pm\left[\sin\gamma_1+\frac{\sqrt{q_1^2-\cos^2\gamma_1}}{q_1}\right].$$

(C)

$$f_1(z_1,z_2)=z_1\cos p_1(z_2)+T_1(z_2),\quad f_2(z_1,z_2)=z_1\cos p_2(z_2)+T_2(z_2), \tag{2.2}$$

where $p_1(z_2)$, $p_2(z_2)$ *are polynomials in* z_2, $T_1(z_2)$, $T_2(z_2)$ *are entire functions in* z_2 *satisfying one of the following.*

(a) $q_1=1$, ξ_1,ξ_2 *are complex constant with* $\xi_1=\xi_2$ *and* $e^{i\xi_1}=1$, q_2, p_1,p_2 *are found from* Lemma 3.1 *and* T_1,T_2 *satisfy*

$$T_1(q_2^2z_2)=T_1(z_2)\quad\text{and}\quad T_2(q_2^2z_2)=T_2(z_2). \tag{2.3}$$

(b) $q_1=-1$, ξ_1,ξ_2 *are complex constant with* $\xi_1=\xi_2$ *and* $e^{i\xi_1}=-1$, q_2, p_1,p_2 *are found from* Lemma 3.1 *and* T_1,T_2 *satisfy*

$$T_1(q_2^2z_2)-T_1(z_2)=2\sin p_1(z_2),\quad T_2(q_2^2z_2)-T_2(z_2)=2\sin p_2(z_2). \tag{2.4}$$

(c) $q_1=1$, $e^{\xi_1}=1$, $e^{i\xi_2}=1$, q_2, p_1,p_2 *are found from* Lemma 3.3 *and* T_1,T_2 *satisfy*

$$T_2(q_2^2z_2)=T_2(z_2),\quad T_1(q_2^2z_2)-T_1(z_2)=2\sin p_1(z_2).$$

(d) $q_1=-1$, $e^{\xi_1}=-1$, $e^{i\xi_2}=-1$, q_2, p_1,p_2 *are found from* Lemma 3.3 *and* T_1,T_2 *satisfy* $T_1(q_2^2z_2)=T_1(z_2)$ *and* $T_2(q_2^2z_2)-T_2(z_2)=2\sin p_2(z_2)$.

(e) $q_1=1$, $\xi_1,\xi_2\in\mathbb{C}$ *with* $\xi_2=-\xi_1$, $e^{\xi_1}=1$, q_2, p_1,p_2 *are found from* Lemma 3.2 *and* T_1,T_2 *satisfy* (2.4).

(f) $q_1 = -1$, $\xi_1, \xi_2 \in \mathbb{C}$ *with* $\xi_2 = -\xi_1$, $e^{\xi_1} = -1$, q_2, p_1, p_2 *are found from Lemma* 3.2 *and* T_1, T_2 *satisfy* (2.3).

(D)

$$\begin{cases} f_1(z_1, z_2) = -\frac{1}{2}\sin(-2z_1 + p_1(z_2)) + T_1(z_2), \\ f_2(z_1, z_2) = -\frac{1}{2}\sin(-2z_1 + p_2(z_2)) + T_2(z_2), \end{cases} \tag{2.5}$$

satisfying one of the following.

(a) $q_1 = -1$, $\xi_1, \xi_2 \in \mathbb{C}$ *such that* $\xi_1 = \xi_2$ *with* $e^{i\xi_1} = 1$, q_2, p_1, p_2 *are found from* Lemma 3.1, *and* T_1, T_2 *satisfy* (2.3).

(b) $q_1 = 1$, $\xi_1, \xi_2 \in \mathbb{C}$ *such that* $\xi_1 = -\xi_2$ *with* $e^{i\xi_1} = -1$, q_2, p_1, p_2 *are found from* Lemma 3.2, *and* T_1, T_2 *satisfy* (2.3).

(E) $q_1 = i$, $\xi_1, \xi_2 \in \mathbb{C}$ *such that* $e^{i\xi_1} = 1$, $e^{i\xi_2} = -1$,

$$\begin{cases} f_1(z_1, z_2) = -\frac{1}{1+i}\sin(-(1+i)z_1 + p_1(z_2)) + T_1(z_2), \\ f_2(z_1, z_2) = -\frac{1}{1-i}\sin(-(1-i)z_1 + p_2(z_2)) + T_2(z_2), \end{cases} \tag{2.6}$$

where q_2, p_1, p_2 *are found from* Lemma 3.3, *and* T_1, T_2 *are entire functions in* z_2 *satisfying* (2.3).

(F) $q_1 = -i$, $\xi_1, \xi_2 \in \mathbb{C}$ *such that* $e^{i\xi_1} = 1$, $e^{i\xi_2} = -1$,

$$\begin{cases} f_1(z_1, z_2) = -\frac{1}{1-i}\sin(-(1-i)z_1 + p_1(z_2)) + T_1(z_2), \\ f_2(z_1, z_2) = -\frac{1}{1+i}\sin(-(1+i)z_1 + p_2(z_2)) + T_2(z_2), \end{cases} \tag{2.7}$$

where q_2, p_1, p_2 *are found from* Lemma 3.3, *and* T_1, T_2 *are entire functions in* z_2 *satisfying* (2.3).

The following examples show that our theorems are precise.

Example 2.1. Let $\xi_1 = \xi_2 = \pi$, $p_1(z_2) = z_2^{15} + \frac{\pi}{3}$, $p_2(z_2) = z_2^{15} + \frac{2\pi}{3}$. Choose $q_2 \in \mathbb{C}$ such that $q_2^{15} = -1$. Then, one can easily see that $\left(\sin\left(\frac{2\pi}{3} - z_1 - z_2^{15}\right), \sin\left(\frac{\pi}{3} - z_1 - z_2^{15}\right)\right)$ is a solution of (1.5).

Example 2.2. Let $\xi_1 = \xi_2 = \pi$, $p_1(z_2) = z_2^9 + z_2^4 + z_2 + \frac{2\pi}{3}$, $p_2(z_2) = z_2^9 - z_2^4 + z_2 + \frac{\pi}{3}$, $q_1 = q_2 = -1$. Then, in view of Theorem 2.1 (a), *we can easily see that* $\left(\sin\left(\frac{\pi}{3} - z_1 - z_2^9 - z_2^4 - z_2\right), \sin\left(\frac{2\pi}{3} - z_1 - z_2^9 + z_2^4 - z_2\right)\right)$ *is a solution of* (1.5).

Example 2.3. Let $\xi_1 = \xi_2 = \pi$, $p_1(z_2) = z_2^{10} + z_2^2 + \frac{2\pi}{3}$, $p_2(z_2) = z_2^{10} + z_2^2 + \frac{\pi}{3}$, $q_1 = -1$, $q_2 = -i$. Then, in view of Theorem 2.1 (a), *one can easily see that* $\left(\sin\left(\frac{\pi}{3} - z_1 - z_2^2 - z_2^{10}\right), \sin\left(\frac{2\pi}{3} - z_1 - z_2^{10} - z_2^2\right)\right)$ *is a solution of* (1.5).

Example 2.4. Let $\xi_1 = \pi$, $\xi_2 = -\pi$, $p_1(z_2) = z_2^6 + \frac{3\pi}{2}$, $p_2(z_2) = z_2^6 + \frac{\pi}{2}$. Choose $q_2 \in \mathbb{C}$ such that $q_2^6 = 1$. Then, in view of Theorem 2.1 (b), *one can easily see that* $\left(\sin\left(\frac{\pi}{2} - z_1 + z_2^6\right), \sin\left(\frac{3\pi}{2} - z_1 + z_2^6\right)\right)$ *is a solution of* (1.5).

Example 2.5. Let $\xi_1 = \xi_2 = \pi$, $p_1(z_2) = z_2^{20} + z_2^2 + \frac{2\pi}{3}$, $p_2(z_2) = -z_2^{20} + z_2^2 + \frac{\pi}{3}$, $q_1 = -1$, $q_2 = i$, $w = 3$. Then, in view of Theorem 2.2, *one can easily see that* (f_1, f_2) *is a solution of* (1.6), *where*

$$f_1(z_1, z_2) = \frac{1}{2\sqrt{2}} \left(\cos\left(\frac{\pi}{3} - iz_1 - z_2^2 - z_2^{10} \right) - i \sin\left(\frac{\pi}{3} - iz_1 - z_2^2 - z_2^{10} \right) \right)$$

and

$$f_2(z_1, z_2) = \frac{1}{2\sqrt{2}} \left(\cos\left(\frac{2\pi}{3} - iz_1 - z_2^2 + z_2^{10} \right) + i \sin\left(\frac{\pi}{3} - iz_1 - z_2^2 + z_2^{10} \right) \right).$$

3. Key Lemmas

To prove our main results, we need the following lemmas.

Lemma 3.1. *Let* $p_1(z) = \sum_{l=0}^{n} a_l z^l$ *and* $p_2(z) = \sum_{l=0}^{n} b_l z^l$ *be two non-constant polynomials of degree* n *in* $\mathbb{C}$ *satisfying*

$$p_1(qz) + p_2(z) = \xi_1, \quad p_2(qz) + p_1(z) = \xi_2, \tag{3.1}$$

where $q \neq 0$, $\xi_1, \xi_2, a_l, b_l \in \mathbb{C}$, $l = 0, 1, 2, \ldots, n$, $a_n \neq 0$, $b_n \neq 0$. *Then,* $\xi_1 = \xi_2 = a_0 + b_0$, $a_j \neq 0$ *if and only if* $b_j \neq 0$ *with* $b_j = \pm a_j$, *where* $1 \leq j < n$. *Further,* $p_1(z)$, $p_2(z)$ *and* q *satisfy one of the following.*

(A) *Let* $p_1(z) = a_n z^n + a_0$ *and* $p_2(z) = b_n z^n + b_0$. *Then,* $q^n = -\frac{b_n}{a_n} = -\frac{a_n}{b_n}$.

(B) *Let* $a_1 \neq 0$. *Then, one of the following cases must occur.*

(*i*) *If* $b_1 = a_1$ *and* $b_n = a_n$, *then* $q = -1$ *and* n *is an odd integer greater than* 1.

(*ii*) *If* $b_1 = a_1$ *and* $b_n = -a_n$, *then* $q = 1$ *and* n *is an even integer.*

Moreover, in both the cases (*i*) *and* (*ii*), *if* $a_j \neq 0$, *where* $1 < j < n$, *then* j *is odd when* $b_j = a_j$ *and* j *is even when* $b_j = -a_j$.

(*iii*) *If* $b_1 = -a_1$, *then* $q = 1$ *and* $b_n = -a_n$, *where* n *is an integer greater than* 1. *Further, if* $a_j \neq 0$, *where* $1 < j < n$, *then* $b_j = -a_j$.

(C) *Let* $a_1 = 0$ *and* $a_2 \neq 0$. *Then, one of the following cases must occur.*

(i) *If* $b_2 = a_2$, *then* $q = \pm i$. *Further, if* $b_n = a_n$, *then* n *is an odd multiple of* 2 *and if* $b_n = -a_n$, *then* n *is an even multiple of* 2. *Moreover, if* $a_j \neq 0$, *where* $2 < j < n$, *then* j *is an odd multiple of* 2 *when* $b_j = a_j$, *and* j *is an even multiple of* 2 *when* $b_j = -a_j$.

(ii) *If* $b_2 = -a_2$, *then* $q = \pm 1$.

Further, we have.

(a) *If* $b_n = a_n$, *then* $q = -1$ *and* n *is an odd integer. Moreover, if* $a_j \neq 0$, *where* $2 < j < n$, *then* j *is odd if* $b_j = a_j$ *and* j *is even if* $b_j = -a_j$.

(b) *If* $b_n = -a_n$ *and* $q = 1$, *then* n *is any integer greater or equal to* 3. *Moreover, if* $a_j \neq 0$ *for* $2 < j < n$, *then* $b_j = -a_j$.

(c) *If* $b_n = -a_n$ *and* $q = -1$, *then* n *is an even integer. Moreover, if* $a_j \neq 0$ *for* $2 < j < n$, *then* j *is odd when* $b_j = a_j$ *and* j *is even when* $b_j = -a_j$.

(D) *Let* k *be the least positive integer such that* $a_k \neq 0$, *where* $3 \leq k < n$. *Then, one of the following assertions holds.*

(i) *If* $b_k = a_k$*, then* $q^k = -1$.

Further, it follows.

(a) *When* $b_n = a_n$ *and* k *is prime, then* n *is an odd multiple of* k*. Moreover, if* $a_j \neq 0$ *for* $k < j < n$*, then* j *is odd multiple of* k *when* $b_j = a_j$*, and* j *is even multiple of* k *when* $b_j = -a_j$.

(b) *When* $b_n = a_n$ *and* k *is odd composite, then* $n = p(p + 2s)$ *when* $q^p + 1$ *is a factor of both* $q^k + 1$ *and* $q^n + 1$*,* p *being a prime factor of* k *and* s *is a natural number chosen in such a way that* $k < n$*. For the other factor of* $q^k + 1$ *after dividing by* $q^p + 1$*, we must have* $n = (2m + 1)k$*,* $m \in \mathbb{N}$*. Moreover, if* $a_j \neq 0$ *for* $k < j < n$*, then* $j = p(p + 2t)$ *when* $b_j = a_j$ *and* $j = 2tk$ *when* $b_j = -a_j$ *and* $q^p + 1$ *is a factor of both* $q^k + 1$ *and* $q^n + 1$*,* $t \in \mathbb{N}$ *such that* $k < n$.

(c) *If* $b_n = a_n$ *and* $k = 4s$*,* $s \in \mathbb{N}$*, then* n *is odd multiple of* k*. Moreover, if* $a_j \neq 0$*,* $k < j < n$*, then* j *is odd multiple of* k *when* $b_j = a_j$*, and* j *is even multiple of* k *when* $b_j = -a_j$.

(d) *If* $k = 4s + 2$*,* $s \in \mathbb{N}$*, then* $\pm i$ *are the roots of* $q^k = -1$*. Further, one obtain the following cases.*

(d_1) *If* $b_n = a_n$ *and* $q = \pm i$*, then* $n = 4m + 2$*, where the integer* m *is chosen so that* $k < n$*. Moreover, if* $a_j \neq 0$ *for* $k < j < n$*, then* $j = 4t + 2$ *when* $b_j = a_j$*, and* $j = 4t$ *when* $b_j = -a_j$*,* t *being an integer.*

(d_2) *If* $b_n = a_n$ *and* $q \neq \pm i$*, then* $n = 2mk$*, where* $m \in \mathbb{N}$ *such that* $k < n$*. Moreover, if* $a_j \neq 0$ *for* $k < j < n$*, then* $j = (2t + 1)k$ *when* $b_j = a_j$*, and* $j = 2tk$ *when* $b_j = -a_j$*,* $t \in \mathbb{N}$.

(e) *If* $b_n = -a_n$ *and* k *be odd, then we have the following cases.*

(e_1) *If* $q = -1$*, then* n *is an even integer. Moreover, if* $a_j \neq 0$ *for* $k < j < n$*, then* j *is odd when* $b_j = a_j$*, and* j *is even when* $b_j = -a_j$.

(e_2) *If* $q \neq -1$*, then* n *is an odd multiple of* k*. Moreover, if* $a_j \neq 0$*, then* j *is odd multiple of* k *when* $b_j = a_j$*, and* j *is even multiple of* k *when* $b_j = -a_j$ *for* $k < j < n$.

(f) *If* $b_n = -a_n$ *and* k *is even, then we have the following conclusions.*

(f_1) *if* $k = 4s$*,* $s \in \mathbb{N}$*, then* n *is even multiple of* k*. Moreover, if* $a_j \neq 0$ *for* $k < j < n$*, then* j *is odd multiple of* k *when* $b_j = a_j$*, whereas* j *is even multiple of* k *when* $b_j = -a_j$.

(f_2) *if* $k = 4s + 2$ *and* $q = \pm i$*, then* $n = 4m$*,* $m \in \mathbb{N}$*. Moreover, if* $a_j \neq 0$ *for* $k < j < n$*, the* $j = 4t + 2$ *when* $b_j = a_j$*, and* $j = 4t$ *when* $b_j = -a_j$*,* $t \in \mathbb{N}$.

(f_3) *if* $k = 4s + 2$ *and* $q \neq \pm i$*, then* n *is an even multiple of* k*. Moreover, if* $a_j \neq 0$*, then* j *is odd multiple of* k *when* $b_j = a_j$*, whereas* j *is an even multiple of* k *when* $b_j = -a_j$.

(ii) *If* $b_k = -a_k$*, then* $q^k = 1$.

Further, we obtain the following cases.

(a) *When* $b_n = a_n$*, then* k *must be an even integer. Moreover, if* $q = -1$*, then* n *must be an odd integer. If* $a_j z^j$*,* $k < j < n$ *be present in* $p_1(z)$*, then* j *is odd when* $b_j = a_j$*, and* j *is even when* $b_j = -a_j$*. If* $q \neq -1$*, then* $k = 4s$*,* $q = \pm i$ *and* $n = 4m+2$*,* *where $s, m \in \mathbb{N}$. In this case if a_jz^j, $k < j < n$ is present in $p_1(z)$, then $j = 4t + 2$ when $b_j = a_j$, whereas $j = 4t$ when $b_j = -a_j$.*

(b) *When $b_n = -a_n$, then one of the following must occur.*

(b_1) *If $q = 1$, then n can be any integer greater than k. Moreover, if a_jz^j, $k < j < n$ is present in $p_1(z)$, then $b_j = -a_j$ and j is any integer.*

(b_2) *If $q \neq 1$ and k is odd, or $q \neq \pm 1$ and k is even, then n is a multiple of k. Moreover, if $a_j \neq 0$, $k < j < n$, then $b_j = -a_j$ and j is a multiple of k.*

(b_3) *If $q = -1$, then k and n both are even. Moreover, if $a_j \neq 0$, $k < j < n$, then j is odd when $b_j = a_j$, whereas j is even when $b_j = -a_j$.*

Proof. In view of the form of $p_1(z)$ and $p_2(z)$, from (3.1), we have $a_0 + b_0 = \xi_1 = \xi_2$ and as a_n, b_n are non zero complex constants, we obtain

$$q^n = -\frac{b_n}{a_n} = -\frac{a_n}{b_n}. \tag{3.2}$$

Further, we have

$$\begin{cases} a_1q + b_1 = 0, \quad a_1 + b_1q = 0, \quad a_2q^2 + b_2 = 0, \quad a_2 + b_2q^2 = 0, \\ \vdots \\ a_rq^r + b_r = 0, \quad a_r + b_rq^r = 0, \\ \vdots \\ a_{n-1}q^{n-1} + b_{n-1} = 0, \quad a_{n-1} + b_{n-1}q^{n-1} = 0, \\ a_nq^n + b_n = 0, \quad a_n + b_nq^n = 0. \end{cases} \tag{3.3}$$

From (3.3), we observe that $b_r \neq 0$ if and only if $a_r \neq 0$ and

$$q^r = -\frac{b_r}{a_r} = -\frac{a_r}{b_r}, \tag{3.4}$$

where $1 \leq r \leq n - 1$ and in this case $b_r = \pm a_r$.

Now we discuss the following three possible cases.

Case 1. Let $a_j = 0$ for all $j = 1, 2, \ldots, n - 1$. Then, equations stated in (3.3) are consistent and we get (3.2).

Case 2. Let $a_j \neq 0$ for at least one j, $j = 1, 2, \ldots, n - 1$.

Subcase 2.1. Let $a_1 \neq 0$. Then, from (3.4), we get $b_1 = \pm a_1$.

Subcase 2.1.1. Let $b_1 = a_1$. Then, by (3.4), it follows that $q = -1$.

Subcase 2.1.1.1. Let $b_n = a_n$. Then, by (3.2), we have $q^n = -1$. As $q = -1$, we must have $n = 2m+1$, where $m = 1, 2, 3, \ldots$. Let a_rz^r be present in $p_1(z)$, i.e., $a_r \neq 0$. Then, b_rz^r is present in $p_2(z)$, i.e., $b_r \neq 0$, $1 < r < n$. If $b_r = a_r$, then $q^r = -1$. As $q = -1$, we conclude that r must be an odd integer. If $b_r = -a_r$, then $q^r = 1$. Since $q = -1$, it follows that r must be an even integer.

Subcase 2.1.1.2. Let $b_n = -a_n$. Then, by (3.2), we see that $q^n = 1$. As $q = -1$, we have n an even integer. Further, if $b_r = a_r \neq 0$, then in view of (3.4), we get $q^r = -1$, and hence r is an odd integer. If $b_r = -a_r \neq 0$, then by (3.4), we get $q^r = 1$, which is a contradiction as $q = -1$, where $1 < r < n$.

Subcase 2.1.2. Let $b_1 = -a_1$. Then, from (3.4), we get $q = 1$. Note that $b_n \neq a_n$. Otherwise, from (3.2), we see that $q^n = -1$, which is not possible for any values of n. Hence $b_n = -a_n$. Therefore, by (3.2), it follows that $q^n = 1$, which implies that n can be any integer greater than or equal to 2. Further, if $a_r \neq 0$ for $1 < r < n$, then from (3.4), we must have $b_r = -a_r$ and r can be any integer satisfying $1 < r < n$.

Subcase 2.2. Let $a_1 = 0$ and $a_2 \neq 0$. Then, by (3.4), we obtain that $b_2 = \pm a_2$.

Subcase 2.2.1. Let $b_2 = a_2$. Then, from (3.4), we get $q = \pm i$.

Subcase 2.2.1.1. Let $b_n = a_n$. Then, (3.2) yields that $q^n = -1$. As $q = \pm i$, we see that $n = 4m + 2$, where $m \in \mathbb{N}$. Moreover, if $a_r z^r$, $2 < r < n$ is present in $p_1(z)$, then $b_r z^r$ is also present in $p_2(z)$. If $b_r = a_r$, then from (3.4), we get $q^r = -1$. Since $q = \pm i$, it follows that r must be the odd multiple of 2. If $b_r = -a_r$, then by (3.4), we get $q^r = 1$, and as $q = \pm i$, we see that r is an even multiple of 2.

Subcase 2.2.1.2. Let $b_n = -a_n$. Then, from (3.2), we get $q^n = 1$. As $q = \pm i$, we must have $n = 4m$, where $m \in \mathbb{N}$. Also, in view of (3.4), we see that if $b_r = a_r \neq 0$, then r must be an odd multiple of 2, and if $b_r = -a_r \neq 0$, then r is an even multiple of 2, where $2 < r < n$.

Subcase 2.2.2. Let $b_2 = -a_2$. Then, in view of (3.4), we obtain $q = \pm 1$.

Subcase 2.2.2.1. Let $b_n = a_n$. Then, by (3.2), we obtain $q^n = -1$. Then clearly $q \neq 1$. Hence $q = -1$ and n is an odd integer. Further, by similar argument as previous if $b_r = a_r \neq 0$, then r is odd and if $b_r = -a_r \neq 0$, then r is even, where $2 < r < n$.

Subcase 2.2.2.2. Let $b_n = -a_n$. Then, by (3.2), we obtain $q^n = 1$. Now if $q = 1$, then n can be any integer greater than or equal to 3. Also, if $a_r z^r$ is present in $p_1(z)$, then $b_r z^r$ is present in $p_2(z)$ and $b_r = -a_r$, and in this case r can be any integer, $2 < r < n$. If $q = -1$, then n must be an even integer. Moreover, if $b_r = a_r$, then r is odd and if $b_r = -a_r$, then r is even, where $2 < r < n$.

Subcase 2.3. Let k be the least positive integer such that $a_k \neq 0$, where $3 \leq k < n$. Then, in view of (3.4), we see that $b_k = \pm a_k$.

Subcase 2.3.1. Let $b_k = a_k$. Then, from (3.4), we get $q^k = -1$.

Subcase 2.3.1.1. Let k be odd. Then, -1 is a solution of $q^k = -1$.

Subcase 2.3.1.1.1. Let $q = -1$. If $b_n = a_n$, then in view of (3.2), we get $q^n = -1$, which yields that n is an odd multiple. Similarly, we easily deduce that n is even, when $b_n = -a_n$. Further, in view of (3.4), we conclude that r is odd when $b_r = a_r \neq 0$, and even when $b_r = -a_r \neq 0$, where $k < r < n$.

Subcase 2.3.1.1.2. Let $q \neq -1$.

Subcase 2.3.1.1.2.1. Let k be prime. If $b_n = a_n$, then in view of (3.2), we conclude that n is an odd multiple of k. If $b_n = -a_n$, then n is even multiple of k. Moreover, using (3.4), we see that r is odd multiple of k when $b_r = a_r \neq 0$, and even multiple of k when $b_r = -a_r$ for $k < r < n$.

Subcase 2.3.1.1.2.2. Suppose k is odd composite. Note that $q^p + 1$ is a factor of $q^k + 1$ for each prime factor p of k.

Subcase 2.3.1.1.2.2.1. Suppose $b_n = a_n$. Then, by (3.2) it follows that $q^n = -1$. If $q^p + 1$ is a factor of $q^n + 1$, then for the zeros of $q^p + 1$, we have $n = p(p + 2m)$, where m is an integer chosen in such a way that $k < n$. Moreover, if $b_r = a_r \neq 0$, $k < r < n$, then $r = p(p + 2t)$ and if $b_r = -a_r$, then $r = 2tk$, where the integer t is so chosen that $k < r < n$. For the zeros of $q^k + 1$ other than that of $q^p + 1$, it must be that n is odd multiple of k. Also observe from (3.4) that if $b_r = a_r \neq 0$, $k < r < n$, then r is odd multiple of k, whereas if $b_r = -a_r$, then r is even multiple of k.

Subcase 2.3.1.1.2.2.2. Suppose $b_n = -a_n$. Then, by (3.2) it follows that $q^n = 1$. This implies that n is an even multiple of k. Further, if $b_r = a_r \neq 0$, then r is odd multiple of k, whereas if $b_r = -a_r \neq 0$, then r is even multiple of k, where $k < r < n$.

Subcase 2.3.1.2. Let k be an even integer. Note that for $k = 4s + 2$, i and $-i$ are the solutions of $q^k = -1$, where $s \in \mathbb{N}$.

Subcase 2.3.1.2.1. Suppose $b_n = a_n$. Then, by (3.2), we see that $q^n = -1$. Now for $k = 4s + 2$ and $q = \pm i$, we have $n = 4m + 2$, where m is an integer chosen in such a way that $k < n$. Further, if $a_r z^r$, $k < r < n$ is present in $p_1(z)$, then $r = 4t + 2$ when $b_r = a_r \neq 0$, whereas $r = 4t$ when $b_r = -a_r \neq 0$, t being an integer chosen in such a way that $k < r < n$. Next, suppose $k = 4s$ or $k = 4s + 2$ and $q \neq \pm i$, where $s \in \mathbb{N}$. Then n must be an odd multiple of k. Furthermore, if $b_r = a_r \neq 0$, by (3.4), we get $q^r = -1$, which implies that r is an odd multiple of k, whereas if $b_r = -a_r \neq 0$, then by (3.4), we have $q^r = 1$, which implies that r is even multiple of k.

Subcase 2.3.1.2.2. Let $b_n = -a_n$. Now for $k = 4s + 2$ and $q = \pm i$, we have $n = 4m$, where m is an integer chosen in such a way that $k < n$. Further, if $a_r z^r$, $k < r < n$ is present in $p_1(z)$, then $r = 4t + 2$ when $b_r = a_r \neq 0$, whereas $r = 4t$ when $b_r = -a_r \neq 0$, where $t \in \mathbb{N}$ chosen in such a way that $k < r < n$. Next, suppose $k = 4s$ or $k = 4s + 2$ and $q \neq \pm i$, where $s \in \mathbb{N}$. Then n must be an even multiple of k. Furthermore, if $b_r = a_r \neq 0$, by (3.4), we get $q^r = -1$, which implies that r is an odd multiple of k, whereas if $b_r = -a_r \neq 0$, then by (3.4), we have $q^r = 1$, which implies that r is even multiple of k.

Subcase 2.3.2. Let $b_k = -a_k$. Then, from (3.4), we get $q^k = 1$. This implies that 1 is a solution of $q^k = 1$.

Subcase 2.3.2.1. Let $q = 1$. If $b_n = a_n$, then by (3.2), we get $q^n = -1$, a contradiction since $q = 1$. Thus, $b_n = -a_n$ and hence in view of (3.2), we obtain $q^n = 1$. This implies that n is any integer greater than k. Further, if $a_r \neq 0$ for $k < r < n$, then by similar argument as above we deduce that $b_r = -a_r$. Therefore, from (3.4), we conclude that r is any integer satisfying $k < r < n$.

Subcase 2.3.2.2. Let $q \neq 1$. We discuss this case as follows.

Subcase 2.3.2.2.1. Let k be an odd integer. Note that $b_n \neq a_n$. Otherwise, from (3.2), it follows that $q^n = -1$, which is not possible as $n > k$ and k is odd. Hence $b_n = -a_n$, and therefore, by (3.2), we have

$$q^n = 1. \tag{3.5}$$

Further, if $a_r z^r$ is present in $p_1(z)$ for $k < r < n$, then by similar argument we get

$b_r = -a_r$, and hence from (3.4), we get

$$q^r = 1. \tag{3.6}$$

If k is odd composite, then note that $q^p - 1$ is a factor of $q^k - 1$, where p is a prime factor of k. If $q^p - 1$ is also a factor of $q^n - 1$, then for the zeros of $q^p - 1$ and in view of (3.5), we have $n = p(p + 2m)$, where the integer m is to be chosen so that $n > k$. Moreover, if $a_r \neq 0$ for $k < r < n$, then in view of (3.6), it follows that $r = p(p + 2t)$, where integer t is chosen so that $k < r < n$. For the zeros of $q^k - 1$ other than those of $q^p - 1$, or if k is prime , then in view of (3.5), we get that n must be a multiple of k. Furthermore, if $a_r \neq 0$, $k < r < n$, the by (3.6), we obtain that r is a multiple of k.

Subcase 2.3.2.2.2. Let k be an even integer. Observe that -1 is a solution of $q^k = 1$.

Subcase 2.3.2.2.2.1. Let $q = -1$. If $b_n = a_n$, then by (3.2), we see that $q^n = -1$. This implies that n is an odd integer. If $b_n = -a_n$, then from (3.2), we get $q^n = 1$, which implies n is even. Further, if $a_r z^r$, $k < r < n$, is present in $p_1(z)$, then in view of (3.4), we conclude that r is odd when $b_r = a_r$, and r is even when $b_r = -a_r$.

Subcase 2.3.2.2.2.2. Let $q \neq -1$. Since k is an even integer, $k = 4s$ or $k = 4s + 2$, where $s \in \mathbb{N}$.

Subcase 2.3.2.2.2.2.1. Let $k = 4s$, where $s \in \mathbb{N}$. Then, $\pm i$ are the solutions of $q^k = 1$.

Subcase 2.3.2.2.2.2.1.1. Let $q = \pm i$. Now if $b_n = a_n$, then in view of (3.2), we get $q^n = -1$, and hence n must be the odd multiple of 2. If $b_n = -a_n$, then by (3.2), we see that $q^n = 1$, which imply that n is an even multiple of 2. Further, if $a_r z^r$, $k < r < n$ be present in $p_1(z)$, then in view of (3.4), we have $r = 4t + 2$, if $b_r = a_r$ and $r = 4t$, if $b_r = -a_r$, where t is an integer.

Subcase 2.3.2.2.2.2.1.2. Let $q \neq \pm i$. We observe that $b_n \neq a_n$. Otherwise, in view of (3.2), we obtain $q^n = -1$. This is not possible as $q^n = -1$, $q^k = 1$ is inconsistent system. Hence, $b_n = -a_n$, and therefore by (3.2), we get $q^n = 1$. This implies that n is a multiple of k. Further, if $a_r z^r$ is present in $p_1(z)$, where $k < r < n$, then by similar argument we conclude that $b_r = -a_r$. Hence by (3.4), it follows that $q^r = 1$. This implies that r is a multiple of k.

Subcase 2.3.2.2.2.2.2. Let $k = 4s + 2$, where $s \in \mathbb{N}$. Then the same conclusion can be obtain as in Subcase 2.3.2.2.2.2.1.2. □

Using the similar arguments as in Lemma 3.1, we obtain the following results.

Lemma 3.2. *Let $p_1(z), p_2(z)$ be two non-constant n-th degree polynomials defined as in Lemma* 3.1 satisfying

$$p_1(qz) - p_2(z) = \xi_1, \quad p_2(qz) - p_1(z) = \xi_2, \tag{3.7}$$

where $q \neq 0$, $\xi_1, \xi_2 \in \mathbb{C}$. Then, $\xi_1 = -\xi_2 = a_0 - b_0$, $a_j \neq 0$ if and only if $b_j \neq 0$ with $b_j = \pm a_j$, where $1 \leq j < n$. Further, $p_1(z)$, $p_2(z)$ and q satisfy one of the following.

(A) Let $p_1(z) = a_n z^n + a_0$ and $p_2(z) = b_n z^n + b_0$, then $q^n = \frac{b_n}{a_n} = \frac{a_n}{b_n}$.

(B) Let $a_1 \neq 0$, then one of the following cases must occur.

(i) If $b_1 = a_1$, then $q = 1$ and $b_n = a_n$, where n is any integer greater than 1. Moreover if $a_j \neq 0$, where $1 < j < n$, then $b_j = a_j$.

(ii) If $b_1 = -a_1$, then $q = -1$. Further, if $b_n = -a_n$, where n is an integer > 1, and if $b_n = -a_n$, then n is an odd integer> 1. Moreover, if $a_j \neq 0$, where $1 < j < n$, then j is even when $b_j = a_j$ and j is odd when $b_j = -a_j$.

(C) Let $a_1 = 0$ and $a_2 \neq 0$, then one of the following cases must occur.

(i) If $b_2 = a_2$, then $q = \pm 1$. Further, if $q = 1$, then $b_n = a_n$, where n is an an integer greater than 2. Moreover if $a_j z^j$ is present in $p_1(z)$ for $2 < j < n$, then $b_j z^j$ is present in $p_2(z)$ with $b_j = a_j$. If $q = -1$ and $b_n = a_n$, then n is an even integer greater than 2, whereas if $q = -1$ and $b_n = -a_n$, then n is an odd integer greater than 2. Moreover, if $a_j z^j$, $2 < j < n$ is present in $p_1(z)$, then j is even when $b_j = a_j$ and j is odd when $b_j = -a_j$.

(ii) If $b_2 = -a_2$, then $q = \pm i$. Further, if $b_n = a_n$, then n is an even multiple of 2, whereas if $b_n = -a_n$, then n is an odd multiple of 2. Moreover, if $a_j \neq 0$, $2 < j < n$, then j is an even multiple of 2 when $q = i$ ans $b_j = a_j$, whereas j is an odd multiple of 2 when $q = -i$ and $b_j = -a_j$.

(D)] Let $a_k \neq 0$, where k be the least integer such that $3 \leq k < n$. Then, one of the following assertions holds.

(i) If $b_k = a_k$, then $q^k = 1$. Further, we have the following cases.

(a) When $q = 1$, then $b_n = a_n$ and n is any integer greater than 3. Moreover, if $a_j \neq 0$, $k < j < n$, then $b_j = a_j$.

(b) When $q \neq 1$ and $b_n = a_n$, then one of the following must occur.

(b_1) When k is prime, n is a multiple of k. Moreover, if $a_j \neq 0$, $k < j < n$, then $b_j = a_j$ and j is a multiple of k.

(b_2) When k is an odd composite number, then $n = p(p + 2m)$, where p is a prime number greater than or equal to 3 and m is an integer. Moreover, if $a_j \neq 0$, $k < j < n$, then $b_j = a_j$, where $j = p(p + 2t)$, where p is prime number greater than or equal to 3 and t, an integer.

(b_3) When $q = -1$, then n is even. Moreover, if $a_j \neq 0$, $k < j < n$, then j is even when $b_j = a_j$, whereas j is odd when $b_j = -a_j$.

(b_4) When $q \neq -1$ and k is even, n must be a multiple of k. Moreover, if $a_j \neq 0$, then $b_j = a_j$, where j is a multiple of k.

(c) When $q \neq 1$ and $b_n = -a_n$, then one of the following holds.

(c_1) When $q = -1$, then n is an odd integer. Moreover, for $k < j < n$, if $b_j = a_j \neq 0$, then j is even and if $b_j = -a_j \neq 0$, then j is odd.

(c_2) When $q = \pm i$, then n must be an odd multiple of 2. Moreover, for $k < j < n$, if $b_j = a_j \neq 0$, then j is an even multiple of 2 and if $b_j = -a_j \neq 0$, then j is odd multiple of 2.

(ii) If $b_k = -a_k$, then $q^k = -1$. Further, we obtain the following.

(a) When k is odd, we have the following conclusions.

(a_1) For the case $q=-1$, n is even when $b_n=a_n$, whereas n is odd when $b_n=-a_n$. Moreover, for $k<j<n$, if $b_j=a_j\neq 0$, then j is an even, and if $b_j=-a_j\neq 0$, then j is odd.

(a_2) For the case $q\neq -1$, n must be an even multiple of k when $b_n=a_n$, whereas n is odd multiple of k when $b_n=-a_n$. Moreover, for $k<j<n$, if $b_j=a_j\neq 0$, then j is an even multiple of k and if $b_j=-a_j\neq 0$, then j is odd multiple of k.

(b) When k is even, the we have the following conclusions.

(b_1) For $k=4s$, $s\in\mathbb{N}$, n must be an even multiple of k if $b_n=a_n$, whereas n is odd multiple of k if $b_n=-a_n$.

(b_2) For $k=4s+2$ and $q=\pm i$, n must be an even multiple of 2 when $b_n=a_n$, whereas n is odd multiple of 2 when $b_n=-a_n$. Moreover, for $k<j<n$, if $b_j=a_j\neq 0$, then j is even multiple of 2, whereas, if $b_j=-a_j\neq 0$, then j is odd multiple of 2.

(b_3) For $k=4s+2$ and $q\neq\pm i$, n is an even multiple of k when $b_n=a_n$, whereas n is an odd multiple of k when $b_n=-a_n$. Moreover, for $k<j<n$, if $b_j=a_j\neq 0$, then j is an even multiple of k. On the other hand if $b_j=-a_j\neq 0$, then j is an odd multiple of k.

Lemma 3.3. *Let $p_1(z),p_2(z)$ be two non-constant n-th degree polynomials defined as in Lemma* 3.1 such that

$$p_1(qz)+p_2(z)=\xi_1,\quad p_2(qz)-p_1(z)=\xi_2, \tag{3.8}$$

where $q\neq 0,\xi_1,\xi_2\in\mathbb{C}$. Then, $\xi_1=a_0+b_0$, $\xi_2=b_0-a_0$ and $a_j\neq 0$ if and only if $b_j\neq 0$ with $b_j=\pm ia_j$, where $1\leq j<n$. Further $p_1(z)$, $p_2(z)$ and q satisfy one of the following.

(A) If $p_1(z)=a_nz^n+a_0$ and $p_2(z)=b_nz^n+b_0$, then $q^n=-\frac{b_n}{a_n}=\frac{a_n}{b_n}$.

(B) When $b_k=ia_k$ for $1\leq k<n$, then we get $n=(4m+1)k$, if $b_n=ia_n$ and $n=(4m-1)k$, if $b_n=-ia_n$, where $m\in\mathbb{N}$. Further, if a_jz^j, $1<j<n$ is present in $p_1(z)$, then b_jz^j is also present in $p_2(z)$ and the form of j is as follows: $j=(4t+1)k$, if $b_j=ia_j$ and $j=(4t-1)k$, if $b_j=-ia_j$, where $t\in\mathbb{N}$.

(C) When $b_k=-ia_k$ for $1\leq k<n$, then we have $n=(4m-1)k$, if $b_n=ia_n$ and $n=(4m+1)k$, if $b_n=-ia_n$, where $m\in\mathbb{N}$. Further if a_jz^j, $1<j<n$ is present in $p_1(z)$, then b_jz^j is also present in $p_2(z)$ and the form of j is as follows: $j=(4t-1)k$, if $b_j=ia_j$ and $j=(4t+1)k$, if $b_j=-ia_j$, where $t\in\mathbb{N}$.

Lemma 3.4 ([12])**.** *Let $f_j\not\equiv 0$ $(j=1,2,3)$ be meromorphic functions on $\mathbb{C}^n$ such that f_1 is not constant, $f_1+f_2+f_3=1$, and such that*

$$\sum_{j=1}^{3}\left\{N_2\left(r,\frac{1}{f_j}\right)+2\overline{N}(r,f_j)\right\}<\lambda T(r,f_j)+O(\log^+T(r,f_j))$$

holds for all r outside possibly a set with finite logarithmic measure, where $\lambda<1$ is a positive number. Then, either $f_2=1$ or $f_3=1$.

Let us define

$$A_1=\frac{1}{2\sqrt{1+w}}+\frac{1}{2i\sqrt{1-w}},\quad A_2=\frac{1}{2\sqrt{1+w}}-\frac{1}{2i\sqrt{1-w}}, \tag{3.9}$$

where $w\in\mathbb{C}$ with $w^2\neq 0,1$.

4. Proof of the Theorems

Proof of Theorem 2.1. Let $(f_1(z_1,z_2),f_2(z_1,z_2))$ be a pair of transcendental entire solutions with finite order of (1.5). Then, by Theorem 1.1, we obtain

$$\begin{cases}\frac{\partial f_1}{\partial z_1}=\cos h_1(z_1,z_2), & f_2(q_1z_1,q_2z_2)=\sin h_1(z_1,z_2),\\ \frac{\partial f_2}{\partial z_1}=\cos h_2(z_1,z_2), & f_1(q_1z_1,q_2z_2)=\sin h_2(z_1,z_2),\end{cases} \tag{4.1}$$

where $h_1(z_1,z_2)$ and $h_2(z_1,z_2)$ are entire functions in $\mathbb{C}^2$. Since $f_1(z_1,z_2),f_2(z_1,z_2)$ are finite order transcendental entire functions, $h_1(z_1,z_2)$ and $h_2(z_1,z_2)$ are non constant polynomials in $\mathbb{C}^2$.

Differentiating fourth equation of (4.1) partially with respect to z_1 and using first equation of (4.1), we easily obtain

$$q_1e^{ih_1(q_1z_1,q_2z_2)+ih_2(z_1,z_2)}+q_1e^{-ih_1(q_1z_1,q_2z_2)+ih_2(z_1,z_2)}-\frac{\partial h_2}{\partial z_1}e^{2ih_2(z_1,z_2)}=\frac{\partial h_2}{\partial z_1}. \tag{4.2}$$

Similarly, from second and third equations of (4.1), we get

$$q_1e^{ih_2(q_1z_1,q_2z_2)+ih_1(z_1,z_2)}+q_1e^{-ih_2(q_1z_1,q_2z_2)+ih_1(z_1,z_2)}-\frac{\partial h_1}{\partial z_1}e^{2ih_1(z_1,z_2)}=\frac{\partial h_1}{\partial z_1}. \tag{4.3}$$

Observe that $\frac{\partial h_1}{\partial z_1}\not\equiv 0$. Otherwise, from (4.3), we have $e^{2ih_2(q_1z_1,q_2z_2)}=-1$. As $q_1\neq 0$, $q_2\neq 0$, and $h_2(z_1,z_2)$ is non constant, we get a contradiction. By similar reason, we get $\frac{\partial h_2}{\partial z_1}\not\equiv 0$. Therefore, using Lemma 3.4 on (4.2) and (4.3), we have either $q_1e^{ih_1(q_1z_1,q_2z_2)+ih_2(z_1,z_2)}=\frac{\partial h_2}{\partial z_1}$ or $q_1e^{-ih_1(q_1z_1,q_2z_2)+ih_2(z_1,z_2)}=\frac{\partial h_2}{\partial z_1}$ and either $q_1e^{ih_2(q_1z_1,q_2z_2)+ih_1(z_1,z_2)}=\frac{\partial h_1}{\partial z_1}$ or $q_1e^{-ih_2(q_1z_1,q_2z_2)+ih_1(z_1,z_2)}=\frac{\partial h_1}{\partial z_1}$. Thus, we can discuss the following four possible cases.

Case 1. Let

$$q_1e^{ih_1(q_1z_1,q_2z_2)+ih_2(z_1,z_2)}=\frac{\partial h_2}{\partial z_1},\quad q_1e^{ih_2(q_1z_1,q_2z_2)+ih_1(z_1,z_2)}=\frac{\partial h_1}{\partial z_1}. \tag{4.4}$$

Now from (4.2), (4.3) and (4.4), we deduce that

$$q_1e^{-ih_1(q_1z_1,q_2z_2)-ih_2(z_1,z_2)}=\frac{\partial h_2}{\partial z_1},\quad q_1e^{-ih_2(q_1z_1,q_2z_2)-ih_1(z_1,z_2)}=\frac{\partial h_1}{\partial z_1}. \tag{4.5}$$

Since $h_1(z_1,z_2)$ and $h_2(z_1,z_2)$ are two non constant polynomials in $\mathbb{C}^2$, from (4.4), we conclude that

$$h_1(q_1z_1,q_2z_2)+h_2(z_1,z_2)=\xi_1,\quad h_2(q_1z_1,q_2z_2)+h_1(z_1,z_2)=\xi_2, \tag{4.6}$$

where ξ_1, ξ_2 are two constants in $\mathbb{C}$. From (4.4), (4.5) and (4.6), we get

$$q_1 e^{i\xi_1} = \frac{\partial h_2}{\partial z_1} = q_1 e^{-i\xi_1} \quad \text{and} \quad q_1 e^{i\xi_2} = \frac{\partial h_1}{\partial z_1} = q_1 e^{-i\xi_2},$$

from which we easily deduce

$$e^{2i\xi_1} = 1, \quad e^{2i\xi_2} = 1, \quad \frac{\partial h_2}{\partial z_1} = q_1 e^{i\xi_1}, \quad \frac{\partial h_1}{\partial z_1} = q_1 e^{i\xi_2}. \tag{4.7}$$

From the last two equations of (4.7), we easily obtain

$$h_1(z_1, z_2) = q_1 e^{i\xi_2} z_1 + p_1(z_2) \quad \text{and} \quad h_2(z_1, z_2) = q_1 e^{i\xi_1} z_1 + p_2(z_2), \tag{4.8}$$

where $p_1(z_2)$ and $p_2(z_2)$ are polynomials in z_2, only. In view of (4.6) and (4.8), we obtain

$$\begin{cases} q_1\left(q_1 e^{i\xi_2} + e^{i\xi_1}\right) z_1 + p_1(q_2 z_2) + p_2(z_2) = \xi_1, \\ q_1\left(q_1 e^{i\xi_1} + e^{i\xi_2}\right) z_1 + p_2(q_2 z_2) + p_1(z_2) = \xi_2, \end{cases}$$

which yield

$$q_1 e^{i\xi_2} + e^{i\xi_1} = 0, \quad q_1 e^{i\xi_1} + e^{i\xi_2} = 0 \tag{4.9}$$

and (3.1). Now, from Lemma 3.1, we see that $\xi_1 = \xi_2 = a_0 + b_0$. Therefore, from (4.9), it follows that $q_1 = -1$. From the first two equations of (4.7), we obtain that $e^{i\xi_1} = e^{i\xi_2} = e^{i(a_0+b_0)} = \pm 1$. Also from Lemma 3.1, we get precise form of $p_1(z_1, z_2)$, $p_2(z_1, z_2)$ and the value of q_2. Thus, from (4.1) and (4.8), we obtain

$$f_1(z_1, z_2) = \sin\left(e^{i\xi_1} z_1 + p_2\left(\frac{z_2}{q_2}\right)\right), \quad f_2(z_1, z_2) = \sin\left(e^{i\xi_2} z_1 + p_1\left(\frac{z_2}{q_2}\right)\right). \tag{4.10}$$

Case 2. Let

$$q_1 e^{ih_1(q_1 z_1, q_2 z_2) + ih_2(z_1, z_2)} = \frac{\partial h_2}{\partial z_1}, \quad q_1 e^{-ih_2(q_1 z_1, q_2 z_2) + ih_1(z_1, z_2)} = \frac{\partial h_1}{\partial z_1}. \tag{4.11}$$

From (4.2), (4.11) and (4.4), we get

$$q_1 e^{-ih_1(q_1 z_1, q_2 z_2) - ih_2(z_1, z_2)} = \frac{\partial h_2}{\partial z_1}, \quad q_1 e^{ih_2(q_1 z_1, q_2 z_2) - ih_1(z_1, z_2)} = \frac{\partial h_1}{\partial z_1}. \tag{4.12}$$

As $h_1(z_1, z_2)$ and $h_2(z_1, z_2)$ are non constant polynomials in $\mathbb{C}^2$, from (4.11), we conclude that

$$h_1(q_1 z_1, q_2 z_2) + h_2(z_1, z_2) = \xi_1, \quad h_2(q_1 z_1, q_2 z_2) - h_1(z_1, z_2) = \xi_2, \tag{4.13}$$

where $\xi_1, \xi_2 \in \mathbb{C}$. From (4.11), (4.12) and (4.13), we get (4.7) and (4.8). Therefore, in view of (4.8) and (4.13), we obtain

$$\begin{cases} q_1\left(q_1 e^{i\xi_2} + e^{i\xi_1}\right) z_1 + p_1(q_2 z_2) + p_2(z_2) = \xi_1, \\ q_1\left(q_1 e^{i\xi_1} - e^{i\xi_2}\right) z_1 + p_2(q_2 z_2) - p_1(z_2) = \xi_2, \end{cases}$$

from which it follows that $q_1 e^{i\xi_2} + e^{i\xi_1} = 0$, $q_1 e^{i\xi_1} - e^{i\xi_2} = 0$ and (3.8). This implies that $e^{i(\xi_2 - \xi_1)} = \pm i$. But this is not possible as we have (4.7).

Case 3. Let

$$q_1e^{-ih_1(q_1z_1,q_2z_2)+ih_2(z_1,z_2)}=\frac{\partial h_2}{\partial z_1},\quad q_1e^{ih_2(q_1z_1,q_2z_2)+ih_1(z_1,z_2)}=\frac{\partial h_1}{\partial z_1}.$$

Then, by similar arguments as used in Case 2, we get a contradiction.

Case 4. Let

$$q_1e^{-ih_1(q_1z_1,q_2z_2)+ih_2(z_1,z_2)}=\frac{\partial h_2}{\partial z_1},\quad q_1e^{-ih_2(q_1z_1,q_2z_2)+ih_1(z_1,z_2)}=\frac{\partial h_1}{\partial z_1}. \tag{4.14}$$

Now from (4.2), (4.3) and (4.14), we obtain

$$q_1e^{ih_1(q_1z_1,q_2z_2)-ih_2(z_1,z_2)}=\frac{\partial h_2}{\partial z_1},\quad q_1e^{ih_2(q_1z_1,q_2z_2)-ih_1(z_1,z_2)}=\frac{\partial h_1}{\partial z_1}. \tag{4.15}$$

Note that $h_1(z_1,z_2)$ and $h_2(z_1,z_2)$ are non constant polynomials in $\mathbb{C}^2$. Then, from (4.15), we get

$$h_1(q_1z_1,q_2z_2)-h_2(z_1,z_2)=\xi_1,\quad h_2(q_1z_1,q_2z_2)-h_1(z_1,z_2)=\xi_2, \tag{4.16}$$

where ξ_1,ξ_2 are two complex constants. Therefore, by similar arguments as in Case 1, we obtain (4.7), (3.7) and

$$q_1e^{i\xi_2}=e^{i\xi_1},\quad q_1e^{i\xi_2}=e^{i\xi_2}.$$

By Lemma 3.2, we see that $\xi_1=-\xi_2=a_0-b_0$. Then, by the above equations and (4.7), we get $q_1=1$. Hence, the form of solutions is given by (4.10), where q_2, $p_1(z_1,z_2)$ and $p_2(z_1,z_2)$ can be found from Lemma 3.2. □

Proof of Theorem 2.2. Suppose $(f_1(z_1,z_2),f_2(z_1,z_2))$ be a pair of transcendental entire solution of finite order of (1.6). Using simple concept of transformation in geometry, we assume

$$\frac{\partial f_1}{\partial z_1}=\frac{1}{\sqrt{2}}(\alpha+\beta),\quad f_2(q_1z_1,q_2z_2)=\frac{1}{\sqrt{2}}(\alpha-\beta), \tag{4.17}$$

where α,β are entire functions in $\mathbb{C}^2$. Therefore, first equation of (1.6) reduces to

$$(\sqrt{1+w}\alpha)^2+(\sqrt{1-w}\beta)^2=1.$$

Therefore, in view of Theorem 1.1, we can have

$$\alpha=\frac{\cos u(z_1,z_2)}{\sqrt{1+w}},\quad \beta=\frac{\sin u(z_1,z_2)}{\sqrt{1-w}}, \tag{4.18}$$

where $u(z_1,z_2)$ is a non-constant entire function. As f_1,f_2 are of finite order, u must be a polynomial in $\mathbb{C}^2$. Hence, by (4.17) and (4.18), we get

$$\frac{\partial f_1}{\partial z_1}=\frac{1}{2}\left(\frac{\cos u}{\sqrt{1+w}}+\frac{\sin u}{\sqrt{1-w}}\right),\quad f_2(q_1z_1,q_2z_2)=\frac{1}{2}\left(\frac{\cos u}{\sqrt{1+w}}-\frac{\sin u}{\sqrt{1-w}}\right). \tag{4.19}$$

In a similar arguments, from the second equation of (1.6), we get

$$\frac{\partial f_2}{\partial z_1}=\frac{1}{2}\left(\frac{\cos v}{\sqrt{1+w}}+\frac{\sin v}{\sqrt{1-w}}\right),\quad f_1(q_1z_1,q_2z_2)=\frac{1}{2}\left(\frac{\cos v}{\sqrt{1+w}}-\frac{\sin v}{\sqrt{1-w}}\right), \tag{4.20}$$

where v is a non-constant polynomial in $\mathbb{C}^2$.

Now differentiating second equation of (4.20) partially with respect to z_1, and using first equation of (4.19), and differentiating second equation of (4.19) partially with respect to z_1, and using first equation of (4.20), we get

(4.21)
$$\begin{cases} q_1 A_1 e^{iu(q_1 z_1, q_2 z_2)+iv(z_1,z_2)} + q_1 A_2 e^{-iu(q_1 z_1,q_2 z_2)+iv(z_1,z_2)} - iA_2 \frac{\partial v}{\partial z_1} e^{2iv(z_1,z_2)} = -iA_1 \frac{\partial v}{\partial z_1}, \\ q_1 A_1 e^{iv(q_1 z_1, q_2 z_2)+iu(z_1,z_2)} + q_1 A_2 e^{-iv(q_1 z_1,q_2 z_2)+iu(z_1,z_2)} - iA_2 \frac{\partial u}{\partial z_1} e^{2iu(z_1,z_2)} = -iA_1 \frac{\partial u}{\partial z_1}. \end{cases}$$

Note that $\frac{\partial v}{\partial z_1} \not\equiv 0$. Otherwise, from first equation of (4.21), we obtain

$$A_1 e^{2iu(q_1 z_1, q_2 z_2)} = -A_2,$$

which implies that u is constant, a contradiction. Similarly, $\frac{\partial u}{\partial z_1} \not\equiv 0$. Therefore, in view of Lemma 3.4, from (4.21), we conclude that either

$$q_1 A_1 e^{iu(q_1 z_1, q_2 z_2)+iv(z_1,z_2)} = -iA_1 \frac{\partial v}{\partial z_1} \quad \text{or} \quad q_1 A_2 e^{-iu(q_1 z_1,q_2 z_2)+iv(z_1,z_2)} = -iA_1 \frac{\partial v}{\partial z_1}$$

and either

$$q_1 A_1 e^{iv(q_1 z_1, q_2 z_2)+iu(z_1,z_2)} = -iA_1 \frac{\partial u}{\partial z_1} \quad \text{or} \quad q_1 A_2 e^{-iv(q_1 z_1,q_2 z_2)+iu(z_1,z_2)} = -iA_1 \frac{\partial u}{\partial z_1}.$$

Now we discuss by considering four possible cases.

Case 1. Let

$$q_1 e^{iu(q_1 z_1, q_2 z_2)+iv(z_1,z_2)} = -i \frac{\partial v}{\partial z_1}, \quad q_1 e^{iv(q_1 z_1, q_2 z_2)+iu(z_1,z_2)} = -i \frac{\partial u}{\partial z_1}. \tag{4.22}$$

From (4.21) and (4.22), we obtain

$$q_1 e^{-iu(q_1 z_1, q_2 z_2)-iv(z_1,z_2)} = i \frac{\partial v}{\partial z_1}, \quad q_1 e^{-iv(q_1 z_1, q_2 z_2)-iu(z_1,z_2)} = i \frac{\partial u}{\partial z_1}. \tag{4.23}$$

Since u, v are non-zero polynomials in $\mathbb{C}^2$, from (4.22), it can be concluded that

$$u(q_1 z_1, q_2 z_2) + v(z_1, z_2) = \xi_1, \quad v(q_1 z_1, q_2 z_2) + u(z_1, z_2) = \xi_2, \tag{4.24}$$

where ξ_1, ξ_2 are constants in $\mathbb{C}$. From (4.22), (4.23) and (4.24), we obtain

$$q_1 A_1 e^{i\xi_1} = -\, iA_1 \frac{\partial v}{\partial z_1}, \quad q_1 A_1 e^{i\xi_2} = -iA_1 \frac{\partial u}{\partial z_1},$$
$$q_1 A_2 e^{-i\xi_1} = iA_2 \frac{\partial v}{\partial z_1}, \quad q_1 A_2 e^{-i\xi_2} = iA_2 \frac{\partial u}{\partial z_1}.$$

From the above four equations, we deduce that

$$\frac{\partial v}{\partial z_1} = iq_1 e^{i\xi_1} = -iq_1 e^{-i\xi_1}, \quad \frac{\partial u}{\partial z_1} = iq_1 e^{i\xi_2} = -iq_1 e^{-i\xi_2},$$

which yield

$$\begin{cases} e^{2i\xi_1} = -1, \quad e^{2i\xi_2} = -1, \\ u(z_1, z_2) = iq_1 e^{i\xi_2} z_1 + p_1(z_2), \quad v(z_1, z_2) = iq_1 e^{i\xi_1} z_1 + p_2(z_2), \end{cases} \tag{4.25}$$

where $p_1(z_2), p_2(z_2)$ are polynomials in z_2, only. Now in view of the last two equations of (4.25) and (4.24), we easily get

$$\begin{cases} e^{i(\xi_1-\xi_2)} = -q_1 = e^{i(\xi_2-\xi_1)}, \\ p_1(q_2z_2) + p_2(z_2) = \xi_1, \\ p_2(q_2z_2) + p_1(z_2) = \xi_2. \end{cases} \tag{4.26}$$

From the last two equation of (4.26) and, by Lemma 3.1, it follows that $\xi_1 = \xi_2$. Therefore, from (4.26), we get $q_1 = -1$. Also, in view of the last two equations of (4.26), it can be concluded that q_2, $p_1(z_1, z_2), p_2(z_1, z_2)$ can be found from Lemma 3.1. Hence, from second equation of (4.19) and (4.20), we obtain (2.1).

Case 2. Let

$$q_1e^{iu(q_1z_1,q_2z_2)+iv(z_1,z_2)} = -i\frac{\partial v}{\partial z_1}, \quad q_1A_2e^{-iv(q_1z_1,q_2z_2)+iu(z_1,z_2)} = -iA_1\frac{\partial u}{\partial z_1}. \tag{4.27}$$

Then, from (4.21), (4.22) and (4.27), we obtain

$$q_1e^{-iu(q_1z_1,q_2z_2)-iv(z_1,z_2)} = i\frac{\partial v}{\partial z_1}, \quad q_1A_1e^{iv(q_1z_1,q_2z_2)-iu(z_1,z_2)} = iA_2\frac{\partial u}{\partial z_1}. \tag{4.28}$$

Since u, v are non constant polynomials in $\mathbb{C}^2$, in view of (4.27), we conclude that

$$u(q_1z_1, q_2z_2) + v(z_1, z_2) = \xi_1, \quad v(q_1z_1, q_2z_2) - u(z_1, z_2) = \xi_2. \tag{4.29}$$

From (4.27), (4.28) and (4.29), we get

$$\frac{\partial u}{\partial z_1} = iq_1\frac{A_2}{A_1}e^{-i\xi_2} = -iq_1\frac{A_1}{A_2}e^{i\xi_2}, \quad \frac{\partial v}{\partial z_1} = iq_1e^{i\xi_1} = -iq_1e^{-i\xi_1},$$

which yield

$$\begin{cases} e^{2i\xi_1} = -1, \quad e^{2i\xi_2} = -\frac{A_2^2}{A_1^2}, \\ u(z_1, z_2) = -iq_1\frac{A_1}{A_2}e^{i\xi_2}z_1 + p_1(z_2), \quad v(z_1, z_2) = iq_1e^{i\xi_1}z_1 + p_2(z_2), \end{cases} \tag{4.30}$$

where $p_1(z_2), p_2(z_2)$ are two polynomials in z_2, only. Therefore, from (4.29) and (4.30), we easily have

$$\begin{cases} -q_1\frac{A_1}{A_2}e^{i\xi_2} + e^{i\xi_1} = 0, \quad q_1e^{i\xi_1} + \frac{A_1}{A_2}e^{i\xi_2} = 0, \\ p_1(q_2z_2) + p_2(z_2) = \xi_1, \quad p_2(q_2z_2) - p_1(z_2) = \xi_2, \end{cases} \tag{4.31}$$

where $q_2 \neq 0, \xi_1, \xi_2 \in \mathbb{C}$. From the first equation of (4.30), we observe that $e^{i\xi_1} = \pm i$. If $e^{i\xi_1} = i$, the from (4.31), we obtain that $q_1 = \pm i$ and $e^{i\xi_2} = \frac{iA_2}{q_1A_1}$. But in view of the second equation of (4.30), we easily get a contradiction. If $e^{i\xi_1} = -i$, then similarly we can get a contradiction.

Case 3. Let

$$q_1A_2e^{-iu(q_1z_1,q_2z_2)+iv(z_1,z_2)} = -iA_1\frac{\partial v}{\partial z_1}, \quad q_1A_1e^{iv(q_1z_1,q_2z_2)+iu(z_1,z_2)} = -iA_1\frac{\partial u}{\partial z_1}.$$

By similar arguments as used in **Case 2**, we can obtain a contradiction.

Case 4. Let

$$q_1A_2e^{-iu(q_1z_1,q_2z_2)+iv(z_1,z_2)}=-iA_1\frac{\partial v}{\partial z_1},\quad q_1A_2e^{-iv(q_1z_1,q_2z_2)+iu(z_1,z_2)}=-iA_1\frac{\partial u}{\partial z_1}. \tag{4.32}$$

Then, by (4.21) and (4.32), we get

$$q_1A_1e^{iu(q_1z_1,q_2z_2)-iv(z_1,z_2)}=iA_2\frac{\partial v}{\partial z_1},\quad q_1A_1e^{iv(q_1z_1,q_2z_2)-iu(z_1,z_2)}=iA_2\frac{\partial u}{\partial z_1}. \tag{4.33}$$

Therefore, by similar arguments as in Case 1, we get

$$u(q_1z_1,q_2z_2)-v(z_1,z_2)=\xi_1,\quad v(q_1z_1,q_2z_2)-u(z_1,z_2)=\xi_2, \tag{4.34}$$

$$\begin{cases} e^{2i\xi_1}=-\frac{A_2^2}{A_1^2}=e^{2i\xi_2},\\ u(z_1,z_2)=-\frac{iq_1A_1}{A_2}e^{i\xi_2}z_1+p_1(z_2),\\ v(z_1,z_2)=-\frac{iq_1A_1}{A_2}e^{i\xi_1}z_1+p_2(z_2), \end{cases} \tag{4.35}$$

where ξ_1,ξ_2 are constants in $\mathbb{C}$. Now from (4.34) and last two equations of (4.35), we obtain

$$\begin{cases} -q_1e^{i\xi_2}+e^{i\xi_1}=0,\quad -q_1e^{i\xi_1}+e^{i\xi_2}=0,\\ p_1(q_2z_2)-p_2(z_2)=\xi_1,\quad p_2(q_2z_2)-p_1(z_2)=\xi_2. \end{cases} \tag{4.36}$$

In view of Lemma 3.2 and (4.36), it follows that $\xi_2=-\xi_1=b_0-a_0$. Therefore, from (4.36), we deduce that $e^{2i\xi_1}=q_1=\frac{1}{q_1}$. This implies that $q_1=\pm1$, and hence $e^{2i\xi_1}=\pm1$. Thus, from (4.35), we get $\frac{A_2^2}{A_1^2}=\pm1$, which is a contradiction. □

Proof of Theorem 2.3. If $(f_1(z_1,z_2),f_2(z_1,z_2))$ be a pair of finite order transcendental entire solution of (1.7). Then by Theorem 1.1, we get

$$\begin{cases} \dfrac{\partial f_1}{\partial z_1}=\cos h_1(z_1,z_2),\quad f_2(q_1z_1,q_2z_2)-f_1(z_1,z_2)=\sin h_1(z_1,z_2),\\ \dfrac{\partial f_2}{\partial z_1}=\cos h_2(z_1,z_2),\quad f_1(q_1z_1,q_2z_2)-f_2(z_1,z_2)=\sin h_2(z_1,z_2), \end{cases} \tag{4.37}$$

where $h_1(z_1,z_2)$ and $h_2(z_1,z_2)$ are entire functions in $\mathbb{C}^2$.

First suppose that $h_1(z_1,z_2)\equiv\gamma_1$ and $h_2(z_1,z_2)\equiv\gamma_2$, where γ_1,γ_2 are constants in $\mathbb{C}$. Then from first and third equations of (4.37), we have

$$f_1(z_1,z_2)=z_1\cos\gamma_1+\phi_1(z_2)\quad\text{and}\quad f_2(z_1,z_2)=z_1\cos\gamma_2+\phi_2(z_2), \tag{4.38}$$

where ϕ_1,ϕ_2 are entire functions in z_2, only. Therefore, from second and fourth equations of (4.37) and (4.38), we obtain

$$\begin{cases} (q_1\cos\gamma_2-\cos\gamma_1)z_1+\phi_2(q_2z_2)-\phi_1(z_2)=\sin\gamma_1,\\ (q_1\cos\gamma_1-\cos\gamma_2)z_1+\phi_1(q_2z_2)-\phi_2(z_2)=\sin\gamma_2, \end{cases}$$

which implies that

$$\begin{cases} q_1\cos\gamma_2-\cos\gamma_1=0, \quad q_1\cos\gamma_1-\cos\gamma_2=0,\\ \phi_2(q_2z_2)-\phi_1(z_2)=\sin\gamma_1, \quad \phi_1(q_2z_2)-\phi_2(z_2)=\sin\gamma_2. \end{cases} \tag{4.39}$$

In view of (4.39), we observe that if $\gamma_1=(2m+1)\pi/2$, then $\gamma_2=(2n+1)\pi/2$, m,n being an integer. In this case (1.7) does not contain the partial derivative term. So, to avoid this situation we assume that γ_1 and γ_2 are not odd multiple of $\pi/2$. Therefore, from first two equations of (4.42), we have $q_1=\pm1$. When $q_1=1$, $\gamma_2=2n\pi\pm\gamma_1$, whereas when $q_1=-1$, $\gamma_2=(2n+1)\pi\pm\gamma_1$, n being an integer.

Next suppose that $h_1(z_1,z_2)\equiv\gamma_1$, a constant and $h_2(z_1,z_2)$ is non constant. Then the form of $f_1(z_1,z_2)$ is given by (4.38). Using this in the first equation of (1.7), we have

$$f_2(z_1,z_2)=\frac{z_1\cos\gamma_1}{q_1}+\phi_1\left(\frac{z_2}{q_2}\right)\pm\sin\gamma_1. \tag{4.40}$$

Using first equation of (4.39) and (4.40) in second equation of (1.7), we get

$$\cos\gamma_1\left(q_1-\frac{1}{q_1}\right)z_1+\phi_1(q_2z_2)-\phi_1\left(\frac{z_2}{q_2}\right)\mp\sin\gamma_1=\pm\frac{\sqrt{q_1^2-\cos^2\gamma_1}}{q_1}.$$

This implies that $q_1\pm1$ and

$$\phi_1(q_2z_2)-\phi_1\left(\frac{z_2}{q_2}\right)=\pm\left[\sin\gamma_1+\frac{\sqrt{q_1^2-\cos^2\gamma_1}}{q_1}\right].$$

Finally, suppose that h_1,h_2 both are nonconstant entire functions in $\mathbb{C}^2$. Then by similar arguments as used in the proof of Theorem 2.1, we can easily deduce that

$$\begin{cases} q_1e^{ih_1(q_1z_1,q_2z_2)+ih_2(z_1,z_2)}+q_1e^{-ih_1(q_1z_1,q_2z_2)+ih_2(z_1,z_2)}-\left(1+\frac{\partial h_2}{\partial z_1}\right)e^{2ih_2}=1+\frac{\partial h_2}{\partial z_1},\\ q_1e^{ih_2(q_1z_1,q_2z_2)+ih_1(z_1,z_2)}+q_1e^{-ih_2(q_1z_1,q_2z_2)+ih_1(z_1,z_2)}-\left(1+\frac{\partial h_1}{\partial z_1}\right)e^{2ih_1}=1+\frac{\partial h_1}{\partial z_1}. \end{cases} \tag{4.41}$$

Note that $1+\frac{\partial h_2}{\partial z_1}\not\equiv0$. Otherwise, from (4.41), we can easily see that $h_1(q_1z_1,q_2z_2)$ is a constant, which is not possible as $h_1(z_1,z_2)$ is a non constant polynomial in $\mathbb{C}^2$. Similarly, we can prove that $1+\frac{\partial h_1}{\partial z_1}\not\equiv0$. Therefore, by Lemma 3.4, it follows that either $q_1e^{ih_1(q_1z_1,q_2z_2)+ih_2(z_1,z_2)}=1+\frac{\partial h_2}{\partial z_1}$ or $q_1e^{-ih_1(q_1z_1,q_2z_2)+ih_2(z_1,z_2)}=1+\frac{\partial h_2}{\partial z_1}$ and either $q_1e^{ih_2(q_1z_1,q_2z_2)+ih_1(z_1,z_2)}=1+\frac{\partial h_1}{\partial z_1}$ or $q_1e^{-ih_2(q_1z_1,q_2z_2)+ih_1(z_1,z_2)}=1+\frac{\partial h_1}{\partial z_1}$.

Now, we can discuss by four possible cases.

Case 1. Let

$$q_1e^{ih_1(q_1z_1,q_2z_2)+ih_2(z_1,z_2)}=1+\frac{\partial h_2}{\partial z_1}, \quad q_1e^{ih_2(q_1z_1,q_2z_2)+ih_1(z_1,z_2)}=1+\frac{\partial h_1}{\partial z_1}. \tag{4.42}$$

From (4.41) and (4.42), we get

$$q_1e^{-ih_1(q_1z_1,q_2z_2)-ih_2(z_1,z_2)}=1+\frac{\partial h_2}{\partial z_1}, \quad q_1e^{-ih_2(q_1z_1,q_2z_2)-ih_1}=1+\frac{\partial h_1}{\partial z_1}. \tag{4.43}$$

Therefore, by the same arguments as used in Case 1 in the proof of Theorem 2.1, we obtain

$$\begin{cases} e^{2i\xi_1} = 1, \quad e^{2i\xi_2} = 1, \\ h_1(z_1,z_2) = \left(q_1 e^{i\xi_2} - 1\right) z_1 + p_1(z_2), \\ h_2(z_1,z_2) = \left(q_1 e^{i\xi_1} - 1\right) z_1 + p_2(z_2), \end{cases} \tag{4.44}$$

and

$$\begin{cases} q_1\left(q_1 e^{i\xi_1} - 1\right) + \left(q_1 e^{i\xi_2} - 1\right) = 0, \quad q_1\left(q_1 e^{i\xi_2} - 1\right) + \left(q_1 e^{i\xi_1} - 1\right) = 0, \\ p_1(q_2 z_2) + p_2(z_2) = \xi_1, \quad p_2(q_2 z_2) + p_1(z_2) = \xi_2, \end{cases} \tag{4.45}$$

where $p_1(z_2), p_2(z_2)$ are polynomials in z_2, only. In view of Lemma 3.1 and (4.45), we see that $\xi_1 = \xi_2 = a_0 + b_0$ and the precise from of q_2, p_1, p_2 are given in Lemma 3.1. From the first equation of (4.44), it follows that $e^{i\xi_1} = \pm 1$.

First suppose $e^{i\xi_1} = 1$, then in view of first two equations of (4.45), we see that $q_1 = \pm 1$. Therefore, for $e^{i\xi_1} = 1$ and $q_1 = 1$, (4.44) yields $h_1(z_1,z_2) = p_1(z_2)$ and $h_2(z_1,z_2) = p_2(z_2)$. Thus from first and third equations of (4.37), we obtain

$$f_1(z_1,z_2) = z_1 \cos p_1(z_2) + T_1(z_2), \quad f_2(z_1,z_2) = z_1 \cos p_2(z_2) + T_2(z_2), \tag{4.46}$$

where T_1, T_2 are entire functions in z_2, only. From (4.46) and second and fourth equations of (4.37), we easily deduce that

$$T_1(q_2^2 z_2) = T_1(z_2), \quad T_2(q_2^2 z_2) = T_2(z_2). \tag{4.47}$$

Next suppose that $e^{i\xi_1} = 1$ and $q_1 = -1$. Then from (4.44), we have $h_1(z_1,z_2) = -2z_1 + p_1(z_2)$ and $h_2(z_1,z_2) = -2z_1 + p_2(z_2)$, and hence from first and third equations of (4.37), we obtain

$$\begin{cases} f_1(z_1,z_2) = -\frac{1}{2}\sin(-2z_1 + p_1(z_2)) + T_1(z_2), \\ f_2(z_1,z_2) = -\frac{1}{2}\sin(-2z_1 + p_2(z_2)) + T_2(z_2), \end{cases} \tag{4.48}$$

where T_1, T_2 are entire functions in z_2, only. From (4.48) and second and fourth equations of (4.37), we get (4.47).

Next suppose that $e^{i\xi_1} = -1$. Then from (4.45), we have $q_1 = -1$. Therefore, from the last two equations of (4.44) and first and third equations of (4.37), we easily get (4.46). Hence, using (4.46) in second and third equations of (4.37), we deduce that

$$T_1(q_2^2 z_2) - T_1(z_2) = 2\sin p_1(z_2) \quad \text{and} \quad T_2(q_2^2 z_2) - T_2(z_2) = 2\sin p_2(z_2). \tag{4.49}$$

Case 2. Let

$$q_1 e^{-ih_2(q_1z_1,q_2z_2)+ih_1(z_1,z_2)} = 1 + \frac{\partial h_1}{\partial z_1}, \quad q_1 e^{ih_1(q_1z_1,q_2z_2)+ih_2(z_1,z_2)} = 1 + \frac{\partial h_2}{\partial z_1}. \tag{4.50}$$

From (4.41) and (4.50), we get

$$q_1 e^{ih_2(q_1z_1,q_2z_2)-ih_1(z_1,z_2)} = 1 + \frac{\partial h_1}{\partial z_1}, \quad q_1 e^{-ih_1(q_1z_1,q_2z_2)-ih_2(z_1,z_2)} = 1 + \frac{\partial h_2}{\partial z_1}. \tag{4.51}$$

By similar argument as in Case 1, we can obtain (4.44) and

$$\begin{cases} q_1\left(q_1e^{i\xi_2}-1\right)+\left(q_1e^{i\xi_1}-1\right)=0, \quad q_1\left(q_1e^{i\xi_1}-1\right)-\left(q_1e^{i\xi_2}-1\right)=0, \\ p_1(q_2z_2)+p_2(z_2)=\xi_1, \quad p_2(q_2z_2)-p_1(z_2)=\xi_2, \end{cases} \tag{4.52}$$

where $p_1(z_2), p_2(z_2)$ are polynomials in z_2, only and hence can be found from Lemma 3.3 along with q_2. In view of the first two equations of (4.44), let us first assume that $e^{i\xi_1}=1$, $e^{i\xi_2}=1$. Then, by (4.52), we get $q_1=1$, and hence by (4.44), we have $h_1(z_1,z_2)=p_1(z_2)$ and $h_2(z_1,z_2)=p_2(z_2)$. Therefore, from first and third equations of (4.37), we obtain (4.46), where T_1, T_2 are entire functions in z_2, only.

Now, by second and third equations of (4.37) and (4.46), we can easily deduce that

$$T_1(q_2^2z_2)-T_1(z_2)=2\sin p_1(z_2) \quad \text{and} \quad T_2(q_2^2z_2)=T_2(z_2).$$

Next suppose that $e^{i\xi_1}=1$ and $e^{i\xi_2}=-1$. Then, from (4.52), it follows that $q_1=\pm i$. If $q_1=i$, then from (4.44), we have $h_1(z_1,z_2)=-(1+i)z_1+p_1(z_2)$ and $h_2(z_1,z_2)=-(1-i)+p_2(z_2)$, and hence from first and third equations of (4.37), we deduce that

$$\begin{cases} f_1(z_1,z_2)=-\frac{1}{1+i}\sin(-(1+i)z_1+p_1(z_2))+T_1(z_2), \\ f_2(z_1,z_2)=-\frac{1}{1-i}\sin(-(1-i)z_1+p_2(z_2))+T_2(z_2), \end{cases} \tag{4.53}$$

where T_1, T_2 are entire functions in z_2. By second and third equations of (4.37) and (4.53), we can prove that T_1, T_2 satisfy (4.47).

If $q_1=-i$, then similarly we can get

$$\begin{cases} f_1(z_1,z_2)=-\frac{1}{1-i}\sin(-(1-i)z_1+p_1(z_2))+T_1(z_2), \\ f_2(z_1,z_2)=-\frac{1}{1+i}\sin(-(1+i)z_1+p_2(z_2))+T_2(z_2), \end{cases}$$

where T_1, T_2 are entire functions in z_2 satisfying (4.47).

If $e^{i\xi_1}=-1$ and $e^{i\xi_2}=1$, then from the first two equations of (4.52), we easily reach to a contradiction.

Finally, let $e^{i\xi_1}=-1$ and $e^{i\xi_2}=-1$. Then from (4.52), we have $q_1=-1$. Therefore, in view of last two equations of (4.44), first and third equations of (4.37), we obtain (4.46). Therefore, from second and fourth equations of (4.37), it can be shown that

$$T_1(q_2^2z_2)=T_1(z_2) \quad \text{and} \quad T_2(q_2^2z_2)-T_2(z_2)=2\sin p_2(z_2).$$

Case 3. Let

$$q_1e^{ih_2(q_1z_1,q_2z_2)+ih_1(z_1,z_2)}=1+\frac{\partial h_1}{\partial z_1}, \quad q_1e^{-ih_1(q_1z_1,q_2z_2)+ih_2(z_1,z_2)}=1+\frac{\partial h_2}{\partial z_1}.$$

Then, after interchanging $p_1(z_2)$ with $p_2(z_2)$, ξ_1 with ξ_2 in (3.8) in Lemma 3.3, we can get the same conclusions as obtained in Case 2. So we omit the details.

Case 4. Let

$$q_1e^{ih_1(z_1,z_2)-ih_2(q_1z_1,q_2z_2)}=1+\frac{\partial h_1}{\partial z_1}, \quad q_1e^{ih_2(z_1,z_2)-ih_1(q_1z_1,q_2z_2)}=1+\frac{\partial h_2}{\partial z_1}. \tag{4.54}$$

From (4.41) and (4.54), we get

$$q_1 e^{ih_2(q_1z_1,q_2z_2)-ih_1(z_1,z_2)} = 1+\frac{\partial h_1}{\partial z_1}, \quad q_1 e^{ih_1(q_1z_1,q_2z_2)-ih_2(z_1,z_2)} = 1+\frac{\partial h_2}{\partial z_1}. \tag{4.55}$$

Now, by similar arguments as used in Case 1, we obtain (4.44), (3.7) and

$$q_1\left(q_1e^{i\xi_2}-1\right)-\left(q_1e^{i\xi_1}-1\right)=0, \quad q_1\left(q_1e^{i\xi_1}-1\right)-\left(q_1e^{i\xi_2}-1\right)=0. \tag{4.56}$$

In view of Lemma 3.2, we see that $\xi_1 = a_0 - b_0 = -\xi_2$. Also the form of the polynomials p_1, p_2 and the value of q_2 can be found from Lemma 3.2. Hence, by first two equations of (4.44), it follows that $e^{i\xi_2} = 1$ when $e^{i\xi_1} = 1$, whereas $e^{i\xi_2} = -1$ when $e^{i\xi_1} = -1$.

First suppose that $e^{i\xi_1} = 1$. Then, by (4.56), we get $q_1 = 1$, and hence in view of (4.44), first and third equation equation of (4.37), we easily have (4.46). Using second and fourth equations of (4.37) and (4.46), we conclude that

$$T_1(q_2^2z_2) - T_1(z_2) = 2\sin p_1(z_2) \quad \text{and} \quad T_2(q_2^2z_2) - T_2(z_2) = 2\sin p_2(z_2).$$

Next suppose $e^{i\xi_1} = -1$. Then, from (4.56), we obtain $q_1 = \pm 1$. If $q_1 = 1$, by (4.44), first and third equations of (4.37), we obtain (4.48). Using (4.48) in second and fourth equations of (4.37), we obtain (4.47). Similarly, if $q_1 = -1$, then we have (4.46), where T_1, T_2 satisfy (4.47). □

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