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INEQUALITIES FOR AVERAGED POWER-TYPE FUNCTIONS UNDER CONVEXITY CONDITIONS

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Abstract. This paper explores certain inequalities involving averages of specific functions defined by combining powers of the input variable. We identify the conditions under which these functions exhibit a shape known as convexity, which roughly means they curve upwards. Using these conditions, we establish precise inequalities that compare the average value of the function over an interval with its values at specific points within that interval. We also provide exact calculations for the averages involved.

1. Introduction

Integral inequalities play a significant role in mathematical analysis, particularly in the context of convex functions, where the classical Hadamard inequality and its generalizations are extensively studied. Hadamard-type inequalities provide sharp estimates on the behavior of a function by comparing its integral average to the function's values at specific points. Such inequalities serve as fundamental tools in optimization, numerical analysis, probability theory, and many other areas of applied mathematics.

In recent years, special versions of Hadamard inequalities have been developed for parametric classes of functions, and the influence of convexity conditions on the validity of these inequalities has been examined in detail. In particular, functions defined as various linear combinations of polynomials and power functions frequently arise in both theoretical and applied analyses. The sign and properties of their second

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derivatives serve as a primary method for investigating convexity. For interested readers, we refer to references [1–8].

In this study, for $p > 2$, we investigate the convexity conditions of functions defined by

$$f_1(t) := g(t) - m\frac{t^p}{p(p-1)}, \quad f_2(t) := M\frac{t^p}{p(p-1)} - g(t), \quad \text{for } t \in [a,b],$$

and establish Hadamard-type integral inequalities under appropriate parameter constraints. The obtained results offer a novel and extended perspective on comparing the integral averages of such functions with their values at boundary and midpoint arguments. Moreover, these inequalities contribute to a deeper understanding of the relationship between structural properties of convex functions and classical integral inequalities.

The remainder of this paper first provides a detailed analysis of the convexity conditions via the second derivatives of the given functions. Subsequently, Hadamard-type inequalities are derived under these conditions, and the integrals involved are explicitly evaluated. Thus, new integral inequalities are presented that are both theoretically sound and practically applicable.

We start with a classical refinement of Hadamard's inequality for convex functions.

Lemma 1.1 (Refined Hadamard Inequality)**.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I, and let $a, b \in I$ with $a < b$. Then,*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right].$$

2. Main Theorems

Theorem 2.1. *Let $g, h : [a,b] \to \mathbb{R}$ be twice differentiable functions such that $h''(t) > 0$ for all $t \in [a,b]$. Suppose there exist real numbers m and M such that*

$$m \le \min_{t\in[a,b]} \frac{g''(t)}{h''(t)} \quad \text{and} \quad M \ge \max_{t\in[a,b]} \frac{g''(t)}{h''(t)}.$$

Then, the following Hadamard-type inequalities hold:

$$\begin{aligned}
&g\left(\frac{a+b}{2}\right) - mh\left(\frac{a+b}{2}\right) + \frac{m}{b-a}\int_a^b h(t)\,dt \\
\le &\frac{1}{b-a}\int_a^b g(t)\,dt \\
\le &\frac{g(a)+g(b)}{2} - m\frac{h(a)+h(b)}{2} + g\left(\frac{a+b}{2}\right) - mh\left(\frac{a+b}{2}\right) + \frac{m}{b-a}\int_a^b h(t)\,dt
\end{aligned}$$

and

$$\frac{g(a)+g(b)}{2} - M\frac{h(a)+h(b)}{2} - Mh\left(\frac{a+b}{2}\right) + g\left(\frac{a+b}{2}\right) + \frac{M}{b-a}\int_a^b h(t)dt$$
$$\leq \frac{1}{b-a}\int_a^b g(t)dt \leq \frac{M}{b-a}\int_a^b h(t)dt - Mh\left(\frac{a+b}{2}\right) + g\left(\frac{a+b}{2}\right).$$

Proof. We define the following two functions:

$$f_1(t) := (g(t) - mh(t)), \quad f_2(t) := (Mh(t) - g(t)).$$

We first analyze the convexity of f_1. Taking the second derivative, we have

$$f_1''(t) = g''(t) - mh''(t).$$

By the hypothesis $m \leq \frac{g''(t)}{h''(t)}$, we get $g''(t) - mh''(t) \geq 0$ for all $t \in [a,b]$, and hence $f_1''(t) \geq 0$. Therefore, f_1 is convex on $[a,b]$. Similarly, for f_2, we compute:

$$f_2''(t) = Mh''(t) - g''(t).$$

By the hypothesis $M \geq \frac{g''(t)}{h''(t)}$, we obtain $Mh''(t) - g''(t) \geq 0$ for all $t \in [a,b]$, so $f_2''(t) \geq 0$, and hence f_2 is also convex on $[a,b]$. Since both f_1 and f_2 are convex functions on $[a,b]$, we can apply the well-known Refined Hadamard-type inequality for f_1, we obtain:

$$f_1\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f_1(t)\,dt \leq \frac{1}{2}\left[\frac{f_1(a)+f_1(b)}{2} + f_1\left(\frac{a+b}{2}\right)\right],$$

which, by inserting the definition of f_1, becomes:

$$\left(g\left(\frac{a+b}{2}\right) - mh\left(\frac{a+b}{2}\right)\right)$$
$$\leq \frac{1}{b-a}\int_a^b [g(t) - mh(t)]\,dt$$
$$\leq \frac{1}{2}\left(g(a) - mh(a) + g(b) - mh(b)\right) + \left(g\left(\frac{a+b}{2}\right) - mh\left(\frac{a+b}{2}\right)\right).$$

Likewise, for f_2, we get:

$$f_2\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f_2(t)\,dt \leq \frac{1}{2}\left[\frac{f_2(a)+f_2(b)}{2} + f_2\left(\frac{a+b}{2}\right)\right],$$

which leads to:

$$\left(Mh\left(\frac{a+b}{2}\right) - g\left(\frac{a+b}{2}\right)\right)$$
$$\leq \frac{1}{b-a}\int_a^b [Mh(t) - g(t)]\,dt$$
$$\leq \frac{1}{2}\left(Mh(a) - g(a) + Mh(b) - g(b)\right) + \left(Mh\left(\frac{a+b}{2}\right) - g\left(\frac{a+b}{2}\right)\right).$$

This completes the proof. □

Theorem 2.2. *Let $a, b > 0$ with $a < b$, $p > 2$, and let $g : [a, b] \to \mathbb{R}$ be a twice continuously differentiable function. Suppose that there exist real constants m and M such that*

$$mt^{p-2} \le g''(t) \le Mt^{p-2}, \quad \text{for all } t \in [a, b].$$

Then, the following Hadamard-type inequalities hold:

$$\begin{aligned} & g\left(\frac{a+b}{2}\right) - \frac{m}{p(p-1)}\left(\frac{a+b}{2}\right)^p + \frac{m}{b-a} \cdot \frac{b^{p+1}-a^{p+1}}{(p+1)p(p-1)} \\ \le & \frac{1}{b-a}\int_a^b g(t)dt \\ \le & \frac{1}{2}\left[\frac{g(a)+g(b)}{2} - \frac{m}{2} \cdot \frac{a^p+b^p}{p(p-1)} + g\left(\frac{a+b}{2}\right) - \frac{m}{p(p-1)}\left(\frac{a+b}{2}\right)^p\right] \end{aligned}$$

and

$$\begin{aligned} & \frac{M}{p(p-1)}\left(\frac{a+b}{2}\right)^p - g\left(\frac{a+b}{2}\right) \\ \le & \frac{m}{b-a} \cdot \frac{b^{p+1}-a^{p+1}}{(p+1)p(p-1)} - \frac{1}{b-a}\int_a^b g(t)dt \\ \le & \frac{1}{2}\left[\frac{M}{2} \cdot \frac{a^p+b^p}{p(p-1)} - \frac{g(a)+g(b)}{2} + \frac{M}{p(p-1)}\left(\frac{a+b}{2}\right)^p - g\left(\frac{a+b}{2}\right)\right]. \end{aligned}$$

Proof. Define the functions

$$f_1(t) := g(t) - m\frac{t^p}{p(p-1)}, \quad f_2(t) := M\frac{t^p}{p(p-1)} - g(t), \quad \text{for } t \in [a, b].$$

The second derivative of f_1 is given by

$$f_1''(t) = g''(t) - mt^{p-2}.$$

By the hypothesis $g''(t) \ge mt^{p-2}$, we have $f_1''(t) \ge 0$, so f_1 is convex on $[a, b]$.

Similarly, the second derivative of f_2 is

$$f_2''(t) = Mt^{p-2} - g''(t).$$

Since $g''(t) \le Mt^{p-2}$, we obtain $f_2''(t) \ge 0$, and thus f_2 is also convex on $[a, b]$.

Since both f_1 and f_2 are convex, the classical Hadamard inequality applies:

$$f_i\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f_i(t)\,dt \le \frac{1}{2}\left[\frac{f_i(a)+f_i(b)}{2} + f_i\left(\frac{a+b}{2}\right)\right], \quad \text{for } i = 1, 2.$$

Substituting the explicit forms of f_1 and f_2 completes the proof. □

Corollary 2.1 (Case $p = 2$). *Let $a, b > 0$ with $a < b$, and let $g : [a, b] \to \mathbb{R}$ be twice continuously differentiable such that there exist $m, M \in \mathbb{R}$ with*

$$m \le g''(t) \le M, \quad \text{for all } t \in [a, b].$$

Then, the following inequalities hold:

$$g\left(\frac{a+b}{2}\right)+\frac{m\,(b-a)^2}{24}\le\frac{1}{b-a}\int_a^b g(t)dt$$
$$\le\frac{1}{2}\left[\frac{g(a)+g(b)}{2}+g\left(\frac{a+b}{2}\right)\right]-\frac{m\,(b-a)^2}{48}$$

and

$$\frac{1}{2}\left[\frac{g(a)+g(b)}{2}+g\left(\frac{a+b}{2}\right)\right]-\frac{M\,(b-a)^2}{48}\le\frac{1}{b-a}\int_a^b g(t)dt$$
$$\le g\left(\frac{a+b}{2}\right)+\frac{M\,(b-a)^2}{24}.$$

Proof. For $p=2$, the hypothesis in the theorem becomes $m\le g''(t)\le M$, which coincides with the hypothesis here.

Define

$$f_1(t):=g(t)-\frac{m}{2}t^2,\quad f_2(t):=\frac{M}{2}t^2-g(t).$$

Then,

$$f_1''(t)=g''(t)-m\ge 0,\quad f_2''(t)=M-g''(t)\ge 0,$$

so f_1 and f_2 are convex on $[a,b]$.

By applying the Hadamard-type inequality for convex functions, the inequalities follow as

$$g\left(\frac{a+b}{2}\right)-\frac{m}{2}\left(\frac{a+b}{2}\right)^2$$
$$\le\frac{1}{b-a}\int_a^b g(t)dt-\frac{m}{b-a}\cdot\frac{b^3-a^3}{6}$$
$$\le\frac{1}{2}\left[\frac{g(a)+g(b)}{2}-\frac{m}{2}\cdot\frac{a^2+b^2}{2}+g\left(\frac{a+b}{2}\right)-\frac{m}{2}\left(\frac{a+b}{2}\right)^2\right]$$

and

$$\frac{M}{2}\left(\frac{a+b}{2}\right)^2-g\left(\frac{a+b}{2}\right)$$
$$\le\frac{m}{b-a}\cdot\frac{b^3-a^3}{6}-\frac{1}{b-a}\int_a^b g(t)dt$$
$$\le\frac{1}{2}\left[\frac{M}{2}\cdot\frac{a^2+b^2}{2}-\frac{g(a)+g(b)}{2}+\frac{M}{2}\left(\frac{a+b}{2}\right)^p-g\left(\frac{a+b}{2}\right)\right],$$

which completes the proof. □

Corollary 2.2 (Weighted double-sided inequality)**.** *Let $a, b > 0$ with $a < b$, and let $g : [a, b] \to \mathbb{R}$ be a twice continuously differentiable function. Suppose there exist constants $m, M \in \mathbb{R}$ such that*

$$m \leq g''(t) \leq M, \quad \text{for all } t \in [a, b].$$

Let $\lambda, \mu, \alpha, \beta \in [0, 1]$ with $\lambda + \mu = 1$ and $\alpha + \beta = 1$. Then the following inequality holds:

$$\begin{aligned}
&\left(\lambda + \frac{\mu}{2}\right) g\left(\frac{a+b}{2}\right) + \frac{\mu}{4}\left[g(a) + g(b)\right] + \left(\lambda \cdot \frac{m(b-a)^2}{24} - \mu \cdot \frac{M(b-a)^2}{48}\right) \\
\leq & \frac{1}{b-a}\int_a^b g(t)\,dt \\
\leq & \left(\frac{\alpha}{2} + \beta\right) g\left(\frac{a+b}{2}\right) + \frac{\alpha}{4}\left[g(a) + g(b)\right] + \left(-\alpha \cdot \frac{m(b-a)^2}{48} + \beta \cdot \frac{M(b-a)^2}{24}\right).
\end{aligned}$$

Proof. From the assumptions $m \leq g''(t) \leq M$ on $[a, b]$, we apply the known bounds of the Hermite-Hadamard and Simpson-type inequalities refined for convexity constraints. Specifically, the following two-sided estimates hold:

$$\begin{aligned}
g\left(\frac{a+b}{2}\right) + \frac{m(b-a)^2}{24} \leq & \frac{1}{b-a}\int_a^b g(t)\,dt \\
\leq & \frac{1}{2}\left[\frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right)\right] - \frac{m(b-a)^2}{48},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\left[\frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right)\right] - \frac{M(b-a)^2}{48} \leq & \frac{1}{b-a}\int_a^b g(t)\,dt \\
\leq & g\left(\frac{a+b}{2}\right) + \frac{M(b-a)^2}{24}.
\end{aligned}$$

We now form convex combinations of the above inequalities. Multiply the first inequality from the top with weights λ and μ such that $\lambda + \mu = 1$

$$\begin{aligned}
&\lambda\left(g\left(\frac{a+b}{2}\right) + \frac{m(b-a)^2}{24}\right) + \mu\left(\frac{1}{2}\left[\frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right)\right] - \frac{M(b-a)^2}{48}\right) \\
=& \left(\lambda + \frac{\mu}{2}\right) g\left(\frac{a+b}{2}\right) + \frac{\mu}{4}[g(a) + g(b)] + \left(\lambda \cdot \frac{m(b-a)^2}{24} - \mu \cdot \frac{M(b-a)^2}{48}\right).
\end{aligned}$$

This gives a lower bound for the integral mean. Similarly, for the upper bound, multiply the second inequality from the bottom by weights α and β satisfying $\alpha + \beta = 1$, we obtain

$$\begin{aligned}
&\alpha\left(\frac{1}{2}\left[\frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right)\right] - \frac{m(b-a)^2}{48}\right) + \beta\left(g\left(\frac{a+b}{2}\right) + \frac{M(b-a)^2}{24}\right) \\
=& \left(\frac{\alpha}{2} + \beta\right) g\left(\frac{a+b}{2}\right) + \frac{\alpha}{4}[g(a) + g(b)] + \left(-\alpha\frac{m(b-a)^2}{48} + \beta\frac{M(b-a)^2}{24}\right).
\end{aligned}$$

Combining both bounds, we arrive at the desired double inequality. □

Remark 2.1 (Special cases: Simpson-type inequalities). Under the assumptions of the previous corollary, the following special cases correspond to classical Simpson-type inequalities.

- **Lower bound with** $g''(t) \geq m$. Choosing $\lambda = 1$, $\mu = 0$ yields
$$g\left(\frac{a+b}{2}\right) + \frac{m(b-a)^2}{24} \leq \frac{1}{b-a}\int_a^b g(t)\,dt.$$
- **Upper bound with** $g''(t) \leq M$. Choosing $\alpha = 0$, $\beta = 1$ gives
$$\frac{1}{b-a}\int_a^b g(t)\,dt \leq g\left(\frac{a+b}{2}\right) + \frac{M(b-a)^2}{24}.$$
- **Symmetric Simpson-type estimate**. Taking $\lambda = \mu = \frac{1}{2}$ and $\alpha = \beta = \frac{1}{2}$, we have
$$\begin{aligned}&\frac{3}{4}g\left(\frac{a+b}{2}\right) + \frac{1}{8}\left[g(a)+g(b)\right] + \left(\frac{m(b-a)^2}{48} - \frac{M(b-a)^2}{96}\right)\\ \leq&\frac{1}{b-a}\int_a^b g(t)\,dt\\ \leq&\frac{3}{4}g\left(\frac{a+b}{2}\right) + \frac{1}{8}\left[g(a)+g(b)\right] + \left(-\frac{m(b-a)^2}{96} + \frac{M(b-a)^2}{48}\right).\end{aligned}$$

Corollary 2.3 (Case $p = 4$)**.** *Let $a, b > 0$ with $a < b$, and let $g : [a,b] \to \mathbb{R}$ be twice continuously differentiable such that there exist $m, M \in \mathbb{R}$ with*
$$mt^2 \leq g''(t) \leq Mt^2, \quad \textit{for all } t \in [a,b].$$
Then, the following inequalities hold:
$$\begin{aligned}&g\left(\frac{a+b}{2}\right) + \frac{m\left(b^4+b^3a+a^2b^2+ba^3\right)}{60} - \frac{m\left(a+b\right)^4}{16}\\ \leq&\frac{1}{b-a}\int_a^b g(t)dt\\ \leq&\frac{1}{2}\left[\frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right)\right] + \frac{m\left(b^4+b^3a+a^2b^2+ba^3\right)}{60} - m\frac{a^4+b^4}{48} - \frac{m\left(a+b\right)^4}{384},\end{aligned}$$
and
$$\begin{aligned}&\frac{1}{2}\left[\frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right)\right] + \frac{M\left(b^4+b^3a+a^2b^2+ba^3\right)}{60} - M\frac{a^4+b^4}{48} - \frac{M\left(a+b\right)^4}{384}\\ \leq&\frac{1}{b-a}\int_a^b g(t)dt \leq g\left(\frac{a+b}{2}\right) + \frac{M\left(b^4+b^3a+a^2b^2+ba^3\right)}{60} - \frac{M\left(a+b\right)^4}{16}.\end{aligned}$$

Proof. For $p = 4$, so the hypothesis becomes $mt^2 \leq g''(t) \leq Mt^2$.

Define
$$f_1(t) := g(t) - m\frac{t^4}{12}, \quad f_2(t) := M\frac{t^4}{12} - g(t), \quad \text{for } t \in [a,b].$$

Then,

$$f_1''(t) = g''(t) - mt^2 \geq 0, \quad f_2''(t) = Mt^2 - g''(t) \geq 0,$$

thus f_1 and f_2 are convex on $[a, b]$.

Applying the Hadamard-type inequality to the convex functions f_1 and f_2 gives the inequalities

$$\begin{aligned} g\left(\frac{a+b}{2}\right) - m\left(\frac{a+b}{2}\right)^4 &\leq \frac{1}{b-a}\int_a^b g(t)dt - \frac{m}{b-a}\cdot\frac{b^5-a^5}{60} \\ &\leq \frac{1}{2}\left[\frac{g(a)+g(b)}{2} - m\frac{a^4+b^4}{24} + g\left(\frac{a+b}{2}\right) - \frac{m}{12}\left(\frac{a+b}{2}\right)^4\right] \end{aligned}$$

and

$$\begin{aligned} M\left(\frac{a+b}{2}\right)^4 - g\left(\frac{a+b}{2}\right) &\leq \frac{m}{b-a}\cdot\frac{b^5-a^5}{60} - \frac{1}{b-a}\int_a^b g(t)dt \\ &\leq \frac{1}{2}\left[M\frac{a^4+b^4}{24} - \frac{g(a)+g(b)}{2} + \frac{M}{12}\left(\frac{a+b}{2}\right)^4 - g\left(\frac{a+b}{2}\right)\right], \end{aligned}$$

which completes the proof. □

Corollary 2.4 (Weighted combination for case $p = 4$)**.** *Let $a, b > 0$ with $a < b$, and let $g : [a, b] \to \mathbb{R}$ be twice continuously differentiable such that there exist constants $m, M \in \mathbb{R}$ satisfying*

$$mt^2 \leq g''(t) \leq Mt^2, \quad \text{for all } t \in [a, b].$$

Let $\lambda, \mu, \alpha, \beta \in [0, 1]$ be such that $\lambda + \mu = 1$ and $\alpha + \beta = 1$. Then, the following double inequality holds:

$$\begin{aligned} &\lambda\left[g\left(\frac{a+b}{2}\right) + \frac{m\left(b^4+b^3a+a^2b^2+ba^3\right)}{60} - \frac{m(a+b)^4}{16}\right] \\ &+\mu\left[\frac{1}{2}\left(\frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right)\right) + \frac{m\left(b^4+b^3a+a^2b^2+ba^3\right)}{60} - \frac{m(a^4+b^4)}{48} - \frac{m(a+b)^4}{384}\right] \\ \leq&\frac{1}{b-a}\int_a^b g(t)\,dt \\ \leq&\alpha\left[\frac{1}{2}\left(\frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right)\right) + \frac{M\left(b^4+b^3a+a^2b^2+ba^3\right)}{60} - \frac{M(a^4+b^4)}{48} - \frac{M(a+b)^4}{384}\right] \\ &+\beta\left[g\left(\frac{a+b}{2}\right) + \frac{M\left(b^4+b^3a+a^2b^2+ba^3\right)}{60} - \frac{M(a+b)^4}{16}\right], \end{aligned}$$

where $\lambda + \mu = 1$ and $\alpha + \beta = 1$.

Proof. From the original corollary for the case $p = 4$, we have the inequalities

$$\begin{aligned}
& g\left(\frac{a+b}{2}\right) + \frac{m\left(b^4 + b^3 a + a^2 b^2 + ba^3\right)}{60} - \frac{m(a+b)^4}{16} \\
\leq & \frac{1}{b-a}\int_a^b g(t)\,dt \\
\leq & \frac{1}{2}\left[\frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right)\right] + \frac{m\left(b^4 + b^3 a + a^2 b^2 + ba^3\right)}{60} - \frac{m(a^4+b^4)}{48} - \frac{m(a+b)^4}{384}
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{2}\left[\frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right)\right] + \frac{M\left(b^4 + b^3 a + a^2 b^2 + ba^3\right)}{60} - \frac{M(a^4+b^4)}{48} - \frac{M(a+b)^4}{384} \\
\leq & \frac{1}{b-a}\int_a^b g(t)\,dt \leq g\left(\frac{a+b}{2}\right) + \frac{M\left(b^4 + b^3 a + a^2 b^2 + ba^3\right)}{60} - \frac{M(a+b)^4}{16}.
\end{aligned}$$

Multiplying the first inequality's right-hand side by λ and left-hand side by μ, and the second inequality's left-hand side by α and right-hand side by β, with $\lambda + \mu = 1$ and $\alpha + \beta = 1$, we obtain

$$\lambda\,(\text{lower bound}_1) + \mu\,(\text{upper bound}_1) \leq \frac{1}{b-a}\int_a^b g(t)\,dt \leq \alpha\,(\text{lower bound}_2) + \beta(\text{upper bound}_2),$$

where $\text{lower bound}_1, \text{upper bound}_1, \text{lower bound}_2, \text{upper bound}_2$ denote the corresponding sides of the original inequalities above.

Since $\lambda + \mu = 1$ and $\alpha + \beta = 1$, the coefficients form convex combinations preserving the inequalities' directions.

Rearranging terms and grouping like terms completes the proof. □

Remark 2.2 (Special cases and interpretations of the weighted inequality for $p = 4$). The corollary presented above admits several notable special cases and interpretations, which are important both theoretically and for practical numerical approximations.

(a) **Case $\lambda = 1$, $\mu = 0$, $\alpha = 1$, $\beta = 0$.** In this case, the weighted inequality reduces exactly to the original inequalities stated in the corollary. This corresponds to considering only the bounds involving the pointwise evaluations and the lower curvature bound m:

$$\begin{aligned}
& g\left(\frac{a+b}{2}\right) + \frac{mQ}{60} - \frac{m(a+b)^4}{16} \\
\leq & \frac{1}{b-a}\int_a^b g(t)\,dt \leq \frac{1}{2}\left[\frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right)\right] + \frac{mQ}{60} - \frac{m(a^4+b^4)}{48} - \frac{m(a+b)^4}{384},
\end{aligned}$$

where $Q = b^4 + b^3 a + a^2 b^2 + ba^3$.

(b) **Case $\lambda = 0,\ \mu = 1,\ \alpha = 0,\ \beta = 1.$** Here, the inequality reduces to the complementary bounds involving the upper curvature bound M:

$$\begin{aligned}&\frac{1}{2}\left[\frac{g(a)+g(b)}{2}+g\left(\frac{a+b}{2}\right)\right]+\frac{MQ}{60}-\frac{M(a^4+b^4)}{48}-\frac{M(a+b)^4}{384}\\&\le\frac{1}{b-a}\int_a^b g(t)\,dt\le g\left(\frac{a+b}{2}\right)+\frac{MQ}{60}-\frac{M(a+b)^4}{16}.\end{aligned}$$

(c) **Case $\lambda = \mu = \frac{1}{2},\ \alpha = \beta = \frac{1}{2}.$** This choice corresponds to averaging the two previous bounds, giving a "smoothed" inequality:

$$\begin{aligned}&\frac{1}{2}\left[g\left(\frac{a+b}{2}\right)+\frac{mQ}{60}-\frac{m(a+b)^4}{16}\right]\\&\quad+\frac{1}{2}\left[\frac{1}{2}\left(\frac{g(a)+g(b)}{2}+g\left(\frac{a+b}{2}\right)\right)+\frac{mQ}{60}-\frac{m(a^4+b^4)}{48}-\frac{m(a+b)^4}{384}\right]\\&\le\frac{1}{b-a}\int_a^b g(t)\,dt\\&\le\frac{1}{2}\left[\frac{1}{2}\left(\frac{g(a)+g(b)}{2}+g\left(\frac{a+b}{2}\right)\right)+\frac{MQ}{60}-\frac{M(a^4+b^4)}{48}-\frac{M(a+b)^4}{384}\right]\\&\quad+\frac{1}{2}\left[g\left(\frac{a+b}{2}\right)+\frac{MQ}{60}-\frac{M(a+b)^4}{16}\right].\end{aligned}$$

This averaged bound may be useful in numerical integration, providing a balanced estimate between lower and upper curvature effects.

(d) **Recovery of Simpson's inequality.** If we let $m = M = 0$, i.e., assume that $g''(t) \equiv 0$ on $[a, b]$, then all curvature-dependent terms vanish, and the inequalities reduce exactly to the classical Simpson quadrature formula:

$$\frac{1}{b-a}\int_a^b g(t)dt=\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right].$$

This shows that the derived inequalities generalize Simpson's formula by including bounds dependent on the curvature of g.

These special cases demonstrate the flexibility of the inequality in controlling the integral average of g based on curvature bounds, and provide a clear link between classical numerical integration and modern integral inequalities involving higher-order derivatives.

Conclusion

In this paper, we have investigated new inequalities involving averages of functions constructed from powers of the input variable. By identifying the convexity properties of these functions, we derived sharp comparisons between their integral means and their values at certain key points within a given interval. Exact expressions for these averages were also computed, enhancing the analytical clarity of the results.

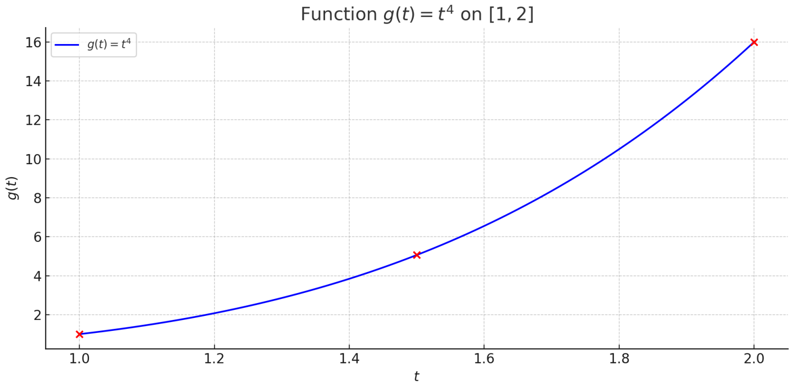

FIGURE 1. The graph of the function $g(t) = t^4$ on the interval $[1, 2]$. The red points correspond to the nodes $a = 1$, $\frac{a+b}{2} = 1.5$, and $b = 2$, which are used in the Simpson-type inequalities. This visualization highlights the smoothness and convexity of g on the interval, a crucial property to apply the integral inequalities derived in Corollary 4.

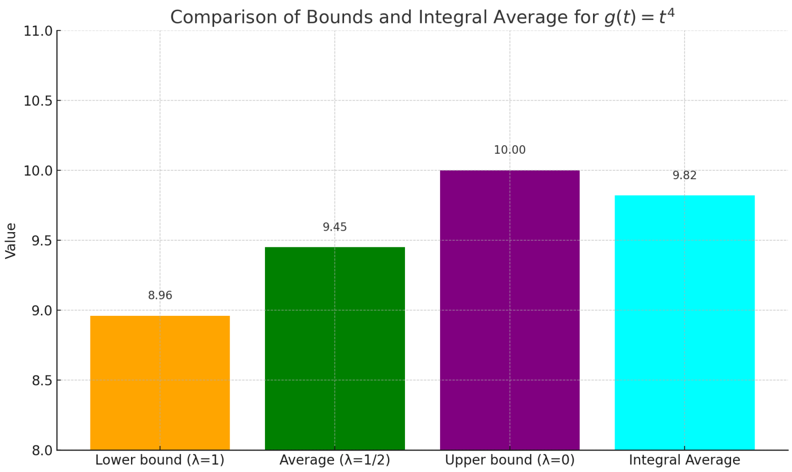

FIGURE 2. Comparison of the numerical values of the bounds and the integral average of $g(t) = t^4$ over $[1, 2]$, corresponding to the inequalities in Corollary 4. The bars represent: the lower bound (when parameter $\alpha = 1$), the average bound (when $\alpha = 1/2$), the upper bound (when $\alpha = 0$), and the actual integral mean value. This bar chart illustrates how the integral average lies tightly between the derived bounds, confirming the sharpness of the inequalities.

The primary contribution lies in the precise characterization of when such power-based combinations exhibit convexity, enabling the derivation of meaningful and applicable inequalities. These results not only deepen the theoretical understanding of such functional structures but also provide computational tools relevant in approximation theory and numerical integration.

There are several potential directions for future research. One natural extension is to consider multidimensional analogues of the presented inequalities, which could lead to new insights in convex analysis on higher-dimensional domains. Another promising avenue involves exploring alternative kernel functions or incorporating fractional integral operators to generalize the results within the framework of fractional calculus. Moreover, applications in optimization, economics, and data science where average behaviors of nonlinear functions are frequently analyzed could benefit from the established inequalities.

Finally, we highlight that some of the structural assumptions made in this work (such as specific power combinations or interval domains) may be relaxed or adapted to broader settings. These adjustments may yield further generalizations and refinements, providing fertile ground for continued investigation.

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