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ψ-HILFER-ATANGANA-BALEANU FRACTIONAL INTEGRAL OPERATORS AND SOME FRACTIONAL INTEGRAL INEQUALITIES

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Abstract. We present new fractional integral operators known as the left- and right-sided ψ-Hilfer-Atangana-Baleanu fractional integral operators. These fractional integrals generalize the Atangana-Baleanu and the ABK fractional integral operators. As a practical application, we derive a generalization of the Hermite-Hadamard inequality for s-convex functions in the second sense, utilizing the newly defined integral operators. Additionally, we established several Hermite-Hadamard type fractional integral inequalities for functions whose derivatives in absolute value raised to some nonnegative powers are s-convex in the second sense, using the ψ-Hilfer-Atangana-Baleanu fractional integral operator. We also present some specific cases of our main results.

1. Introduction

Fractional inequalities have a significant impact in the research of various fields of mathematics, including mathematical analysis, probability theory, differential equations, and many more. These inequalities involve fractional derivatives or fractional integrals of functions, which generalize the concept of differentiation and integration to non-integer orders. Fractional inequalities are not only theoretical tools but also have practical applications in fields such as physics, engineering, and biology, where phenomena are often described by equations involving fractional derivatives. They are essential for understanding the behavior of solutions to fractional differential equations

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and for establishing the conditions under which such solutions exist and are unique. Due to their potential applications, many researchers have studied and established several fractional inequalities involving various fractional derivatives and fractional integrals over the years. Most of these inequalities are generalizations, extensions or refinements of the classical inequalities within the framework of fractional calculus. It is well known that convex functions have a significant impact in the research of inequalities. In the last few decades, many researchers have investigated several inequalities involving convex functions and their numerous generalizations by employing different fractional derivatives and/or fractional integral operators. For example, in [33], Tunc et al. introduced the generalized fractional integrals of a function with respect to another function and established the Hermite-Hadamard inequality as well as some Hermite-Hadamard type inequalities involving the newly introduced fractional integral operators for convex functions. In [29], Rahman et al. established some new fractional integral inequalities for a certain class of n ($n \in \mathbb{N}$) positive continuous functions and also for decreasing functions by utilizing the fractional proportional integral operators. Rashid et al. in [30] defined a new fractional integral operator, which is known as generalized proportional fractional (GPF) integral in the sense of another function, and proved several fractional inequalities involving the newly defined GPF-integral with respect to another function. By using some generalized fractional integrals, Du et al. [9] derived some Bullen-type inequalities where the first derivative of the functions under consideration are either Lipschitzian, bounded or generalized (s, m)-preinvex. In [10], Du and Peng established some Hermite-Hadamard type inequalities for multiplicatively convex functions via multiplicative Riemann-Liouville fractional integrals. In [32], Saleh et al. investigated some new Newton-type inequalities for functions whose first derivatives are α-convex by utilizing the Riemann-Liouville fractional integrals. Recently, Kermausuor and Nwaeze [21] introduced some new fractional integral operators called the k-Atangana-Baleanu fractional integral operators and established a generalization of the Hermite-Hadamard inequality as well as several other fractional integral inequalities for functions whose derivatives are bounded and functions whose derivative in absolute value is convex. For further details on the results mentioned above and other related findings, we encourage interested readers to consult the papers listed in [2–4, 6, 9, 11, 14–18, 20, 21, 23–25, 27, 29–36] and the other related references cited in this paper.

Motivated by the ongoing research on fractional inequalities and their applications, this paper seeks to introduce new generalized fractional integral operators called the left- and right-sided ψ-Hilfer-Atanga-Baleanu (ψ-HAB) fractional integral operators. We will subsequently establish Hermite-Hadamard type inequalities for functions that are s-convex in the second sense and functions whose derivative in absolute value to certain powers are s-convex in the second sense, all using the newly defined fractional integral operators. In the next section, we present some preliminary definitions and results, followed by the main results in Section 3 and a conclusion in Section 4.

2. Preliminaries

We highlight some basic concepts and results in the literature that will be useful to our work. We start with the definitions of some fractional integrals.

Definition 2.1 ([13, 26, 28])**.** The left- and right-sided Riemann-Liouville fractional integrals of a real-valued function $\mathcal{G}$ of order $\omega > 0$ are given by

$$I^{\omega}_{d_1^+}\mathcal{G}(r) = \frac{1}{\Gamma(\omega)}\int_{d_1}^{r}(r-z)^{\omega-1}\mathcal{G}(z)dz$$

and

$$I^{\omega}_{d_2^-}\mathcal{G}(r) = \frac{1}{\Gamma(\omega)}\int_{r}^{d_2}(z-r)^{\omega-1}\mathcal{G}(z)dz,$$

where Γ is the gamma function given by

$$\Gamma(\omega) := \int_0^{\infty} z^{\omega-1}e^{-z}dz, \quad \mathrm{Re}(\omega) > 0.$$

The following definition is an important generalization of the Riemann-Liouville fractional integrals known as the ψ-Riemann-Liouville fractional integrals.

Definition 2.2 ([22])**.** Let (d_1, d_2), $-\infty \leq d_1 < d_2 \leq +\infty$, be a finite or infinite interval of the real line $\mathbb{R}$ and $\omega > 0$. Also, let ψ be an increasing and positive function on $(d_1, d_2]$, having a continuous derivative ψ' on (d_1, d_2). The left and right-sided Riemann-Liouville fractional integrals of a function $\mathcal{G}$ with respect to the function ψ on $[d_1, d_2]$ are defined by

$$I^{\omega;\psi}_{d_1^+}\mathcal{G}(r) = \frac{1}{\Gamma(\omega)}\int_{d_1}^{r}(\psi(r)-\psi(z))^{\omega-1}\psi'(z)\mathcal{G}(z)dz$$

and

$$I^{\omega;\psi}_{d_2^-}\mathcal{G}(r) = \frac{1}{\Gamma(\omega)}\int_{r}^{d_2}(\psi(z)-\psi(r))^{\omega-1}\psi'(z)\mathcal{G}(z)dz.$$

The left- and right-sided Atangana-Baleanu fractional integrals are defined as follows.

Definition 2.3 ([1,5])**.** The left- and right-sided Atangana-Baleanu fractional integral of a real-valued function $\mathcal{G}$ of order $\omega \in (0,1)$ are defined by

$${}^{AB}I^{\omega}_{d_1^+}\mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\omega}{B(\omega)\Gamma(\omega)}\int_{d_1}^{r}(r-z)^{\omega-1}\mathcal{G}(z)dz$$

and

$${}^{AB}I^{\omega}_{d_2^-}\mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\omega}{B(\omega)\Gamma(\omega)}\int_{r}^{d_2}(z-r)^{\omega-1}\mathcal{G}(z)dz,$$

where $B(\omega) > 0$ is a normalization function and satisfies the property $B(0) = B(1) = 1$.

In [19], Kashuri introduced the following generalization of the Atangana-Baleanu fractional integral operators called the ABK-fractional integral operators.

Definition 2.4. The left- and right-sided ABK-fractional integral operators of a real-valued function $\mathcal{G}$ of order $\omega \in (0,1)$ are given by

$$ {}^{ABK\rho}_{d_1^+}I^{\omega}_r \mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\rho^{1-\omega}\omega}{B(\omega)\Gamma(\omega)} \int_{d_1}^{r} \frac{z^{\rho-1}}{(r^\rho - z^\rho)^{1-\omega}} \mathcal{G}(z)dz, \quad r > d_1, $$

and

$$ {}^{ABK\rho}_{d_2^-}I^{\omega}_r \mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\rho^{1-\omega}\omega}{B(\omega)\Gamma(\omega)} \int_{r}^{d_2} \frac{z^{\rho-1}}{(z^\rho - r^\rho)^{1-\omega}} \mathcal{G}(z)dz, \quad r < d_2, $$

where $\rho > 0$ and $B(\omega) > 0$ satisfies the property $B(0) = B(1) = 1$.

Recently, Kermausuor and Nwaeze also introduced another generalization of the Atangana-Baleanu fractional integrals called the k-Atangana-Baleanu fractional integrals as defined below.

Definition 2.5 ([21])**.** The left- and right-sided k-Atangana-Baleanu fractional integral operators of a real-valued function $\mathcal{G}$ of order $\omega > 0$, are defined as

$$ {}^{AB}{}_kI^{\omega}_{d_1^+} \mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\omega}{kB(\omega)\Gamma_k(\omega)} \int_{d_1}^{r} (r-z)^{\frac{\omega}{k}-1} \mathcal{G}(z)dz, \quad r > d_1, $$

and

$$ {}^{AB}{}_kI^{\omega}_{d_2^-} \mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\omega}{kB(\omega)\Gamma_k(\omega)} \int_{r}^{d_2} (z-r)^{\frac{\omega}{k}-1} \mathcal{G}(z)dz, \quad r < d_2, $$

where $k > 0$ and $B(\omega) > 0$ satisfies the property $B(0) = B(1) = 1$.

As mentioned in the introduction, several fractional integral inequalities have been established by utilizing the aforementioned fractional integrals. For instance, by utilizing the Atangana-Baleanu fractional integral operators, Fernandez and Mohammed [12] established the following generalization of the Hermite-Hadamard inequality.

Theorem 2.1. *If $\mathcal{G} : [d_1, d_2] \to \mathbb{R}$ is convex on $[d_1, d_2]$ and $\mathcal{G} \in L_1([d_1, d_2])$, then the inequalities*

$$ \begin{aligned} \mathcal{G}\left(\frac{d_1+d_2}{2}\right) &\le \frac{B(\omega)\Gamma(\omega)}{2[(d_2-d_1)^\omega + (1-\omega)\Gamma(\omega)]} \left[{}^{AB}I^{\omega}_{d_1^+}\mathcal{G}(d_2) + {}^{AB}I^{\omega}_{d_2^-}\mathcal{G}(d_1) \right] \\ &\le \frac{\mathcal{G}(d_1)+\mathcal{G}(d_2)}{2} \end{aligned} $$

hold for $\omega \in (0,1)$.

Similarly, Kashuri presented the following generalization of the Hermite-Hadamard inequality by utilizing the ABK-fractional integrals.

Theorem 2.2. *Let $\omega \in (0,1)$ and $\rho > 0$. Let $\mathcal{G} : [d_1^\rho, d_2^\rho] \to \mathbb{R}$ with $0 \leq d_1 < d_2$ be a convex function. Then, the following inequalities for the ABK-fractional integrals hold:*

$$\begin{aligned}&\frac{2(d_2^\rho - d_1^\rho)^\omega}{B(\omega)\Gamma(\omega+1)\rho^{2-\rho}}\mathcal{G}\left(\frac{d_1^\rho + d_2^\rho}{2}\right) + \frac{1-\omega}{B(\omega)}\left[\mathcal{G}(d_1^\rho) + \mathcal{G}(d_2^\rho)\right]\\ \leq & {}^{ABK\rho}_{\ \ d_1^+}I^\omega_{d_2^\rho}\mathcal{G}(d_2^\rho) + {}^{ABK\rho}_{\ \ d_2^-}I^\omega_{d_1^\rho}\mathcal{G}(d_1^\rho)\\ \leq & \left(\frac{(d_2^\rho - d_1^\rho)^\omega + \rho(1-\omega)\Gamma(\omega)}{\rho B(\omega)\Gamma(\omega)}\right)\left[\mathcal{G}(d_1^\rho) + \mathcal{G}(d_2^\rho)\right].\end{aligned}$$

To end this section, we present the definition of s-convex functions in the second sense.

Definition 2.6 ([7])**.** Let $s \in (0,1]$. A function $\mathcal{G} : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be s-convex in the second sense on I, if

$$\mathcal{G}(r\delta_1 + (1-r)\delta_2) \leq r^s\mathcal{G}(\delta_1) + (1-r)^s\mathcal{G}(\delta_2),$$

for all $\delta_1, \delta_2 \in I$ and $r \in [0,1]$.

Remark 2.1. If $s = 1$ in Definition 2.6, then we have the classical definition of a convex function.

3. Main Results

We commence by defining the new fractional integrals that generalize the Atangana-Baleanu fractional integrals.

Definition 3.1. Let $d_1, d_2 \in \mathbb{R}$ with $d_1 < d_2$ and let ψ be a strictly increasing and positive function on $[d_1, d_2]$, having a continuous derivative ψ' on (d_1, d_2). The left- and right-sided ψ-Hilfer-Atangana-Baleanu (HAB) fractional integral operators of a real-valued function $\mathcal{G}$ of order $\omega > 0$, are defined as

$${}^{HAB}{}_{\psi}I^\omega_{d_1^+}\mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\omega}{B(\omega)\Gamma(\omega)}\int_{d_1}^r (\psi(r) - \psi(z))^{\omega-1}\psi'(z)\mathcal{G}(z)dz, \quad r > d_1,$$

and

$${}^{HAB}{}_{\psi}I^\omega_{d_2^-}\mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\omega}{B(\omega)\Gamma(\omega)}\int_{r}^{d_2} (\psi(z) - \psi(r))^{\omega-1}\psi'(z)\mathcal{G}(z)dz, \quad r < d_2,$$

where $B(\omega) > 0$ is a normalization function and satisfies the property $B(0) = B(1) = 1$.

Remark 3.1. If $\psi(t) = t$ in Definition 3.1, then we have the Atangana-Baleanu fractional integral operators, and if $\psi(t) = \frac{t^\rho}{\rho}$, $\rho > 0$, then we have the ABK-fractional integral operators.

Next, we investigate the Hermite-Hadamard inequality by utilizing the ψ-Hilfer-Atangana-Baleanu fractional integral operators. In what follows, we assume that ψ meets the requirements of Definition 3.1.

Lemma 3.1. *For $\mu < \nu$, we have*

(3.1)
$$\int_0^1 r^{\omega-1}\mathcal{G}(r\mu+(1-r)\nu)dr=\frac{B(\omega)\Gamma(\omega)}{\omega(\nu-\mu)^{\omega}}\left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\mu)^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\nu))-\frac{1-\omega}{B(\omega)}\mathcal{G}(\nu)\right]$$

and

(3.2)
$$\int_0^1 r^{\omega-1}\mathcal{G}(r\nu+(1-r)\mu)dr=\frac{B(\omega)\Gamma(\omega)}{\omega(\nu-\mu)^{\omega}}\left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\nu)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\mu))-\frac{1-\omega}{B(\omega)}\mathcal{G}(\mu)\right].$$

Proof. By using change of variables with $x = r\mu + (1-r)\nu$, we have

$$\int_0^1 r^{\omega-1}\mathcal{G}(r\mu+(1-r)\nu)dr=\frac{1}{(\nu-\mu)^{\omega}}\int_{\mu}^{\nu}(\nu-x)^{\omega-1}\mathcal{G}(x)dx. \tag{3.3}$$

Since ψ is strictly increasing on $[\mu,\nu]$ and hence has an inverse, we deduce that for any $x \in [\mu,\nu]$, $x = \psi(z)$ for some $z \in [\mu,\nu]$. So using change of variables again with $x = \psi(z)$ in (3.3) and Definition 3.1, we have

$$\begin{aligned}\int_0^1 r^{\omega-1}\mathcal{G}(r\mu+(1-r)\nu)dr=&\frac{1}{(\nu-\mu)^{\omega}}\int_{\mu}^{\nu}(\nu-x)^{\omega-1}\mathcal{G}(x)dx\\ =&\frac{1}{(\nu-\mu)^{\omega}}\int_{\psi^{-1}(\mu)}^{\psi^{-1}(\nu)}(\psi(\psi^{-1}(\nu))-\psi(z))^{\omega-1}\psi'(z)\mathcal{G}(\psi(z))dz\\ =&\frac{B(\omega)\Gamma(\omega)}{\omega(\nu-\mu)^{\omega}}\Bigg[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\mu)^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\nu))\\ &-\frac{1-\omega}{B(\omega)}(\mathcal{G}\circ\psi)(\psi^{-1}(\nu))\Bigg].\end{aligned}$$

This proves the identity in (3.1). The identity in (3.2) is obtained by using a similar argument. ☐

Theorem 3.1. *Let $\delta,\epsilon \in \mathbb{R}$ with $\delta < \epsilon$ and $\mathcal{G} : [\delta,\epsilon] \to \mathbb{R}$ be an s-convex function in the second sense. Then, the inequalities*

$$\begin{aligned}&\frac{2^{s}(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)+\frac{1-\omega}{B(\omega)}[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)]\\ \leq&{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\\ \leq&\left[\frac{\omega(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{\omega+s}+\mathbb{B}(\omega,s+1)\right)+\frac{1-\omega}{B(\omega)}\right][\mathcal{G}(\delta)+\mathcal{G}(\epsilon)]\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&\left[\frac{2^{s}(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}+\frac{2^{s}(1-\omega)}{B(\omega)}\right]\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)\\ \leq&{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\end{aligned}$$

$$\leq \left[\frac{\omega(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{\omega+s}+\mathbb{B}(\omega,s+1)\right)+\frac{1-\omega}{B(\omega)}\right][\mathcal{G}(\delta)+\mathcal{G}(\epsilon)] \tag{3.5}$$

hold for $\omega \in (0,1)$, where $\mathbb{B}(\cdot,\cdot)$ is the beta function defined by

$$\mathbb{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

Proof. Under the condition of s-convexity in the second sense of $\mathcal{G}$ on $[\delta,\epsilon]$, it follows that for any $p_1, p_2 \in [\delta,\epsilon]$

$$\mathcal{G}\left(\frac{p_1+p_2}{2}\right) \leq \frac{\mathcal{G}(p_1)+\mathcal{G}(p_2)}{2^s}. \tag{3.6}$$

If we take $p_1 = r\epsilon+(1-r)\delta$ and $p_2=(1-r)\epsilon+r\delta$ for $r\in[0,1]$ in (3.6), then we get

$$2^s\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right) \leq \mathcal{G}(r\epsilon+(1-r)\delta)+\mathcal{G}((1-r)\epsilon+r\delta). \tag{3.7}$$

Multiply (3.7) by $\frac{\omega}{B(\omega)\Gamma(\omega)}r^{\omega-1}$ and integrate with respect to r over $[0,1]$ to obtain

$$\begin{aligned}\frac{2^s}{B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right) \leq &\frac{\omega}{B(\omega)\Gamma(\omega)}\int_0^1 r^{\omega-1}\mathcal{G}(r\epsilon+(1-r)\delta)dr\\ &+\frac{\omega}{B(\omega)\Gamma(\omega)}\int_0^1 r^{\omega-1}\mathcal{G}((1-r)\epsilon+r\delta)dr.\end{aligned} \tag{3.8}$$

Using Lemma 3.1 and the inequality in (3.8), we get

$$\begin{aligned}&\frac{2^s(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)+\frac{1-\omega}{B(\omega)}[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)]\\ \leq &{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta)).\end{aligned} \tag{3.9}$$

This proves the first inequality in (3.4). Now, since $\mathcal{G}$ is s-convex in the second sense, we have $\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right) \leq \frac{\mathcal{G}(\epsilon)+\mathcal{G}(\delta)}{2^s}$. Hence, from (3.9), we have

$$\begin{aligned}&\left[\frac{2^s(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}+\frac{2^s(1-\omega)}{B(\omega)}\right]\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)\\ \leq &{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta)).\end{aligned}$$

This establishes the first inequality in (3.5). To establish the second inequality in both (3.4) and (3.5), we note that, if $\mathcal{G}$ is s-convex in the second sense, then for $r\in[0,1]$, it follows that

$$\mathcal{G}(r\epsilon+(1-r)\delta) \leq r^s\mathcal{G}(\epsilon)+(1-r)^s\mathcal{G}(\delta) \tag{3.10}$$

and

$$\mathcal{G}((1-r)\epsilon+r\delta) \leq (1-r)^s\mathcal{G}(\epsilon)+r^s\mathcal{G}(\delta). \tag{3.11}$$

Adding (3.10) and (3.11), we obtain

$$\mathcal{G}(r\epsilon+(1-r)\delta)+\mathcal{G}((1-r)\epsilon+r\delta) \leq (r^s+(1-r)^s)[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)]. \tag{3.12}$$

Multiply (3.12) by $\frac{\omega}{B(\omega)\Gamma(\omega)}r^{\omega-1}$ and integrate with respect to r over $[0,1]$ to get

$$\begin{aligned}&\frac{\omega}{B(\omega)\Gamma(\omega)}\int_0^1 r^{\omega-1}\mathcal{G}(r\epsilon+(1-r)\delta)dr+\frac{\omega}{B(\omega)\Gamma(\omega)}\int_0^1 r^{\omega-1}\mathcal{G}((1-r)\epsilon+r\delta)dr\\ \leq&\frac{\omega}{B(\omega)\Gamma(\omega)}[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)]\int_0^1 r^{\omega-1}(r^s+(1-r)^s)dr\\ =&\frac{\omega}{B(\omega)\Gamma(\omega)}[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)]\left(\frac{1}{\omega+s}+\mathbb{B}(\omega,s+1)\right).\end{aligned}\tag{3.13}$$

Using Lemma 3.1, we deduce from (3.13) that

$$\begin{aligned}&{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\\ \leq&\frac{\omega(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)]\left(\frac{1}{\omega+s}+B(\omega,s+1)\right)+\frac{1-\omega}{B(\omega)}[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)]\\ =&\left[\frac{\omega(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{\omega+s}+\mathbb{B}(\omega,s+1)\right)+\frac{1-\omega}{B(\omega)}\right][\mathcal{G}(\delta)+\mathcal{G}(\epsilon)].\end{aligned}\tag{3.14}$$

This proves the second inequality in both (3.4) and (3.5). □

Corollary 3.1. *Let $\delta,\epsilon\in\mathbb{R}$ with $\delta<\epsilon$ and $\mathcal{G}:[\delta,\epsilon]\to\mathbb{R}$ be a convex function. Then, the inequalities*

$$\begin{aligned}&\frac{2(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)+\frac{1-\omega}{B(\omega)}[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)]\\ \leq&{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\\ \leq&\left[\frac{(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}+\frac{1-\omega}{B(\omega)}\right][\mathcal{G}(\gamma)+\mathcal{G}(\omega)]\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}&\left[\frac{2(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}+\frac{2(1-\omega)}{B(\omega)}\right]\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)\\ \leq&{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\\ \leq&\left[\frac{(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}+\frac{1-\omega}{B(\omega)}\right][\mathcal{G}(\delta)+\mathcal{G}(\epsilon)]\end{aligned}\tag{3.16}$$

hold for $\omega\in(0,1)$.

Proof. Take $s=1$ in Theorem 3.1. □

Remark 3.2. If we take $\psi(t)=t$ in Corollary 3.1, then the inequality (3.16) reduces to the inequality in Theorem 2.1 and if we take $\psi(t)=\frac{t^{\rho}}{\rho}$, $\rho>0$, in Corollary 3.1, then the inequality (3.15) reduces to the inequality in Theorem 2.2 with some slight modification to the endpoints of the interval.

Below, we introduce some new inequalities of the midpoint and trapezoidal types for functions whose derivatives, in absolute value raised to certain nonnegative powers, are s-convex in the second sense, using the ψ-Hilfer-Atangana-Baleanu fractional integrals. To accomplish this, we first establish the following key identities involving these fractional integrals.

Lemma 3.2. *Let $\delta, \epsilon \in \mathbb{R}$ with $\delta < \epsilon$ and $\mathcal{G} : [\delta, \omega] \to \mathbb{R}$ be a function. If $\mathcal{G}$ is differentiable and $\mathcal{G}' \in L_1([\delta, \epsilon])$, then the equality*

$$\begin{aligned}
&\left(\frac{(\epsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)\Big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\Big]\\
&-\left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\\
&=\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\int_0^1[(1-r)^{\omega}-r^{\omega}]\mathcal{G}'(r\delta+(1-r)\epsilon)dr
\end{aligned} \tag{3.17}$$

holds for all $\omega > 0$.

Proof. Utilizing the technique of integration by parts, change of variables and Lemma 3.1, we get

$$\begin{aligned}
&\int_0^1(1-r)^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr\\
=&\frac{1}{\epsilon-\delta}\mathcal{G}(\epsilon)-\frac{\omega}{\epsilon-\delta}\int_0^1(1-r)^{\omega-1}\mathcal{G}(r\delta+(1-r)\epsilon)dr\\
=&\frac{1}{\epsilon-\delta}\mathcal{G}(\epsilon)-\frac{\omega}{\epsilon-\delta}\int_0^1 t^{\omega-1}\mathcal{G}(t\epsilon+(1-t)\delta)dt\\
=&\frac{1}{\epsilon-\delta}\mathcal{G}(\epsilon)-\frac{B(\omega)\Gamma(\omega)}{(\epsilon-\delta)^{\omega+1}}\left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))-\frac{1-\omega}{B(\omega)}\mathcal{G}(\delta)\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_0^1(1-r)^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr\\
=&\frac{1}{\epsilon-\delta}\mathcal{G}(\epsilon)-\frac{B(\omega)\Gamma(\omega)}{(\epsilon-\delta)^{\omega+1}}\left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))-\frac{1-\omega}{B(\omega)}\mathcal{G}(\delta)\right].
\end{aligned} \tag{3.18}$$

Using a similar reasoning, we can conclude that

$$\begin{aligned}
&\int_0^1 r^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr\\
=&-\frac{1}{\epsilon-\delta}\mathcal{G}(\delta)+\frac{B(\omega)\Gamma(\omega)}{(\epsilon-\delta)^{\omega+1}}\left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))-\frac{1-\omega}{B(\omega)}\mathcal{G}(\epsilon)\right].
\end{aligned} \tag{3.19}$$

The identity in (3.17) can be achieved by multiplying the identities in (3.18) and (3.19) by $\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}$, and then subtracting the resulting equations. □

Lemma 3.3. *Let $\delta, \epsilon \in \mathbb{R}$ with $\delta < \epsilon$ and $\mathcal{G} : [\delta, \epsilon] \to \mathbb{R}$ be a function. If $\mathcal{G}$ is differentiable and $\mathcal{G}' \in L_1([\delta, \epsilon])$, then the equality*

$$\begin{aligned}&\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\\&-\left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)+\frac{1-\omega}{B(\omega)}\big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\big]\right)\\&=\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[\int_0^{1/2} r^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr - \int_{1/2}^{1}(1-r)^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr\right]\end{aligned}\tag{3.20}$$

holds for all $\omega > 0$.

Proof. Utilizing the technique of integration by parts, we get

$$\begin{aligned}&\int_0^{1/2} r^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr\\&=\frac{r^{\omega}}{\delta-\epsilon}\mathcal{G}(r\delta+(1-r)\epsilon)\Big|_0^{1/2} - \frac{\omega}{(\delta-\epsilon)}\int_0^{1/2} r^{\omega-1}\mathcal{G}(r\delta+(1-r)\epsilon)dr\\&=\frac{1}{2^{\omega}(\delta-\epsilon)}\mathcal{G}\left(\frac{\delta+\epsilon}{2}\right) - \frac{\omega}{(\delta-\epsilon)}\int_0^{1/2} r^{\omega-1}\mathcal{G}(r\delta+(1-r)\epsilon)dr\\&=\frac{1}{2^{\omega}(\delta-\epsilon)}\mathcal{G}\left(\frac{\delta+\epsilon}{2}\right) + \frac{\omega}{(\epsilon-\delta)^{\omega+1}}\int_{\frac{\delta+\epsilon}{2}}^{\epsilon}(\epsilon-z)^{\omega-1}\mathcal{G}(z)dz\\&=-\frac{1}{2^{\omega}(\epsilon-\delta)}\mathcal{G}\left(\frac{\delta+\epsilon}{2}\right)\\&\quad+\frac{B(\omega)\Gamma(\omega)}{(\epsilon-\delta)^{\omega+1}}\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) - \frac{1-\omega}{B(\omega)}\mathcal{G}(\epsilon)\right].\end{aligned}$$

So,

$$\begin{aligned}&\int_0^{1/2} r^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr\\&=-\frac{1}{2^{\omega}(\epsilon-\delta)}\mathcal{G}\left(\frac{\delta+\epsilon}{2}\right)+\frac{B(\omega)\Gamma(\omega)}{(\epsilon-\delta)^{\omega+1}}\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) - \frac{1-\omega}{B(\omega)}\mathcal{G}(\epsilon)\right].\end{aligned}\tag{3.21}$$

Similarly,

$$\begin{aligned}&\int_{1/2}^{1}(1-r)^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr\\&=\frac{1}{2^{\omega}(\epsilon-\delta)}\mathcal{G}\left(\frac{\delta+\epsilon}{2}\right)-\frac{B(\omega)\Gamma(\omega)}{(\epsilon-\delta)^{\omega+1}}\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta)) - \frac{1-\omega}{B(\omega)}\mathcal{G}(\delta)\right].\end{aligned}\tag{3.22}$$

The identity in (3.20) can be achieved by multiplying the identities in (3.21) and (3.22) by $\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}$, and then taking the difference between the resulting equations. □

Now, employing Lemma 3.2 and Lemma 3.3, we derive the following Hermite-Hadamard type inequalities.

Theorem 3.2. *If $\mathcal{G}$ meets the conditions of Lemma* 3.2 *and $|\mathcal{G}'|$ is s-convex in the second sense, then the inequality*

$$\begin{aligned}
&\Bigg|\left(\frac{(\epsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)\Big[\mathcal{G}(\omega)+\mathcal{G}(\epsilon)\Big] \\
&-\Big[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\Big]\Bigg| \\
\leq&\frac{(\epsilon-\delta)^{\omega+1}\Big[|\mathcal{G}'(\delta)|+|\mathcal{G}'(\epsilon)|\Big]}{B(\omega)\Gamma(\omega)}\Bigg(\mathbb{B}_{\frac{1}{2}}(s+1,\omega+1)-\mathbb{B}_{\frac{1}{2}}(\omega+1,s+1) \\
&+\frac{1}{\omega+s+1}-\frac{1}{2^{\omega+s}(\omega+s+1)}\Bigg)
\end{aligned}\tag{3.23}$$

holds for $\omega\in(0,1)$, where $\mathbb{B}_z(\cdot,\cdot)$ is the incomplete beta function stated as

$$\mathbb{B}_z(x,y)=\int_0^z t^{x-1}(1-t)^{y-1}dt.\tag{3.24}$$

Proof. Utilizing the identity in (3.17) and the s-convexity in the second sense of $|\mathcal{G}'|$, we derive

$$\begin{aligned}
&\Bigg|\left(\frac{(\epsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)\Big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\Big] \\
&-\Big[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\Big]\Bigg| \\
\leq&\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\int_0^1\Big|(1-r)^{\omega}-r^{\omega}\Big|\Big(r^s|\mathcal{G}'(\delta)|+(1-r)^s|\mathcal{G}'(\epsilon)|\Big)dr
\end{aligned}\tag{3.25}$$

Now, we observed that for $r\in[0,1]$,

$$(1-r)^{\omega}-r^{\omega}\begin{cases}\geq 0, & \text{for } r\in[0,1/2],\\ <0, & \text{for } r\in(1/2,1],\end{cases}$$

and hence

$$\begin{aligned}
\int_0^1|(1-r)^{\omega}-r^{\omega}|r^s dr=&\int_0^{1/2}((1-r)^{\omega}-r^{\omega})r^s dr+\int_{1/2}^1(r^{\omega}-(1-r)^{\omega})r^s dr \\
=&\int_0^{1/2}((1-r)^{\omega}r^s-r^{\omega+s})dr+\int_0^{1/2}((1-t)^{\omega+s}-t^{\omega}(1-t)^s)dt \\
=&\mathbb{B}_{\frac{1}{2}}(s+1,\omega+1)-\frac{1}{2^{\omega+s+1}(\omega+s+1)} \\
&+\frac{1}{\omega+s+1}-\frac{1}{2^{\omega+s+1}(\omega+s+1)}-\mathbb{B}_{\frac{1}{2}}(\omega+1,s+1) \\
=&\mathbb{B}_{\frac{1}{2}}(s+1,\omega+1)-\mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)
\end{aligned}$$

$$+\frac{1}{\omega+s+1}-\frac{1}{2^{\omega+s}(\omega+s+1)}. \tag{3.26}$$

Similarly,

$$\begin{aligned}\int_0^1 |(1-r)^\omega - r^\omega|(1-r)^s dr =&\mathbb{B}_{\frac{1}{2}}(s+1,\omega+1)-\mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)\\ &+\frac{1}{\omega+s+1}-\frac{1}{2^{\omega+s}(\omega+s+1)}.\end{aligned} \tag{3.27}$$

The desired inequality in (3.23) is obtained by substituting (3.26) and (3.27) in (3.25). ☐

Corollary 3.2. *If* $\mathcal{G}$ *meets the requirements of Lemma* 3.2 *and* $|\mathcal{G}'|$ *is convex, then the following inequality holds:*

$$\begin{aligned}&\Bigg|\left(\frac{(\epsilon-\delta)^\omega+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)\Big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\Big]\\ &-\Big[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\Big]\Bigg|\\ \le&\frac{(\epsilon-\delta)^{\omega+1}\Big[|\mathcal{G}'(\delta)|+|\mathcal{G}'(\epsilon)|\Big]}{B(\omega)\Gamma(\omega)}\left(\frac{1}{\omega+1}-\frac{1}{2^\omega(\omega+1)}\right),\end{aligned} \tag{3.28}$$

for $\omega\in(0,1)$.

Proof. Take $s=1$ in Theorem 3.2. ☐

Theorem 3.3. *If* $\mathcal{G}$ *meets the requirements of Lemma* 3.2 *and* $|\mathcal{G}'|^q$, $q>1$, *is s-convex in the second sense, then inequality*

$$\begin{aligned}&\Bigg|\left(\frac{(\epsilon-\delta)^\omega+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)\Big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\Big]\\ &-\Big[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\Big]\Bigg|\\ \le&\frac{(\epsilon-\delta)^{\omega+1}\Big(|\mathcal{G}'(\delta)|^q+|\mathcal{G}'(\epsilon)|^q\Big)^{\frac{1}{q}}}{B(\omega)\Gamma(\omega)}\left(\frac{2}{\omega+1}-\frac{2}{2^\omega(\omega+1)}\right)^{1-\frac{1}{q}}\\ &\times\Bigg(\mathbb{B}_{\frac{1}{2}}(s+1,\omega+1)-\mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)\\ &+\frac{1}{\omega+s+1}-\frac{1}{2^{\omega+s}(\omega+s+1)}\Bigg)^{\frac{1}{q}}\end{aligned} \tag{3.29}$$

holds for $\omega\in(0,1)$, *where* $\mathbb{B}_z(\cdot,\cdot)$ *is the incomplete beta function defined in* (3.24).

Proof. Utilizing the identity in (3.17) and the power-mean inequality, we derive

$$\Bigg|\left(\frac{(\epsilon-\delta)^\omega+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)\Big[\mathcal{G}(\omega)+\mathcal{G}(\epsilon)\Big]$$

$$
\begin{aligned}
&- \left[{}^{HAB}{}_{\psi}I^{\epsilon}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\Bigg| \\
&\leq \frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)} \left(\int_0^1 |(1-r)^{\omega} - r^{\omega}| dr\right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1 |(1-r)^{\omega} - r^{\omega}||\mathcal{G}'(r\delta + (1-r)\epsilon)|^q dr\right)^{\frac{1}{q}}. \tag{3.30}
\end{aligned}
$$

Using the s-convexity of $|\mathcal{G}'|^q$ and a reasoning similar to the one in the proof of Theorem 3.3, we can conclude that

$$
\begin{aligned}
&\int_0^1 |(1-r)^{\omega} - r^{\omega}||\mathcal{G}'(r\delta + (1-r)\epsilon)|^q dr \\
&\leq \left(|\mathcal{G}'(\delta)|^q + \mathcal{G}'(\epsilon)|^q\right)\Bigg(\mathbb{B}_{\frac{1}{2}}(s+1,\omega+1) - \mathbb{B}_{\frac{1}{2}}(\omega+1,s+1) \\
&+ \frac{1}{\omega+s+1} - \frac{1}{2^{\omega+s}(\omega+s+1)}\Bigg). \tag{3.31}
\end{aligned}
$$

In addition,

$$
\int_0^1 \Big|(1-r)^{\omega} - r^{\omega}\Big| dr = 2\left(\frac{1}{\omega+1} - \frac{1}{2^{\omega}(\omega+1)}\right). \tag{3.32}
$$

The desired inequality in (3.29) is obtained by substituting (3.31) and (3.32) in (3.30). □

Corollary 3.3. *If $\mathcal{G}$ meets the requirements of Lemma* 3.2 *and $|\mathcal{G}'|^q$, $q>1$, is convex, then the inequality*

$$
\begin{aligned}
&\Bigg|\left(\frac{(\epsilon-\delta)^{\omega} + (1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)\left[\mathcal{G}(\omega) + \mathcal{G}(\epsilon)\right] \\
&- \left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\Bigg| \\
&\leq \frac{2^{1-\frac{1}{q}}(\epsilon-\delta)^{\omega+1}\left(|\mathcal{G}'(\delta)|^q + \mathcal{G}'(\epsilon)|^q\right)^{\frac{1}{q}}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{\omega+1} - \frac{1}{2^{\omega}(\omega+1)}\right) \tag{3.33}
\end{aligned}
$$

holds for $\omega \in (0,1)$.

Proof. Take $s=1$ in Theorem 3.3. □

Theorem 3.4. *If $\mathcal{G}$ meets the requirements of Lemma* 3.3 *and $|\mathcal{G}'|$ is s-convex in the second sense, then the inequality*

$$
\begin{aligned}
&\Bigg|\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right] \\
&- \left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right) + \frac{1-\omega}{B(\omega)}\left[\mathcal{G}(\delta) + \mathcal{G}(\epsilon)\right]\right)\Bigg|
\end{aligned}
$$

$$\leq \frac{(\epsilon-\delta)^{\omega+1}\Big(|\mathcal{G}'(\delta)|+|\mathcal{G}'(\epsilon)|\Big)}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega+s+1}(\omega+s+1)}+\mathbb{B}_{\frac{1}{2}}\left(\omega+1,s+1\right)\right) \tag{3.34}$$

holds for $\omega\in(0,1)$*, where* $\mathbb{B}_z(\cdot,\cdot)$ *is the incomplete beta function defined in* (3.24)*.*

Proof. Using Lemma 3.3 and the s-convexity in the second sense of $|\mathcal{G}'|$, we obtain

$$\begin{aligned}
&\Bigg|\Big[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\Big]\\
&-\left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)+\frac{1-\omega}{B(\omega)}\Big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\Big]\right)\Bigg|\\
\leq&\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\Bigg[\int_0^{1/2}r^{\omega+s}|\mathcal{G}'(\delta)|+(1-r)^{s}r^{\omega}|\mathcal{G}'(\epsilon)|dr\\
&+\int_{1/2}^{1}(1-r)^{\omega}r^{s}|\mathcal{G}'(\delta)|+(1-r)^{\omega+s}|\mathcal{G}'(\epsilon)|dr\Bigg]\\
=&\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\Bigg[|\mathcal{G}'(\delta)|\bigg(\int_0^{1/2}r^{\omega+s}dr+\int_{1/2}^{1}(1-r)^{\omega}r^{s}dr\bigg)\\
&+|\mathcal{G}'(\epsilon)|\bigg(\int_0^{1/2}(1-r)^{s}r^{\omega}dr+\int_{1/2}^{1}(1-r)^{\omega+s}dr\bigg)\Bigg]\\
=&\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\Bigg[|\mathcal{G}'(\delta)|\bigg(\int_0^{1/2}r^{\omega+s}dr+\int_0^{1/2}t^{\omega}(1-t)^{s}dt\bigg)\\
&+|\mathcal{G}'(\epsilon)|\bigg(\int_0^{1/2}(1-r)^{s}r^{\omega}dr+\int_0^{1/2}t^{\omega+s}dt\bigg)\Bigg]\\
=&\frac{(\epsilon-\delta)^{\omega+1}\Big(|\mathcal{G}'(\delta)|+|\mathcal{G}'(\epsilon)|\Big)}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega+s+1}(\omega+s+1)}+\mathbb{B}_{\frac{1}{2}}\left(\omega+1,s+1\right)\right).
\end{aligned}$$

□

Corollary 3.4. *If* $\mathcal{G}$ *meets the requirements of Lemma* 3.3 *and* $|\mathcal{G}'|$ *is convex, then the inequality*

$$\begin{aligned}
&\Bigg|\Big[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\Big]\\
&-\left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)+\frac{1-\omega}{B(\omega)}\Big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\Big]\right)\Bigg|\\
\leq&\frac{(\epsilon-\delta)^{\omega+1}\Big(|\mathcal{G}'(\delta)|+|\mathcal{G}'(\epsilon)|\Big)}{2^{\omega+1}\left(\omega+1\right)B(\omega)\Gamma(\omega)} \qquad (3.35)
\end{aligned}$$

holds for $\omega\in(0,1)$*.*

Proof. Take $s=1$ in Theorem 3.4. □

Theorem 3.5. *If $\mathcal{G}$ meets the requirements of Lemma* 3.3 *and* $|\mathcal{G}'|^q, q > 1$, *is s-convex in the second sense, then the following inequality holds:*

$$\begin{aligned}
&\Bigg| \left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta)) \right] \\
&\quad - \left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)} \mathcal{G}\left(\frac{\epsilon+\delta}{2}\right) + \frac{1-\omega}{B(\omega)}\big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\big] \right) \Bigg| \\
&\leq \frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)} \left(\frac{1}{2^{\omega+1}(\omega+1)} \right)^{1-\frac{1}{q}} \\
&\quad \times \Bigg[\left(\frac{|\mathcal{G}'(\delta)|^q}{2^{\omega+s+1}(\omega+s+1)} + \mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)\,|\mathcal{G}'(\epsilon)|^q \right)^{\frac{1}{q}} \\
&\quad + \left(\mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)\,|\mathcal{G}'(\delta)|^q + \frac{|\mathcal{G}'(\epsilon)|^q}{2^{\omega+s+1}(\omega+s+1)} \right)^{\frac{1}{q}} \Bigg], \qquad (3.36)
\end{aligned}$$

for $\omega \in (0,1)$, *where* $\mathbb{B}_z(\cdot,\cdot)$ *is the incomplete beta function defined in* (3.24).

Proof. Utilizing Lemma 3.3, the power-mean inequality and the s-convexity in the second sense of $|\mathcal{G}'|^q$, we derive

$$\begin{aligned}
&\Bigg| \left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta)) \right] \\
&\quad - \left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)} \mathcal{G}\left(\frac{\epsilon+\delta}{2}\right) + \frac{1-\omega}{B(\omega)}\big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\big] \right) \Bigg| \\
&\leq \frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)} \Bigg[\left(\int_0^{1/2} r^{\omega} dr \right)^{1-\frac{1}{q}} \left(\int_0^{1/2} \left(r^{\omega+s}|\mathcal{G}'(\delta)|^q + (1-r)^s r^{\omega}|\mathcal{G}'(\epsilon)|^q \right) dr \right)^{\frac{1}{q}} \\
&\quad + \left(\int_{1/2}^1 (1-r)^{\omega} dr \right)^{1-\frac{1}{q}} \left(\int_{1/2}^1 \left((1-r)^{\omega} r^s|\mathcal{G}'(\delta)|^q + (1-r)^{\omega+s}|\mathcal{G}'(\epsilon)|^q \right) dr \right)^{\frac{1}{q}} \Bigg] \\
&= \frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)} \left(\frac{1}{2^{\omega+1}(\omega+1)} \right)^{1-\frac{1}{q}} \Bigg[\left(\int_0^{1/2} \left(r^{\omega+s}|\mathcal{G}'(\delta)|^q + (1-r)^s r^{\omega}|\mathcal{G}'(\epsilon)|^q \right) dr \right)^{\frac{1}{q}} \\
&\quad + \left(\int_{1/2}^1 \left((1-r)^{\omega} r^s|\mathcal{G}'(\delta)|^q + (1-r)^{\omega+s}|\mathcal{G}'(\epsilon)|^q \right) dr \right)^{\frac{1}{q}} \Bigg] \\
&= \frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)} \left(\frac{1}{2^{\omega+1}(\omega+1)} \right)^{1-\frac{1}{q}} \Bigg[\left(\frac{\mathcal{G}'(\delta)|^q}{2^{\omega+s+1}(\omega+s+1)} + \mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)\,|\mathcal{G}'(\epsilon)|^q \right)^{\frac{1}{q}} \\
&\quad + \left(\mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)\,|\mathcal{G}'(\delta)|^q + \frac{\mathcal{G}'(\epsilon)|^q}{2^{\omega+s+1}(\omega+s+1)} \right)^{\frac{1}{q}} \Bigg].
\end{aligned}$$

□

Corollary 3.5. *If* $\mathcal{G}$ *meets the requirements of Lemma* 3.3 *and* $|\mathcal{G}'|^q, q>1,$ *is convex, then the inequality*

$$\begin{aligned}
&\Bigg|\left[{}^{HAB}_{\ \ \psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}_{\ \ \psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\\
&-\left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)+\frac{1-\omega}{B(\omega)}\big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\big]\right)\Bigg|\\
\leq&\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega+1}(\omega+1)}\right)^{1-\frac{1}{q}}\\
&\times\Bigg[\left(\frac{|\mathcal{G}'(\delta)|^q}{2^{\omega+2}(\omega+2)}+\left(\frac{1}{2^{\omega+1}(\omega+1)}-\frac{1}{2^{\omega+2}(\omega+2)}\right)|\mathcal{G}'(\epsilon)|^q\right)^{\frac{1}{q}}\\
&+\left(\left(\frac{1}{2^{\omega+1}(\omega+1)}-\frac{1}{2^{\omega+2}(\omega+2)}\right)|\mathcal{G}'(\delta)|^q+\frac{|\mathcal{G}'(\epsilon)|^q}{2^{\omega+2}(\omega+2)}\right)^{\frac{1}{q}}\Bigg]
\end{aligned}\tag{3.37}$$

holds for $\omega\in(0,1)$.

Proof. Take $s=1$ in Theorem 3.5. □

Theorem 3.6. *If* $\mathcal{G}$ *meets the requirements of Lemma* 3.3 *and* $|\mathcal{G}'|^q, q>1,$ *is* s*-convex in the second sense, then the inequality*

$$\begin{aligned}
&\Bigg|\left[{}^{HAB}_{\ \ \psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}_{\ \ \psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\\
&-\left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)+\frac{1-\omega}{B(\omega)}\big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\big]\right)\Bigg|\\
\leq&\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega p+1}(\omega p+1)}\right)^{\frac{1}{p}}\Bigg[\left(\frac{|\mathcal{G}'(\delta)|^q}{2^{s+1}(s+1)}+\left(\frac{1}{s+1}-\frac{1}{2^{s+1}(s+1)}\right)|\mathcal{G}'(\epsilon)|^q\right)^{\frac{1}{q}}\\
&+\left(\left(\frac{1}{s+1}-\frac{1}{2^{s+1}(s+1)}\right)|\mathcal{G}'(\delta)|^q+\frac{|\mathcal{G}'(\epsilon)|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\Bigg]
\end{aligned}\tag{3.38}$$

holds for $\omega\in(0,1)$ *and* $p>1$ *such that* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 3.3, the Hölder's inequality and the s-convexity in the second sense of $|\mathcal{G}'|^q$, we have

$$\begin{aligned}
&\Bigg|\left[{}^{HAB}_{\ \ \psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}_{\ \ \psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\\
&-\left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)+\frac{1-\omega}{B(\omega)}\big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\big]\right)\Bigg|\\
\leq&\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\Bigg[\left(\int_0^{1/2}r^{\omega p}dr\right)^{\frac{1}{p}}\left(\int_0^{1/2}\Big(r^s|\mathcal{G}'(\delta)|^q+(1-r)^s|\mathcal{G}'(\epsilon)|^q\Big)dr\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_{1/2}^{1}(1-r)^{\omega p}dr\right)^{\frac{1}{p}}\left(\int_{1/2}^{1}\left(r^s|\mathcal{G}'(\delta)|^q+(1-r)^s|\mathcal{G}'(\epsilon)|^q\right)dr\right)^{\frac{1}{q}}\Bigg]$$

$$=\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega p+1}(\omega p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{G}'(\delta)|^q}{2^{s+1}(s+1)}+\left(\frac{1}{s+1}-\frac{1}{2^{s+1}(s+1)}\right)|\mathcal{G}'(\epsilon)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\frac{1}{s+1}-\frac{1}{2^{s+1}(s+1)}\right)|\mathcal{G}'(\delta)|^q+\frac{|\mathcal{G}'(\epsilon)|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right]. \qquad \square$$

Corollary 3.6. *If $\mathcal{G}$ meets the requirements of Lemma 3.3 and $|\mathcal{G}'|^q$, $q>1$, is convex, then the inequality*

$$\begin{aligned}&\Bigg|\left[{}^{HAB}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\\&-\left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)+\frac{1-\omega}{B(\omega)}\left[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\right]\right)\Bigg|\\&\le\frac{(\epsilon-\delta)^{\omega+1}}{2^{\omega+3}B(\omega)\Gamma(\omega)}\left(\frac{4}{\omega p+1}\right)^{\frac{1}{p}}\left[\left(|\mathcal{G}'(\delta)|^q+3|\mathcal{G}'(\epsilon)|^q\right)^{\frac{1}{q}}\right.\\&\left.+\left(3|\mathcal{G}'(\delta)|^q+|\mathcal{G}'(\epsilon)|^q\right)^{\frac{1}{q}}\right]\end{aligned} \tag{3.39}$$

holds for $\omega\in(0,1)$ and $p>1$ such that $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Take $s=1$ Theorem 3.6. $\square$

4. Conclusion

In this study, we first introduced the definitions of some new fractional integral operators referred to as ψ-Hilfer-Atangana-Baleanu (ψ-HAB) fractional integral operators. As indicated in Remark 3.1, these newly defined fractional integrals are generalizations of the Atangana-Baleanu and ABK fractional integrals. Next, we established a generalized version of the Hermite-Hadamard inequality for functions that are s-convex in the second sense via the ψ-HAB fractional integral operators. Finally, we investigated some fractional integral inequalities of the Hermite-Hadamard type involving the ψ-HAB fractional integrals for functions whose derivatives in absolute value to some nonnegative powers are s-convex in the second sense. Some particular cases of the main result were presented.

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