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A NOTE ON DISCRETE CLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract. We introduce the concept of $D_{w,p}$-classical orthogonal polynomials, where $D_{w,p}$ is the lowering operator given by $D_{w,p} := \frac{\tau_{-w} - \tau_{-p}}{w-p}$, $w, p \in \mathbb{C}$, with $\tau_{-w} f(x) := f(x+w)$. We conclude that these polynomials are the shifted discrete classical orthogonal polynomials.

1. Introduction

An orthogonal sequence of polynomials $\{p_n\}_{n\geq 0}$ is called classical if $\{p_n'\}_{n\geq 0}$ is also orthogonal. This characterization is essentially the Hahn-Sonine characterization (see [11, 19]) of the classical orthogonal polynomials. In [12], Hahn proved similar characterizations for orthogonal sequences of polynomials p_n such that $D_w p_n$ or $H_q p_n$ $(n \geq 1)$ are again orthogonal sequences. Here, D_w is the difference operator and H_q is the q-difference operator given, respectively, by $D_w f(x) = \frac{f(x+w)-f(x)}{w}$, $w \neq 0$ and $H_q f(x) = \frac{f(qx)-f(x)}{(q-1)x}$, $q \neq 1$. Note that differentiation, difference, and q-difference are lowering operators as they reduce the degree of a polynomial by exactly one.

The concept of O-classical orthogonal polynomials, where O is an operator on the space of polynomials, has been studied by many authors in the literature (see [1–14]).

The aim of the present paper is to pick up orthogonal sequences of polynomials under a lowering operator denoted by $D_{w,p}$, where $D_{w,p} f(x) := \frac{f(x+w)-f(x+p)}{w-p}$, generalizing the difference operator $D_w f(x) := \frac{f(x+w)-f(x)}{w}$ (see [1]).

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The structure of this paper is as follows. In Section 2, we present the terminology and basic definitions that will be used later on. In Section 3, we give some properties of the $D_{w,p}$-classical orthogonal polynomials.

2. Preliminaries

Let $\mathcal{P}$ be the linear space of polynomials in one variable with complex coefficients and $\mathcal{P}'$ its dual space, whose elements are *forms*. We denote by $\langle u, p\rangle$ the action of $u \in \mathcal{P}'$ on $p \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle,\ n \geq 0$, the moments of u.

Let us introduce some useful operations in $\mathcal{P}'$. For any $u \in \mathcal{P}',\ g \in \mathcal{P},\ a \in \mathbb{C}\backslash\{0\}$, and $b \in \mathbb{C}$, we let $Du = u'$, gu, $h_a u$ and $\tau_b u$, be the forms defined by duality [15]

$$\langle u', f\rangle := -\langle u, f'\rangle, \quad \langle gu, f\rangle := \langle u, gf\rangle, \quad f \in \mathcal{P}$$
$$\langle h_a u, f\rangle := \langle u, h_a f\rangle = \langle u, f(ax)\rangle, \quad \langle \tau_{-b} u, f\rangle := \langle u, \tau_b f\rangle = \langle u, f(x-b)\rangle, \quad f \in \mathcal{P}.$$

A form u is called normalized if it satisfies $(u)_0 = 1$. We assume that the forms used in this paper are normalized.

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials (MPS) with $\deg P_n = n$ and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$, defined by $\langle u_n, P_m\rangle = \delta_{n,m}$ with $n,\ m \geq 0$. Note that u_0 is said to be the canonical functional associated with the MPS $\{P_n\}_{n\geq 0}$.

Let us recall the following result.

Lemma 2.1 ([15,16]). *For any $u \in \mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent:*

(i) $\langle u, P_{m-1}\rangle \neq 0,\ \langle u, P_n\rangle = 0,\ n \geq m$;

(ii) *exist* $\lambda_\nu \in \mathbb{C},\ 0 \leq \nu \leq m-1,\ \lambda_{m-1} \neq 0$ *such that* $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$.

The form u is called *regular* if we can associate with it a sequence $\{P_n\}_{n\geq 0}$ such that

$$\langle u, P_n P_m\rangle = r_n \delta_{n,m}, \quad r_n \neq 0, n, m \geq 0.$$

The sequence $\{P_n\}_{n\geq 0}$ is then called an *orthogonal* sequence of monic polynomials (MOPS) with respect to u. Note that $u = (u)_0 u_0 = u_0$. When u is regular, let F be a polynomial such that if $Fu = 0$, then $F = 0$ (see [18]).

Proposition 2.1 ([15,16]). *Let $\{P_n\}_{n\geq 0}$ be an MPS with* $\deg P_n = n,\ n \geq 0$, *and let $\{u_n\}_{n\geq 0}$ be its dual sequence. The following statements are equivalent.*

(i) $\{P_n\}_{n\geq 0}$ *is orthogonal with respect to* u_0.

(ii) *For all* $n \geq 0$

$$u_n = \langle u_0, P_n^2\rangle^{-1} P_n u_0.$$

(iii) $\{P_n\}_{n\geq 0}$ *satisfies the three-term recurrence relation*

$$\text{(TTRR)} \ : \begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1} P_n(x), \quad n \geq 0, \end{cases}$$

where $\beta_n = \langle u_0, xP_n^2\rangle\langle u_0, P_n^2\rangle^{-1},\ n \geq 0$ *and* $\gamma_{n+1} = \langle u_0, P_{n+1}^2\rangle\langle u_0, P_n^2\rangle^{-1} \neq 0,\ n \geq 0.$

If $\{P_n\}_{n\geq 0}$ is a MOPS with respect to the regular form u_0, then $\{\tilde{P}_n\}_{n\geq 0}$, where $\tilde{P}_n(x) = a^{-n}P_n(ax),\ n \geq 0,\ a \neq 0$, is a MOPS with respect to the regular form $\tilde{u}_0 = h_{a^{-1}}u_0$, and satisfies [16]

$$\begin{cases} \tilde{P}_0(x) = 1, \quad \tilde{P}_1(x) = x - \tilde{\beta}_0, \\ \tilde{P}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\tilde{P}_{n+1}(x) - \tilde{\gamma}_{n+1}\tilde{P}_n(x), \quad n \geq 0, \end{cases}$$

where $\tilde{\beta}_n = a^{-1}\beta_n$ and $\tilde{\gamma}_{n+1} = a^{-2}\gamma_{n+1}$.

Recall the operator

$$(D_{w,p}f)(x) := \frac{f(x+w) - f(x+p)}{w-p}, \quad f \in \mathcal{P}, w, p \in \mathbb{C}.$$

The transposition ${}^tD_{w,p}$ of $D_{w,p}$ is $-D_{-w,-p}$, with a slight abuse of notation which is harmless. Thus,

$$\langle D_{-w,-p}u, f\rangle = -\langle u, D_{w,p}f\rangle, \quad u \in \mathcal{P}', f \in \mathcal{P}, w, p \in \mathbb{C}.$$

Note that $D_{w,0}$ reduces to the operator D_w where $(D_wf)(x) = \frac{f(x+w)-f(x)}{w}$ (see [1]).

Lemma 2.2. *The following formulas hold*

$$(D_{w,p}fg)(x) = (\tau_{-p}f)(x)(D_{w,p}g)(x) + (\tau_{-w}g)(x)(D_{w,p}f)(x), \quad f, g \in \mathcal{P}, \tag{2.1}$$

$$(D_{w,p}f(\tau_w g))(x) = (\tau_{-p}f)(x)(D_{w,p}(\tau_w g))(x) + g(x)(D_{w,p}f)(x), \quad f, g \in \mathcal{P}, \tag{2.2}$$

$$(\tau_{-w}fg)(x) = (\tau_{-w}f)(x)(\tau_{-w}g)(x), \quad f, g \in \mathcal{P}, \tag{2.3}$$

$$(\tau_{-w}gu) = (\tau_{-w}g)(\tau_{-w}u), \quad g \in \mathcal{P},\ u \in \mathcal{P}', \tag{2.4}$$

$$D_{-w,-p}(gu) = (\tau_w g)(D_{-w,-p}u) + (D_{-w,-p}g)(\tau_p u), \quad g \in \mathcal{P},\ u \in \mathcal{P}', \tag{2.5}$$

$$\begin{aligned} &(\tau_b \circ D_{w,p})(f) = (D_{w,p} \circ \tau_b)(f), \\ &(\tau_b \circ D_{w,p})(u) = (D_{w,p} \circ \tau_b)(u), \quad f \in \mathcal{P},\ u \in \mathcal{P}',\ b \in \mathbb{C}, \end{aligned} \tag{2.6}$$

$$(h_a \circ D_{w,p})(u) = (aD_{aw,ap} \circ h_a)(u), \quad u \in \mathcal{P}',\ a \in \mathbb{C} \setminus \{0\}, \tag{2.7}$$

$$(h_a \circ D_{w,p})(f) = (a^{-1}D_{a^{-1}w,a^{-1}p} \circ h_a)(f), \quad f \in \mathcal{P},\ a \in \mathbb{C} \setminus \{0\}. \tag{2.8}$$

The relations (2.1)–(2.4) are evident. Further, we have

$$\begin{aligned} \langle D_{-w,-p}(gu), f\rangle &= -\langle u, g(D_{w,p}f)\rangle = -\langle u, D_{w,p}(f(\tau_w g)) - (\tau_{-p}f)D_{w,p}(\tau_w g)\rangle \\ &\quad \text{(from (2.2))} \\ &= \langle (\tau_w g)(D_{-w,-p}u) + \tau_p((D_{w,p}\tau_w g)u), f\rangle, \end{aligned}$$

but

$$\begin{aligned} \tau_p\Big((D_{w,p}\tau_w g)u\Big) &= \Big(\tau_p \circ D_{w,p} \circ \tau_w g\Big)(\tau_p u) \quad \text{(from(2.3))} \\ &= (D_{-w,-p}g)(\tau_p u) \quad \text{(following the definitions.)} \end{aligned}$$

Hence, we have (2.5).

The proofs of (2.6)–(2.8) follow easily from the definitions.

Now, consider $\{P_n\}_{n\geq 0}$ as above in Section 1 and let

$$\widehat{P}_n(x) = \frac{1}{n+1}(D_{w,p}P_{n+1})(x), \quad n \geq 0. \tag{2.9}$$

Denoting by $\{\widehat{u}_n\}_{n\geq 0}$ the dual sequence of $\{\widehat{P}_n\}_{n\geq 0}$, we have the following result.

Lemma 2.3.

$$D_{-w,-p}(\widehat{u}_n) = -(n+1)u_{n+1}, \quad n \geq 0. \tag{2.10}$$

Indeed, from the definition $\langle \widehat{u}_n, \widehat{P}_m\rangle = \delta_{n,m}$, $n, m \geq 0$, we have $-\langle D_{-w,-p}(\widehat{u}_n), P_{m+1}\rangle = (m+1)\delta_{n,m}$, therefore

$$\begin{aligned}\langle D_{-w,-p}(\widehat{u}_n), P_m\rangle =& 0, \quad m \geq n+2, n \geq 0,\\ \langle D_{-w,-p}(\widehat{u}_n), P_{n+1}\rangle =& -(n+1), \quad n \geq 0.\end{aligned}$$

By virtue of Lemma 2.1,

$$D_{-w,-p}(\widehat{u}_n) = \sum_{\mu=0}^{n+1} \lambda_{n,\mu} u_\mu.$$

However, $\langle D_{-w,-p}(\widehat{u}_n), P_\mu\rangle = \lambda_{n,\mu}$, $0 \leq \mu \leq n+1$ and $\lambda_{n,\mu} = 0$, $0 \leq \mu \leq n$, $\lambda_{n,n+1} = -(n+1)$, $n \geq 0$. Hence, we have (2.10).

Let ϕ and ψ be two polynomials with ϕ monic, and $\deg \phi = t$, $\deg \psi = q \geq 1$. We suppose that the pair (ϕ, ψ) is admissible, i.e., when $q = t-1$, writing $\psi(x) = a_q x^q + \cdots$, then a_q is not a positive integer.

Definition 2.1. A form u is called $D_{w,p}$-semi-classical when it is regular and satisfies

$$D_{-w,-p}(\phi u) + \psi u = 0, \tag{2.11}$$

where the pair (ϕ, ψ) is admissible. The corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ is called $D_{w,p}$-semi-classical.

Lemma 2.4. *Consider the sequence $\{\tilde{P}_n\}_{n\geq 0}$ obtained by shifting P_n, i.e.,*

$$\tilde{P}_n(x) = a^{-n}P_n(ax+b) = a^{-n}(h_a \circ \tau_{-b}P_n)(x), \quad n \geq 0, a \neq 0.$$

If u_0 satisfies (2.11), then $\tilde{u}_0 = (h_{a^{-1}} \circ \tau_{-b})u_0$ fulfils the equation

$$D_{-wa^{-1},-pa^{-1}}(\tilde{\phi}\tilde{u}_0) + \tilde{\psi}\tilde{u}_0 = 0, \tag{2.12}$$

where $\tilde{\phi}(x) = a^{-t}\phi(ax+b)$ and $\tilde{\psi}(x) = a^{1-t}\psi(ax+b)$.

We need the following formulas, which are easy to prove.

$$\begin{cases} g(\tau_b u) = \tau_b((\tau_{-b}g)u), & g \in \mathcal{P}, u \in \mathcal{P}', b \in \mathbb{C},\\ g(h_a u) = h_a((h_a g)u), & g \in \mathcal{P}, u \in \mathcal{P}', a \in \mathbb{C}\backslash\{0\}.\end{cases} \tag{2.13}$$

Let $u_0 = (\tau_b \circ h_a)\tilde{u}_0$ and $v = h_a\tilde{u}_0$. From (2.13) we have

$$\begin{aligned}\psi u_0 =& \psi(\tau_b v) = \tau_b((\tau_{-b}\psi)v)\\ =& \tau_b((\tau_{-b}\psi)(h_a\tilde{u}_0)) = (\tau_b \circ h_a)(h_a \circ \tau_{-b}\psi)\tilde{u}_0 = (\tau_b \circ h_a)(\psi(ax+b)\tilde{u}_0).\end{aligned}$$

Further, by using (2.13) and (2.7) we get

$$\begin{aligned} D_{-w,-p}(\phi u_0) =& D_{-w,-p}(\phi(\tau_b v)) = D_{-w,-p}(\tau_b((\tau_{-b}\phi)v)) \\ =& \tau_b D_{-w,-p}\Big((\tau_{-b}\phi)(h_a\tilde{u}_0)\Big) = \tau_b D_{-w,-p}\Big(h_a\left((h_a\circ\tau_{-b}\phi)\,\tilde{u}_0\right)\Big) \\ =& a^{-1}(\tau_b\circ h_a)D_{-wa^{-1},-pa^{-1}}\Big(\phi(ax+b)\tilde{u}_0\Big). \end{aligned}$$

Equation (2.11) becomes

$$(\tau_b\circ h_a)\Big(D_{-wa^{-1},-pa^{-1}}\Big(\phi(ax+b)\tilde{u}_0\Big) + a\psi(ax+b)\tilde{u}_0\Big) = 0.$$

Hence, we have the desired result.

Regarding general semi-classical sequences, we have the following statement that we give for the sake of completeness [17, 18].

Proposition 2.2. *For any monic polynomial ϕ and any orthogonal sequence $\{P_n\}_{n\geq 0}$, the following statements are equivalent.*

a) *There exists an integer $s \geq 0$ such that*

$$\phi(x)\widehat{P}_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu}P_\nu(x), \quad n \geq s,$$
$$\lambda_{n,n-s} \neq 0, \quad n \geq s+1.$$

b) *There exists a polynomial ψ, $deg\psi = q \geq 1$ such that*

$$D_{-w,-p}(\phi u_0) + \psi u_0 = 0, \tag{2.14}$$

where the pair (ϕ,ψ) is admissible.

Remark 2.1. (a) We also have the following statement: the form u_0 is $D_{w,p}$-semi-classical if and only if the sequence $\{\widehat{P}_n\}_{n\geq 0}$ is quasi-orthogonal of order s with respect to ϕu_0.

(b) When $\{P_n\}_{n\geq 0}$ is orthogonal, it fulfils the standard recurrence relation

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad \gamma_{n+1} \neq 0,\ n \geq 0. \end{cases}$$

Likewise, when $\{\widehat{P}_n\}_{n\geq 0}$ is orthogonal $(s = 0)$, it fulfils the recurrence relation

$$\begin{cases} \widehat{P}_0(x) = 1,\ \widehat{P}_1(x) = x - \widehat{\beta}_0, \\ \widehat{P}_{n+2}(x) = (x-\widehat{\beta}_{n+1})\widehat{P}_{n+1}(x) - \widehat{\gamma}_{n+1}\widehat{P}_n(x), \quad \widehat{\gamma}_{n+1} \neq 0,\ n \geq 0. \end{cases}$$

3. The $D_{w,p}$-Classical Orthogonal Polynomials

When $s = 0$, the sequence $\{P_n\}_{n\geq 0}$ is called $D_{w,p}$-classical (discrete classical orthogonal polynomials), moreover, we have the more accurate following statements.

Proposition 3.1. *For any orthogonal sequence $\{P_n\}_{n\geq 0}$, the following statements are equivalent.*

a) *The sequence $\{P_n\}_{n\geq 0}$ is $D_{w,p}$-classical.*

b) *The sequence* $\{\widehat{P}_n\}_{n\geq 0}$ *is orthogonal.*
c) *There are two polynomials,* ϕ *which is monic with degree at most* 2*, and* ψ *with degree* 1*, along with a sequence* $\{\lambda_n\}_{n\geq 0}$*, where each* λ_n *is nonzero for* $n \geq 0$*, such that*

$$\phi(x)(D_{w,p} \circ D_{-w,-p}P_{n+1})(x) - \psi(x)(D_{-w,-p}P_{n+1})(x) + \lambda_n P_{n+1}(x) = 0, \quad n \geq 0.$$

Proof. a) ⇒ b). From (2.14) and Lemma 2.2, we have

$$\begin{aligned}
\langle u_0, \phi P_m \widehat{P}_n\rangle =& \frac{1}{n+1}\langle P_m\phi u_0, D_{w,p}P_{n+1}\rangle\\
=& -\frac{1}{n+1}\langle D_{-w,-p}(P_m\phi u_0), P_{n+1}\rangle\\
=& -\frac{1}{n+1}\langle (\tau_w P_m)D_{-w,-p}(\phi u_0) + (D_{-w,-p}P_m)\tau_p(\phi u_0), P_{n+1}\rangle\\
=& \frac{1}{n+1}\langle (\tau_w P_m)\psi u_0 - (D_{-w,-p}P_m)\tau_p(\phi u_0), P_{n+1}\rangle\\
=& \frac{1}{n+1}\left\langle u_0, ((\tau_w P_m)\psi P_{n+1} - \phi\tau_{-p}((D_{-w,-p}P_m))P_{n+1})\right\rangle.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle \phi u_0, P_m\widehat{P}_n\rangle =& 0, \quad 0 \leq m \leq n-1, n \geq 1,\\
\langle \phi u_0, (\widehat{P}_n)^2\rangle =& \frac{1}{n+1}\left(\psi'(0) - \frac{1}{2}\phi''(0)n\right)\langle u_0, P_{n+1}^2\rangle \neq 0, \quad n \geq 0,
\end{aligned}$$

since (ϕ, ψ) is admissible.

b) ⇒ c). From (2.10) and the assumptions,

$$D_{-w,-p}(\widehat{P}_n\widehat{u}_0) = -\mathcal{X}_n P_{n+1}u_0, \quad n \geq 0, \tag{3.1}$$

with

$$\mathcal{X}_n = (n+1)\frac{\langle \widehat{u}_0, \widehat{P}_n^2\rangle}{\langle u_0, P_{n+1}^2\rangle}, \quad n \geq 0.$$

For $n = 0$ in (3.1), we obtain

$$D_{-w,-p}(\widehat{u}_0) = -\gamma_1^{-1}P_1u_0. \tag{3.2}$$

In accordance with Lemma 2.2, we have

$$D_{-w,-p}(\widehat{P}_n\widehat{u}_0) = (\tau_w\widehat{P}_n)(D_{-w,-p}\widehat{u}_0) + (D_{-w,-p}\widehat{P}_n)(\tau_p\widehat{u}_0),$$

therefore, on account of (3.2),

$$-\mathcal{X}_0P_1(\tau_w\widehat{P}_n)u_0 + (D_{-w,-p}\widehat{P}_n)(\tau_p\widehat{u}_0) = -\mathcal{X}_nP_{n+1}u_0, \quad n \geq 0. \tag{3.3}$$

Putting $n = 1$, we get

$$\tau_p(\widehat{u}_0) = \gamma_1^{-1}\kappa\phi u_0, \tag{3.4}$$

where $\kappa\phi(x) = P_1(x)(\tau_w\widehat{P}_1)(x) - 2\widehat{\gamma}_1\gamma_2^{-1}P_2(x)$ (ϕ monic). So, Equations (3.3), (3.4) and the regularity of u_0 imply

$$\phi(x)(D_{-w,-p}\widehat{P}_n)(x) - \psi(x)(\tau_w\widehat{P}_n)(x) + \gamma_1\kappa^{-1}\mathfrak{X}_n P_{n+1}(x) = 0, \quad n \geq 0,$$

with $\psi(x) = \kappa^{-1}P_1(x)$. Comparing the degrees, we obtain

$$\frac{1}{2}\phi''(0)n - \psi'(0) + \gamma_1\kappa^{-1}\mathfrak{X}_n = 0, \quad n \geq 0,$$

which means that the pair (ϕ, ψ) is admissible. Finally, we have the desired second-order difference equation with $\lambda_n = \gamma_1\kappa^{-1}(n+1)\mathfrak{X}_n,\ n \geq 0$. In fact, we also have proved that b) $\Rightarrow$ c).

c) $\Rightarrow$ a). From the given equation, we get

$$\langle u_0, \phi(D_{w,p} \circ D_{-w,-p}P_{n+1}) - \psi(D_{-w,-p}P_{n+1})\rangle = 0, \quad n \geq 0.$$

Hence

$$\langle D_{w,p}\Big(D_{-w,-p}(\phi u_0) + \psi u_0\Big), P_{n+1}\rangle = 0, \quad n \geq 0.$$

Since $\Big\langle D_{w,p}\Big(D_{-w,-p}(\phi u_0) + \psi u_0\Big), 1\Big\rangle = 0$, we get

$$D_{w,p}\Big(D_{-w,-p}(\phi u_0) + \psi u_0\Big) = 0.$$

Hence, (2.14) where the pair (ϕ, ψ) is admissible on account of $\lambda_n \neq 0,\ n \geq 0$. $\square$

Remark 3.1. (a) In the case $s = 0$, when the pair (ϕ, ψ) is not admissible, then the solution u of (2.11) is not regular. In other words, when the solution u of (2.11) is regular, then the pair (ϕ, ψ) is necessarily admissible.

(b) Necessarily, we have

$$\begin{aligned}\kappa\phi(x) =& (1 - 2\widehat{\gamma}_1\gamma_2^{-1})x^2 + (2\widehat{\gamma}_1\gamma_2^{-1}(\beta_0 + \beta_1) - \beta_0 - \widehat{\beta}_0 - w)x \\ &+ \beta_0(\widehat{\beta}_0 + w) - 2\widehat{\gamma}_1\gamma_2^{-1}(\beta_0\beta_1 - \gamma_1), \\ k\psi(x) =& P_1(x).\end{aligned}$$

Proposition 3.2. *If $\{P_n\}_{n\geq 0}$ is $D_{w,p}$-classical, the sequence $\{\widehat{P}_n\}_{n\geq 0}$ is $D_{w-p,0}$-classical and we have*

$$D_{p-w,0}(\phi_1\widehat{u}_0) + \psi_1\widehat{u}_0 = 0, \tag{3.5}$$

with $\phi_1(x) = (\tau_{-w}\phi)(x)$ and $\psi_1(x) = \Big(\kappa^{-1}(\tau_{-p}P_1) - (D_{w,p}\phi)\Big)(x)$.

Proof. When $\{P_n\}_{n\geq 0}$ is $D_{w,p}$-classical, we have (3.2)

$$D_{-w,-p}(\widehat{u}_0) = -\gamma_1^{-1}P_1u_0.$$

Multiplying this equation by ϕ, we get

$$\phi D_{-w,-p}(\widehat{u}_0) = -\gamma_1^{-1}\phi P_1u_0.$$

But by (2.5), $\phi D_{-w,-p}(\widehat{u}_0) = D_{-w,-p}\Big((\tau_{-w}\phi)\widehat{u}_0\Big) - \Big(D_{-w,-p} \circ \tau_{-w}\phi\Big)\tau_p\widehat{u}_0$, so we have

$$D_{-w,-p}\Big((\tau_{-w}\phi)\widehat{u}_0\Big) - \Big(D_{-w,-p} \circ \tau_{-w}\phi\Big)\tau_p\widehat{u}_0 = -\gamma_1^{-1}\phi P_1u_0$$

and

$$\tau_{-p} \circ D_{-w,-p}\Big((\tau_{-w}\phi)\widehat{u}_0\Big) - (D_{w,p}\phi)\widehat{u}_0 = -\gamma_1^{-1}\tau_{-p}(P_1)\tau_{-p}\Big(\phi u_0\Big).$$

By (3.4), we have $\widehat{u}_0 = \gamma_1^{-1}\kappa\tau_{-p}\Big(\phi u_0\Big)$, and so we now get

$$D_{-w+p,0}\Big((\tau_{-w}\phi)\widehat{u}_0\Big) + \Big(\kappa^{-1}(\tau_{-p}P_1) - (D_{w,p}\phi)\Big)\widehat{u}_0 = 0.$$

This completes the proof. □

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