ON PERFECT CO-ANNIHILATING-IDEAL GRAPH OF A COMMUTATIVE ARTINIAN RING

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Abstract. Let $R$ be a commutative ring with identity. The co-annihilating-ideal graph of $R$, denoted by $A_R$, is a graph whose vertex set is the set of all non-zero proper ideals of $R$ and two distinct vertices $I$ and $J$ are adjacent whenever $\text{Ann}(I) \cap \text{Ann}(J) = (0)$. In this paper, we characterize all Artinian rings for which both of the graphs $A_R$ and $\overline{A_R}$ (the complement of $A_R$), are chordal. Moreover, all Artinian rings whose $A_R$ (and thus $\overline{A_R}$) is perfect are characterized.

1. Introduction

Assigning a graph to a ring gives us the ability to translate algebraic properties of rings into graph-theoretic language and vice versa. It leads to arising interesting algebraic and combinatorics problems. Therefore, the study of graphs associated with rings has attracted many researches. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory; for instance see [2, 3, 5, 6, 10, 11] and [12].

Throughout this paper, all rings are assumed to be commutative with identity. We denote by $Z(R)$, $\text{Max}(R)$, $\text{Nil}(R)$ and $J(R)$ the set of all zero-divisor elements of $R$, the set of all maximal ideals of $R$, the set of all nilpotent elements of $R$ and jacobson radical of $R$, respectively. We call an ideal $I$ of $R$, an annihilating-ideal if there exists $r \in R \setminus \{0\}$ such that $Ir = (0)$. The set of all annihilating-ideals of $R$ is denote by $A(R)$. Let $I$ be an ideal of $R$. We denote by $A(I)$ the set of all ideals of $R$ contained in $I$. The ring $R$ is said to be reduced if it has no non-zero nilpotent element. For every ideal $I$ of $R$, we denote the annihilator of $I$ by $\text{Ann}(I)$. We let $A^* = A \setminus \{0\}$. For any undefined notation or terminology in ring theory, we refer the reader to [4, 7].

Key words and phrases. Co-annihilating-ideal graph, perfect graph, chordal graph.

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We use the standard terminology of graphs following [13]. Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, we mean the complement graph of $G$. We write $u - v$, to denote an edge with ends $u, v$. A graph $H = (V_0, E_0)$ is called a subgraph of $G$ if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, $H$ is called an induced subgraph by $V_0$, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$. Also $G$ is called a null graph if it has no edge. A complete graph of $n$ vertices is denoted by $K_n$. An $n$-part graph is one whose vertex set can be partitioned into $n$ subsets, so that no edge has both ends in any one subset. 

A complete $n$-partite graph is an $n$-part graph such that every pair of graph vertices in the $n$ sets are adjacent. In a graph $G$, a vertex $x$ is isolated, if no vertices of $G$ is adjacent to $x$. Let $G_1$ and $G_2$ be two disjoint graphs. The join of $G_1$ and $G_2$, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. For a graph $G$, $S \subseteq V(G)$ is called a clique if the subgraph induced on $S$ is complete. The number of vertices in the largest clique of graph $G$ is called the clique number of $G$ and is often denoted by $\omega(G)$. For a graph $G$, let $\chi(G)$ denote the chromatic number of $G$, i.e., the minimal number of colors which can be assigned to the vertices of $G$ in such a way that every two adjacent vertices have different colors. Clearly, for every graph $G$, $\omega(G) \leq \chi(G)$. A graph $G$ is said to be weakly perfect if $\omega(G) = \chi(G)$. A perfect graph $G$ is a graph in which every induced subgraph is weakly perfect. A chord of a cycle $C$ is an edge which is not in $C$ but has both its endvertices in $C$. A graph $G$ is chordal if every cycle of length at least 4 has a chord.

Let $R$ be a commutative ring with identity. The co-annihilating-ideal graph of $R$, denoted by $A_R$, is a graph whose vertex set is the set of all non-zero proper ideals of $R$ and two distinct vertices $I$ and $J$ are adjacent whenever $\text{Ann}(I) \cap \text{Ann}(J) = (0)$. This graph was first introduced and studied in [1] and many interesting properties of this graph were explored by the authors. In [1, Theorem 17], it was proved $A_R$ is a weakly perfect graph, if $R$ is an Artinian ring. In this paper, we continue study the perfectness of $A_R$. Indeed, we characterize all Artinian rings for which both of the graphs $A_R$ and $\overline{A_R}$, are chordal. Moreover, all Artinian rings whose $A_R$ is perfect are given.

2. When $A_R$ and $\overline{A_R}$ are chordal?

In this section, we characterize all Artinian rings $R$, for which $A_R$ and $\overline{A_R}$ are chordal. We begin with the following lemmas.

**Lemma 2.1.** Let $R$ be an Artinian ring. Then there exists a positive integer $n$ such that $R \cong R_1 \times \cdots \times R_n$, where $R_i$ is an Artinian local ring, for every $1 \leq i \leq n$.

*Proof.* See [4, Theorem 8.7].

**Lemma 2.2.** Let $R$ be an Artinian ring and $I$ be a non-zero ideal of $R$. Then $I$ is a nilpotent ideal of $R$ if and only if $I$ is an isolated vertex in $A_R$.
Proof. Assume that $I$ is a non-zero nilpotent ideal of $R$. First, we show that $\text{Ann}(I)$ is an essential ideal of $R$. Suppose to the contrary, there exists an ideal $J$ such that $J \cap \text{Ann}(I) = (0)$. Thus $KI \neq (0)$, for every $K \subseteq J$. Obviously, $KI \subseteq J$ and so $(KI)I = KI^2 \neq (0)$. By continuing this procedure, $KI^n \neq 0$, for every positive integer $n$, a contradiction. Hence $\text{Ann}(I)$ is an essential ideal of $R$ and so $\text{Ann}(I) \cap \text{Ann}(J) \neq (0)$, for every $J \in A(R)^*$. Therefore, $I$ is an isolated vertex in $A_R$.

Conversely, suppose that $I$ is an isolated vertex in $A_R$. If $I$ is not a nilpotent ideal of $R$, then $I \not\subseteq J(R)$, i.e., there exists $m \in \text{Max}(R)$ such that $I + m = R$, and so $I$ is adjacent to $m$, a contradiction. Thus $I$ is a nilpotent ideal of $R$.

Next we need to study the structure of $A_R$, where $R$ is an Artinian ring with at most two maximal ideals.

**Theorem 2.1.** Let $R$ be an Artinian ring. Then the following statements are equivalent:

1. $|\text{Max}(R)| = 1$;
2. $A_R = K_n$, where $n = |A(R)^*|$.

Proof. (1) $\Rightarrow$ (2) Since $R$ is an Artinian local ring, every ideal of $A(R)^*$ is a nilpotent ideal of $R$ and thus by Lemma 2.2, $A_R$ is a null graph.

(2) $\Rightarrow$ (1) is obtained by Lemma 2.2. □

**Theorem 2.2.** Let $R$ be an Artinian ring. Then the following statements are equivalent:

1. $|\text{Max}(R)| = 2$;
2. $A_R = K_{n_1} + K_{n_2,n_3}$, where $n_1 = |A(\text{Nil}(R))^*|$, $n_2 = |A(m_1)^*| - n_1$, $n_3 = |A(m_2)^*| - n_1$ and $m_1, m_2 \in \text{Max}(R)$.

Proof. (1) $\Rightarrow$ (2) Let $\text{Max}(R) = \{m_1, m_2\}$. Since $m_1 \cap m_2 = \text{Nil}(R)$, Lemma 2.2 implies that $A_R[A(\text{Nil}(R))^*]$ is a null graph. Let $A = \{I \in A(m_1) \setminus A(\text{Nil}(R))\}$ and $B = \{I \in A(m_2) \setminus A(\text{Nil}(R))\}$. If $I \in A$ and $J \in B$, then $I + J = R$, and thus $I$ is adjacent to $J$. Moreover, $A_R[A]$ and $A_R[B]$ are null graphs. This means that $A_R[A \cup B] = K_{|A|,|B|}$. Since $A \cup B \cup A(\text{Nil}(R))^* = A(R)^*$, we deduce that $A_R = K_{n_1} + K_{n_2,n_3}$, where $n_1 = |A(\text{Nil}(R))^*|$, $n_2 = |A(m_1)^*| - n_1$, $n_3 = |A(m_2)^*| - n_1$ and $m_1, m_2 \in \text{Max}(R)$.

(2) $\Rightarrow$ (1) By Theorem 2.1, $|\text{Max}(R)| \geq 2$. If $|\text{Max}(R)| \geq 3$, then $A_R$ has a cycle of length 3, as $A_R[\text{Max}(R)]$ is a complete graph, a contradiction. Thus $|\text{Max}(R)| = 2$. □

We are now in a position to characterize all Artinian rings for which both of the graphs $A_R$ and $\overline{A_R}$ are chordal.

**Theorem 2.3.** Let $R$ be an Artinian ring. Then

1. $A_R$ is chordal if and only if one of the following statements holds:
   (i) $R$ is local;
   (ii) $R \cong F \times S$, where $F$ is a field and $S$ is local;
(iii) $R \cong F_1 \times F_2 \times F_3$, where $F_i$ is a field for every $1 \leq i \leq 3$;

(2) $\overline{A_R}$ is chordal if and only if $|\text{Max}(R)| \leq 3$.

Proof. (1) Let $A_R$ be chordal. First we show that $|\text{Max}(R)| \leq 3$. If $|\text{Max}(R)| \geq 4$, then Figure 1 is a cycle of length 4,

![Figure 1](image)

where

\[
I_1 = (0) \times R_2 \times R_3 \times (0) \times R_5 \times \cdots \times R_n,
\]
\[
I_2 = R_1 \times (0) \times (0) \times R_4 \times R_5 \times \cdots \times R_n,
\]
\[
I_3 = R_1 \times R_2 \times R_3 \times (0) \times R_5 \times \cdots \times R_n,
\]
\[
I_4 = R_1 \times (0) \times R_3 \times R_4 \times R_5 \times \cdots \times R_n.
\]

Thus $|\text{Max}(R)| \leq 3$. If $|\text{Max}(R)| = 3$, then $R \cong R_1 \times R_2 \times R_3$, where $R_i$ is an Artinian local ring, for every $1 \leq i \leq n$. If $R_1$ is not field, then consider $I \in A(\text{Nil}(R_1))^*$ and thus Figure 2 is a cycle of length 4,

![Figure 2](image)

where

\[
I_1 = R_1 \times (0) \times (0),
\]
\[
I_2 = (0) \times R_2 \times R_3,
\]
\[
I_3 = R_1 \times R_2 \times (0),
\]
\[
I_4 = I \times R_2 \times R_3.
\]

Hence $R_1$ is a field. Similarly, $R_2$ and $R_3$ are fields. Let $|\text{Max}(R)| = 2$. Then $R \cong R_1 \times R_2$, where $R_i$ is an Artinian local ring, for every $1 \leq i \leq 2$. We show that
one of the rings $R_1$ and $R_2$ is a field. If $I, J$ are non-zero proper ideals of $R_1$ and $R_2$, respectively, then Figure 3 is a cycle of length 4, where

$$I_1 = I \times R_2,$$
$$I_2 = R_1 \times J,$$
$$I_3 = (0) \times R_2,$$
$$I_4 = R_1 \times (0).$$

\[
\begin{array}{c}
I_1 \quad I_2 \\
I_4 \quad I_3
\end{array}
\]

**Figure 3.** A cycle of length 4 in $A_R$

This means that one of the rings $R_1$ and $R_2$ is a field. Thus in this case $R \cong F \times S$, where $F$ is a field and $S$ is local. Clearly, if $|\text{Max}(R)| = 1$, $R$ is local.

Conversely, suppose that one of the conditions (i), (ii), (iii) is satisfied. Condition (i) implies that $A_R$ is a null graph by Theorem 2.1, and thus $A_R$ is chordal. If (ii) holds, then by Theorem 2.2, $A_R = \overline{K_n} + K_{1,n+1}$ where $n = |A(\text{Nil}(R))^*|$. This implies that $A_R$ is chordal. If (iii) holds, then Figure 4 shows that $A_R$ is chordal where

\[
\begin{array}{c}
I_1 \quad I_2 \\
I_5 \quad I_3 \quad I_4 \\
I_6
\end{array}
\]

**Figure 4.** $A_{F_1 \times F_2 \times F_3}$

$$I_1 = (0) \times (0) \times F_3,$$
$$I_2 = F_1 \times F_2 \times (0),$$
$$I_3 = F_1 \times (0) \times F_3,$$
$$I_4 = (0) \times F_2 \times (0),$$
$$I_5 = (0) \times F_2 \times F_3.$$
First suppose that we consider the following cases.

(2) First suppose that $\overline{A_R}$ is chordal. If $|\text{Max}(R)| \geq 4$, then we put

\begin{align*}
I_1 &= (0) \times R_2 \times R_3 \times (0) \times R_5 \times \ldots \times R_n, \\
I_2 &= (0) \times R_2 \times (0) \times R_4 \times R_5 \times \ldots \times R_n, \\
I_3 &= R_1 \times (0) \times (0) \times R_4 \times R_5 \times \ldots \times R_n, \\
I_4 &= R_1 \times (0) \times R_3 \times (0) \times R_5 \times \ldots \times R_n.
\end{align*}

Now, it is not hard to see that $I_1 - I_2 - I_3 - I_4 - I_1$ is a cycle of length 4, a contradiction. Thus $|\text{Max}(R)| \leq 3$.

Conversely, suppose that $|\text{Max}(R)| \leq 3$. We show that $\overline{A_R}$ is chordal. To see this, we consider the following cases.

**Case 1.** $|\text{Max}(R)| = 1$. In this case, $R$ is local and thus by Theorem 2.1, $\overline{A_R}$ is a complete graph. Hence $\overline{A_R}$ is chordal.

**Case 2.** $|\text{Max}(R)| = 2$. By Theorem 2.2, $\overline{A_R} = K_{n_1} \lor (K_{n_2} + K_{n_3})$, where $n_1 = |A(\text{Nil}(R))^*|$, $n_2 = |A(m_1)^*| - n_1$, $n_3 = |A(m_2)^*| - n_1$ and $m_2, m_2 \in \text{Max}(R)$. Thus every cycle is a triangle, i.e, $\overline{A_R}$ is chordal.

**Case 3.** $|\text{Max}(R)| = 3$. In this case, $R \cong R_1 \times R_2 \times R_3$. Let $I_i$ be an ideal of $R_i$, for every $1 \leq i \leq 3$. Suppose that

\begin{align*}
A_1 &= \{ I_1 \times I_2 \times I_3 \mid I_i \subseteq \text{Nil}(R_i), \text{ for } i = 1, 2, 3 \} \setminus \{(0) \times (0) \times (0)\}, \\
A_2 &= \{ R_1 \times I_2 \times I_3 \mid I_i \subseteq \text{Nil}(R_i), \text{ for } i = 2, 3 \}, \\
A_3 &= \{ I_1 \times R_2 \times I_3 \mid I_i \subseteq \text{Nil}(R_i), \text{ for } i = 1, 3 \}, \\
A_4 &= \{ I_1 \times I_2 \times R_3 \mid I_i \subseteq \text{Nil}(R_i), \text{ for } i = 1, 2 \}, \\
B_1 &= \{ R_1 \times R_2 \times I_3 \mid I_3 \subseteq \text{Nil}(R_3) \}, \\
B_2 &= \{ R_1 \times I_2 \times R_3 \mid I_2 \subseteq \text{Nil}(R_2) \}, \\
B_3 &= \{ I_1 \times R_2 \times R_3 \mid I_1 \subseteq \text{Nil}(R_1) \}.
\end{align*}

Let $A = \bigcup_{i=1}^4 A_i$ and $B = \bigcup_{i=1}^3 B_i$. One may check that $A \cap B = \emptyset$ and $V(\overline{A_R}) = A \cup B$ and so $\{A, B\}$ is a partition of $V(\overline{A_R})$. We claim that $\overline{A_R}$ contains no induced cycle of length at least 4. Assume to the contrary, $a_1 - a_2 - \cdots - a_n - a_1$ is an induced cycle of length at least 4 in $\overline{A_R}$. We show that

\[ \{a_1, a_2, \ldots, a_n\} \cap B_1 = \emptyset. \]

Suppose to the contrary (and with no loss of generality), $a_1 \in B_1$. Thus $a_1 = R_1 \times R_2 \times I_3$, where $I_3 \subseteq \text{Nil}(R_3)$. Since $a_2$ and $a_n$ are adjacent to $a_1$, we conclude that the third components of $a_2$ and $a_n$ must be nilpotent ideals of $R_3$. This implies that $a_2$ and $a_n$ are adjacent, a contradiction. Hence,

\[ \{a_1, a_2, \ldots, a_n\} \cap B_1 = \emptyset. \]

Similarly,

\[ \{a_1, a_2, \ldots, a_n\} \cap B_2 = \{a_1, a_2, \ldots, a_n\} \cap B_3 = \emptyset. \]
This means that
\[ \{a_1, a_2, \ldots, a_n\} \subseteq A. \]
But this contradicts the fact that $\overline{A_R}[A]$ is a complete graph, and so $\overline{A_R}$ contains no induced cycle of length at least 4. Thus $\overline{A_R}$ is chordal.

\[ \square \]

3. When $A_R$ is Perfect?

In this section, we characterize all Artinian rings $R$ whose $A_R$ is Perfect. First, we need two celebrate results.

**Theorem 3.1** (The Strong Perfect Graph Theorem [8]). A graph $G$ is perfect if and only if neither $G$ nor $\overline{G}$ contains an induced odd cycle of length at least 5.

In light of Theorem 3.1, we have the following corollary.

**Corollary 3.1.** Let $G$ be a graph. Then the following statements hold.
1. $G$ is a perfect graph if and only if $\overline{G}$ is a perfect graph.
2. If $G$ is a complete bipartite graph, then $G$ is a perfect graph.

**Theorem 3.2.** [9] Every chordal graph is perfect.

**Lemma 3.1.** Let $n$ be a positive integer and $R \cong R_1 \times \cdots \times R_n$, where $R_i$ is an Artinian ring for every $1 \leq i \leq n$. Let $I = I_1 \times \cdots \times I_n, J = J_1 \times \cdots \times J_n$ be two distinct ideals of $R$ and $n \geq 2$. Then $I - J$ is an edge of $A_R$ if and only if for every $1 \leq i \leq n, I_i \not\in A(\text{Nil}(R_i))$ or $J_i \not\in A(\text{Nil}(R_i))$.

**Proof.** Let $I - J$ be an edge of $A_R$. If there exists $1 \leq i \leq n$ such that $I_i, J_i \in A(\text{Nil}(R_i))$, then by Lemma 2.2, $\text{Ann}(I_i) \cap \text{Ann}(J_i) \neq (0)$. So if $0 \neq a_i \in \text{Ann}(I_i) \cap \text{Ann}(J_i)$, then $(0) \times \cdots \times (0) \times R_i a_i \times (0) \times \cdots \times (0) \subseteq \text{Ann}(I) \cap \text{Ann}(J)$ and thus $I - J$ is not an edge of $A_R$, a contradiction.

Conversely, suppose that $I_i \not\in A(\text{Nil}(R_i))$ or $J_i \not\in A(\text{Nil}(R_i))$, for every $1 \leq i \leq n$. Thus $I_i = R_i$ or $J_i = R_i$, for every $1 \leq i \leq n$. This implies that $\text{Ann}(I) \cap \text{Ann}(J) = (0)$. Hence $I - J$ is an edge of $A_R$. \[ \square \]

We are now in a position to state our main result in this paper.

**Theorem 3.3.** Let $R$ be an Artinian rings. Then $\overline{A_R}$ is a perfect graph if and only if $|\text{Max}(R)| \leq 4$.

**Proof.** First suppose $\overline{A_R}$ is perfect. Since $R$ is an Artinian ring, there exists a positive integer $n = |\text{Max}(R)|$ such that $R \cong R_1 \times \cdots \times R_n$, where $R_i$ is an Artinian local ring, for every $1 \leq i \leq n$, by Lemma 2.1. If $n \geq 5$, then we put
\[
\begin{align*}
I_1 &= (0) \times R_2 \times R_3 \times (0) \times R_5 \times R_6 \times \cdots \times R_n, \\
I_2 &= (0) \times R_2 \times (0) \times R_4 \times R_5 \times R_6 \times \cdots \times R_n, \\
I_3 &= R_1 \times (0) \times R_4 \times R_5 \times R_6 \times \cdots \times R_n, \\
I_4 &= R_1 \times (0) \times R_3 \times R_4 \times (0) \times R_6 \times \cdots \times R_n,
\end{align*}
\]

Thus $\overline{A_R}$ is chordal.
\(I_5 = R_1 \times R_2 \times R_3 \times (0) \times R_6 \times \cdots \times R_n.\)

Then it is easily seen that

\[I_1 - I_2 - I_3 - I_4 - I_5 - I_1\]

is a cycle of length 5 in \(\overline{A_R}\), a contradiction (by Theorem 3.1). So \(n \leq 4\).

Conversely, suppose that \(|\text{Max}(R)| \leq 3\). We show that \(\overline{A_R}\) is a perfect graph. If \(|\text{Max}(R)| \leq 3\), then by part (2) of Theorem 2.3, \(\overline{A_R}\) is chordal and thus by Theorem 3.2, \(\overline{A_R}\) is a perfect graph. Therefore, we need only to check the case \(|\text{Max}(R)| = 4\).

Let \(R \cong R_1 \times R_2 \times R_3 \times R_4\). We have the following claims.

**Claim 1.** \(\overline{A_R}\) contains no induced odd cycle of length at least 5. We consider the following partition for \(V(\overline{A_R})\):

\[
A = \{I_1 \times I_2 \times I_3 \times I_4 \mid I_i \in A(R_i)\} \text{ for every } 1 \leq i \leq 4 \text{ and } I_4 \in A(\text{Nil}(R_4))
\]

\[
B = \{I_1 \times I_2 \times I_3 \times R_4 \mid I_i \in A(R_i)\} \text{ for every } 1 \leq i \leq 3 \text{ and } I_3 \in A(\text{Nil}(R_3))
\]

\[
C = \{I_1 \times I_2 \times R_3 \times R_4 \mid I_i \in A(R_i)\} \text{ for every } 1 \leq i \leq 2 \text{ and } I_2 \in A(\text{Nil}(R_2))
\]

\[
D = \{R_1 \times I_2 \times R_3 \times R_4, I_1 \times R_2 \times R_3 \times R_4 \mid \text{ for every } 1 \leq i \leq 2 \text{ } I_i \in A(\text{Nil}(R_i))\}
\]

Now, assume to the contrary, \(a_1 - a_2 - \cdots - a_n - a_1\) is an induced odd cycle of length at least 5 in \(\overline{A_R}\). We consider the following cases.

**Case 1.** \(\{a_1, a_2, \ldots, a_n\} \cap D = \emptyset\). Let \(a_i \in \{a_1, a_2, \ldots, a_n\} \cap D\), for some \(1 \leq i \leq n\). Then we can let \(a_i = I_1 \times R_2 \times R_3 \times R_4\) or \(a_i = R_1 \times I_2 \times R_3 \times R_4\). If \(a_i = I_1 \times R_2 \times R_3 \times R_4\), then the first components of \(a_{i-1}\) and \(a_{i+1}\) must be in \(A(\text{Nil}(R_i))\) and \(A(\text{Nil}(R_i))\), respectively. So by Lemma 3.1, \(a_{i-1}\) is adjacent to \(a_{i+1}\), a contradiction. Thus, \(a_i \neq I_1 \times R_2 \times R_3 \times R_4\). Similarly, \(a_i \neq R_1 \times I_2 \times R_3 \times R_4\). This means that \(\{a_1, a_2, \ldots, a_n\} \cap D = \emptyset\).

**Case 2.** \(\{a_1, a_2, \ldots, a_n\} \cap C = \emptyset\). First we show that \(|\{a_1, a_2, \ldots, a_n\} \cap C| \leq 1\). Let \(a, b \in \{a_1, a_2, \ldots, a_n\} \cap C\). Then we can easily check that if there exists \(x \in V(\overline{A_R})\) such that \(\text{Ann}(x) \cap \text{Ann}(a) \neq (0)\), then \(\text{Ann}(x) \cap \text{Ann}(b) \neq (0)\). This means that if \(x\) is adjacent to \(a\), then \(x\) is adjacent to \(b\), a contradiction. So \(\{a_1, a_2, \ldots, a_n\} \cap C| \leq 1\). This together with the fact that \(\overline{A_R}[A]\) and \(\overline{A_R}[B]\) are complete subgraphs, imply that \(n = 5\) and \(|\{a_1, a_2, \ldots, a_n\} \cap C| = 2\). Hence \(\{a_1, a_2, \ldots, a_n\} \cap C = 1\), and thus we can let \(a \in \{a_1, a_2, \ldots, a_n\} \cap C\). Since \(a\) is adjacent to all vertices of \(B \setminus \{R_1 \times R_2 \times I_3 \times R_4 \mid I_3 \subseteq \text{Nil}(R_3)\}\) and \(\overline{A_R}[B]\) is a complete subgraph, \(a_i \in \{a_1, a_2, \ldots, a_n\} \cap \{R_1 \times R_2 \times I_3 \times R_4 \mid I_3 \subseteq \text{Nil}(R_3)\}\), for some \(1 \leq i \leq n\). We can let \(a_i = R_1 \times R_2 \times I_3 \times R_4\). Since only one of the components of \(a_i\) is a nilpotent ideal of \(R_i\), by a similar argument to that of case 1, we get a contradiction. Hence, \(\{a_1, a_2, \ldots, a_n\} \cap C = \emptyset\).

By the above cases, \(\{a_1, a_2, \ldots, a_n\} \subseteq A \cup B\), but this contradicts the fact \(\overline{A_R}[A]\) and \(\overline{A_R}[B]\) are complete graphs, and thus \(\overline{A_R}\) contains no induced odd cycle of length at least 5.
Claim 2. \( A_R \) contains no induced odd cycle of length at least 5. We consider the following partition for \( V(A_R) \):

- \( A_1 = \{ I_1 \times R_2 \times R_3 \times R_4 \mid I_1 \in A(\text{Nil}(R_1)) \} \),
- \( A_2 = \{ R_1 \times I_2 \times R_3 \times R_4 \mid I_2 \in A(\text{Nil}(R_2)) \} \),
- \( A_3 = \{ R_1 \times R_2 \times I_3 \times R_4 \mid I_3 \in A(\text{Nil}(R_3)) \} \),
- \( A_4 = \{ R_1 \times R_2 \times R_3 \times I_4 \mid I_4 \in A(\text{Nil}(R_4)) \} \),
- \( B_1 = \{ I_1 \times I_2 \times R_3 \times R_4 \mid I_1 \in A(\text{Nil}(R_1)), I_2 \in A(\text{Nil}(R_2)) \} \),
- \( B_2 = \{ R_1 \times R_2 \times I_3 \times I_4 \mid I_3 \in A(\text{Nil}(R_3)), I_4 \in A(\text{Nil}(R_4)) \} \),
- \( B_3 = \{ I_1 \times R_2 \times I_3 \times R_4 \mid I_1 \in A(\text{Nil}(R_1)), I_3 \in A(\text{Nil}(R_3)) \} \),
- \( B_4 = \{ R_1 \times I_2 \times R_3 \times I_4 \mid I_2 \in A(\text{Nil}(R_2)), I_4 \in A(\text{Nil}(R_4)) \} \),
- \( B_5 = \{ I_1 \times I_2 \times R_3 \times I_4 \mid I_1 \in A(\text{Nil}(R_1)), I_4 \in A(\text{Nil}(R_4)) \} \),
- \( B_6 = \{ R_1 \times I_2 \times I_3 \times R_4 \mid I_2 \in A(\text{Nil}(R_2)), I_3 \in A(\text{Nil}(R_3)) \} \),
- \( C_1 = \{ R_1 \times I_2 \times I_3 \times I_4 \mid I_2 \in A(\text{Nil}(R_2)), I_3 \in A(\text{Nil}(R_3)), I_4 \in A(\text{Nil}(R_4)) \} \),
- \( C_2 = \{ I_1 \times R_2 \times I_3 \times I_4 \mid I_1 \in A(\text{Nil}(R_1)), I_3 \in A(\text{Nil}(R_3)), I_4 \in A(\text{Nil}(R_4)) \} \),
- \( C_3 = \{ I_1 \times I_2 \times R_3 \times I_4 \mid I_1 \in A(\text{Nil}(R_1)), I_2 \in A(\text{Nil}(R_2)), I_4 \in A(\text{Nil}(R_4)) \} \),
- \( C_4 = \{ I_1 \times I_2 \times I_3 \times R_4 \mid I_1 \in A(\text{Nil}(R_1)), I_2 \in A(\text{Nil}(R_2)), I_3 \in A(\text{Nil}(R_3)) \} \),
- \( D = \{ I_1 \times I_2 \times I_3 \times I_4 \mid I_1 \in A(\text{Nil}(R_1)), I_2 \in A(\text{Nil}(R_2)), I_3 \in A(\text{Nil}(R_3)), I_4 \in A(\text{Nil}(R_4)) \} \).

If we put \( A = \bigcup_{i=1}^{4} A_i \), \( B = \bigcup_{i=1}^{6} B_i \), and \( C = \bigcup_{i=1}^{4} C_i \), then one may check that \( \{ A, B, C, D \} \) is a partition of \( V(A_R) \). We show that \( A_R \) contains no induced odd cycle of length at least 5. Assume to the contrary, \( a_1 - a_2 - \cdots - a_n - a_1 \) is a induced odd cycle of length at least 5 in \( A_R \). By Lemma 2.2, every vertex in \( D \) is an isolated vertex in \( A_R \) and thus \( \{ a_1, a_2, \ldots, a_n \} \cap D = \emptyset \). Next, we show that

\[ \{ a_1, a_2, \ldots, a_n \} \cap C_1 = \emptyset. \]

To see this, if \( a_i \in \{ a_1, a_2, \ldots, a_n \} \cap C_1 \), for some \( 1 \leq i \leq n \), then with no loss of generality, assume that \( a_1 \in C_1 \). Since every vertex of \( C_1 \) is adjacent only to vertices of \( A_1, a_2, a_n \in A_1 \). This is impossible, as every vertex of \( A_R \) is adjacent to \( a_2 \) if and only if it is adjacent to \( a_n \). Therefore

\[ \{ a_1, a_2, \ldots, a_n \} \cap C_1 = \emptyset. \]

Similarly,

\[ \{ a_1, a_2, \ldots, a_n \} \cap C_2 = \{ a_1, a_2, \ldots, a_n \} \cap C_3 = \{ a_1, a_2, \ldots, a_n \} \cap C_4 = \emptyset. \]

Thus

\[ \{ a_1, a_2, \ldots, a_n \} \cap C = \emptyset. \]

Finally, we show that

\[ \{ a_1, a_2, \ldots, a_n \} \cap B_1 = \emptyset. \]
Assume to the contrary and with no loss of generality, \( a_1 \in B_1 \). As \( a_1 \) is adjacent only to vertices of \( B_2 \cup A_3 \cup A_4 \), \( \{a_2, a_n\} \subseteq B_2 \cup A_3 \cup A_4 \). If \( a_2 \in B_2 \), then \( a_3 \) is adjacent to \( a_n \) (since if \( a \) is adjacent to \( a_2 \) and \( b \) is adjacent to \( a_1 \), \( a \) is adjacent to \( b \)), a contradiction. Thus \( a_2 \notin B_2 \). Similarly, \( a_n \notin B_2 \) and so \( \{a_2, a_n\} \subseteq A_3 \cup A_4 \). Since \( A_R[A_3 \cup A_4] \) is a complete bipartite graph, we conclude that \( \{a_2, a_n\} \subseteq A_3 \) or \( \{a_2, a_n\} \subseteq A_4 \). With no loss of generality, we may assume that \( \{a_2, a_n\} \subseteq A_3 \). This implies that \( a_3 \) is adjacent to \( a_2 \) and \( a_n \) (since a vertex is adjacent to \( a_2 \) if and only if it is adjacent to \( a_n \)), a contradiction. Hence,
\[
\{a_1, a_2, \ldots, a_n\} \cap B_1 = \emptyset.
\]
Similarly, for every \( 2 \leq i \leq 6 \)
\[
\{a_1, a_2, \ldots, a_n\} \cap B_i = \emptyset.
\]
This means that
\[
\{a_1, a_2, \ldots, a_n\} \subseteq A.
\]
But \( A_R[A] \) is a complete 4-partite graph with parts \( A_i \) for \( 1 \leq i \leq 4 \), a contradiction. Therefore, \( A_R \) contains no induced odd cycle of length at least 5 and thus by Claim 1, Claim 2 and Theorem 3.1, we have \( A_R \) is a perfect graph. \( \square \)

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