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ON PERFECT CO-ANNIHILATING-IDEAL GRAPH OF A COMMUTATIVE ARTINIAN RING

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Abstract. Let R be a commutative ring with identity. The co-annihilating-ideal graph of R, denoted by $\mathcal{A}_R$, is a graph whose vertex set is the set of all non-zero proper ideals of R and two distinct vertices I and J are adjacent whenever $\operatorname{Ann}(I) \cap \operatorname{Ann}(J) = (0)$. In this paper, we characterize all Artinian rings for which both of the graphs $\mathcal{A}_R$ and $\overline{\mathcal{A}_R}$ (the complement of $\mathcal{A}_R$), are chordal. Moreover, all Artinian rings whose $\mathcal{A}_R$ (and thus $\overline{\mathcal{A}_R}$) is perfect are characterized.

1. Introduction

Assigning a graph to a ring gives us the ability to translate algebraic properties of rings into graph-theoretic language and vice versa. It leads to arising interesting algebraic and combinatorics problems. Therefore, the study of graphs associated with rings has attracted many researches. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory; for instance see [2, 3, 5, 6, 10, 11] and [12].

Throughout this paper, all rings are assumed to be commutative with identity. We denote by $Z(R)$, $\operatorname{Max}(R)$, $\operatorname{Nil}(R)$ and $J(R)$ the set of all zero-divisor elements of R, the set of all maximal ideals of R, the set of all nilpotent elements of R and jacobson radical of R, respectively. We call an ideal I of R, an *annihilating-ideal* if there exists $r \in R \setminus \{0\}$ such that $Ir = (0)$. The set of all annihilating-ideals of R is denote by $A(R)$. Let I be an ideal of R. We denote by $\mathcal{A}(I)$ the set of all ideals of R contained in I. The ring R is said to be *reduced* if it has no non-zero nilpotent element. For every ideal I of R, we denote the *annihilator* of I by $\operatorname{Ann}(I)$. We let $A^* = A \setminus \{0\}$. For any undefined notation or terminology in ring theory, we refer the reader to [4, 7].

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We use the standard terminology of graphs following [13]. Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, we mean the complement graph of G. We write $u - v$, to denote an edge with ends u, v. A graph $H = (V_0, E_0)$ is called a *subgraph of* G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph by* V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$. Also G is called a *null graph* if it has no edge. A complete graph of n vertices is denoted by K_n. An *n-part graph* is one whose vertex set can be partitioned into n subsets, so that no edge has both ends in any one subset. A *complete n-partite graph* is an n-part graph such that every pair of graph vertices in the n sets are adjacent. In a graph G, a vertex x is *isolated*, if no vertices of G is adjacent to x. Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. For a graph G, $S \subseteq V(G)$ is called a *clique* if the subgraph induced on S is complete. The number of vertices in the largest clique of graph G is called the *clique number* of G and is often denoted by $\omega(G)$. For a graph G, let $\chi(G)$ denote the *chromatic number* of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Clearly, for every graph G, $\omega(G) \leq \chi(G)$. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. A *perfect graph* G is a graph in which every induced subgraph is weakly perfect. A *chord* of a cycle C is an edge which is not in C but has both its endvertices in C. A graph G is *chordal* if every cycle of length at least 4 has a chord.

Let R be a commutative ring with identity. The co-annihilating-ideal graph of R, denoted by A_R, is a graph whose vertex set is the set of all non-zero proper ideals of R and two distinct vertices I and J are adjacent whenever $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) = (0)$. This graph was first introduced and studied in [1] and many interesting properties of this graph were explored by the authors. In [1, Theorem 17], it was proved A_R is a weakly perfect graph, if R is an Artinian ring. In this paper, we continue study the perfectness of A_R. Indeed, we characterize all Artinian rings for which both of the graphs A_R and $\overline{A_R}$, are chordal. Moreover, all Artinian rings whose A_R is perfect are given.

2. When A_R and $\overline{A_R}$ are Chordal?

In this section, we characterize all Artinian rings R, for which A_R and $\overline{A_R}$ are chordal. We begin with the following lemmas.

Lemma 2.1. *Let R be an Artinian ring. Then there exists a positive integer n such that $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$.*

Proof. See [4, Theorem 8.7]. □

Lemma 2.2. *Let R be an Artinian ring and I be a non-zero ideal of R. Then I is a nilpotent ideal of R if and only if I is an isolated vertex in A_R.*

Proof. Assume that I is a non-zero nilpotent ideal of R. First, we show that $\mathrm{Ann}(I)$ is an essential ideal of R. Suppose to the contrary, there exists an ideal J such that $J \cap \mathrm{Ann}(I) = (0)$. Thus $KI \neq (0)$, for every $K \subseteq J$. Obviously, $KI \subseteq J$ and so $(KI)I = KI^2 \neq (0)$. By continuing this procedure, $KI^n \neq 0$, for every positive integer n, a contradiction. Hence $\mathrm{Ann}(I)$ is an essential ideal of R and so $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) \neq (0)$, for every $J \in A(R)^*$. Therefore, I is an isolated vertex in A_R.

Conversely, suppose that I is an isolated vertex in A_R. If I is not a nilpotent ideal of R, then $I \nsubseteq J(R)$, i.e, there exists $\mathfrak{m} \in \mathrm{Max}(R)$ such that $I + \mathfrak{m} = R$, and so I is adjacent to $\mathfrak{m}$, a contradiction. Thus I is a nilpotent ideal of R. □

Next we need to study the structure of A_R, where R is an Artinian ring with at most two maximal ideals.

Theorem 2.1. *Let R be an Artinian ring. Then the following statements are equivalent:*

(1) $|\mathrm{Max}(R)| = 1$*;*

(2) $A_R = \overline{K_n}$*, where* $n = |A(R)^*|$.

Proof. (1) ⇒ (2) Since R is an Artinian local ring, every ideal of $A(R)^*$ is a nilpotent ideal of R and thus by Lemma 2.2, A_R is a null graph.

(2) ⇒ (1) is obtained by Lemma 2.2. □

Theorem 2.2. *Let R be an Artinian ring. Then the following statements are equivalent:*

(1) $|\mathrm{Max}(R)| = 2$*;*

(2) $A_R = \overline{K_{n_1}} + K_{n_2,n_3}$*, where* $n_1 = |A(\mathrm{Nil}(R))^*|$, $n_2 = |A(\mathfrak{m}_1)^*| - n_1$, $n_3 = |A(\mathfrak{m}_2)^*| - n_1$ *and* $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{Max}(R)$.

Proof. (1) ⇒ (2) Let $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. Since $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \mathrm{Nil}(R)$, Lemma 2.2 implies that $A_R[A(\mathrm{Nil}(R))^*]$ is a null graph. Let $A = \{I \in A(\mathfrak{m}_1) \setminus A(\mathrm{Nil}(R))\}$ and $B = \{I \in A(\mathfrak{m}_2) \setminus A(\mathrm{Nil}(R))\}$. If $I \in A$ and $J \in B$, then $I + J = R$, and thus I is adjacent to J. Moreover, $A_R[A]$ and $A_R[B]$ are null graphs. This means that $A_R[A \cup B] = K_{|A|,|B|}$. Since $A \cup B \cup A(\mathrm{Nil}(R))^* = A(R)^*$, we deduce that $A_R = \overline{K_{n_1}} + K_{n_2,n_3}$, where $n_1 = |A(\mathrm{Nil}(R))^*|$, $n_2 = |A(\mathfrak{m}_1)^*| - n_1$, $n_3 = |A(\mathfrak{m}_2)^*| - n_1$ and $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{Max}(R)$.

(2) ⇒ (1) By Theorem 2.1, $|\mathrm{Max}(R)| \geq 2$. If $|\mathrm{Max}(R)| \geq 3$, then A_R has a cycle of length 3, as $A_R[\mathrm{Max}(R)]$ is a complete graph, a contradiction. Thus $|\mathrm{Max}(R)| = 2$. □

We are now in a position to characterize all Artinian rings for which both of the graphs A_R and $\overline{A_R}$ are chordal.

Theorem 2.3. *Let R be an Artinian ring. Then*

(1) A_R *is chordal if and only if one of the following statements holds:*

 (i) R *is local;*

 (ii) $R \cong F \times S$*, where F is a field and S is local;*

(iii) $R \cong F_1 \times F_2 \times F_3$, *where* F_i *is a field for every* $1 \leq i \leq 3$;

(2) $\overline{A_R}$ *is chordal if and only if* $|\mathrm{Max}(R)| \leq 3$.

Proof. (1) Let A_R be chordal. First we show that $|\mathrm{Max}(R)| \leq 3$. If $|\mathrm{Max}(R)| \geq 4$, then Figure 1 is a cycle of length 4,

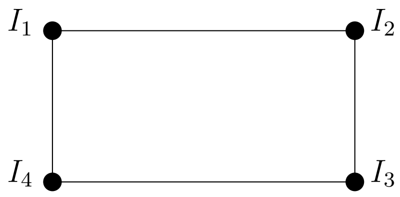

FIGURE 1. A cycle of length 4 in A_R

where

$$\begin{aligned}
I_1 &= (0) \times R_2 \times R_3 \times (0) \times R_5 \times \cdots \times R_n, \\
I_2 &= R_1 \times (0) \times (0) \times R_4 \times R_5 \times \cdots \times R_n, \\
I_3 &= R_1 \times R_2 \times R_3 \times (0) \times R_5 \times \cdots \times R_n, \\
I_4 &= R_1 \times (0) \times R_3 \times R_4 \times R_5 \times \cdots \times R_n.
\end{aligned}$$

Thus $|\mathrm{Max}(R)| \leq 3$. If $|\mathrm{Max}(R)| = 3$, then $R \cong R_1 \times R_2 \times R_3$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$. If R_1 is not field, then consider $I \in A(\mathrm{Nil}(R_1))^*$ and thus Figure 2 is a cycle of length 4,

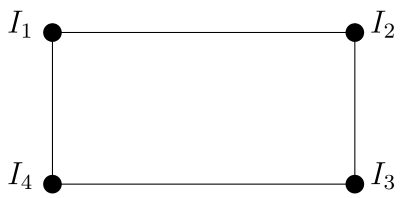

FIGURE 2. A cycle of length 4 in A_R

where

$$\begin{aligned}
I_1 &= R_1 \times (0) \times (0), \\
I_2 &= (0) \times R_2 \times R_3, \\
I_3 &= R_1 \times R_2 \times (0), \\
I_4 &= I \times R_2 \times R_3.
\end{aligned}$$

Hence R_1 is a field. Similarly, R_2 and R_3 are fields. Let $|\mathrm{Max}(R)| = 2$. Then $R \cong R_1 \times R_2$, where R_i is an Artinian local ring, for every $1 \leq i \leq 2$. We show that one of the rings R_1 and R_2 is a field. If I, J are non-zero proper ideals of R_1 and R_2, respectively, then Figure 3 is a cycle of length 4, where

$$\begin{aligned} I_1 &= I \times R_2, \\ I_2 &= R_1 \times J, \\ I_3 &= (0) \times R_2, \\ I_4 &= R_1 \times (0). \end{aligned}$$

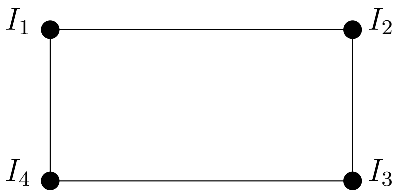

FIGURE 3. A cycle of length 4 in A_R

This means that one of the rings R_1 and R_2 is a field. Thus in this case $R \cong F \times S$, where F is a field and S is local. Clearly, if $|\mathrm{Max}(R)| = 1$, R is local.

Conversely, suppose that one of the conditions (i), (ii), (ii) is satisfied. Condition (i) implies that A_R is a null graph by Theorem 2.1, and thus A_R is chordal. If (ii) holds, then by Theorem 2.2, $A_R = \overline{K_n} + K_{1,n+1}$ where $n = |A(\mathrm{Nil}(R))^*|$. This implies that A_R is chordal. If (iii) holds, then Figure 4 shows that A_R is chordal where

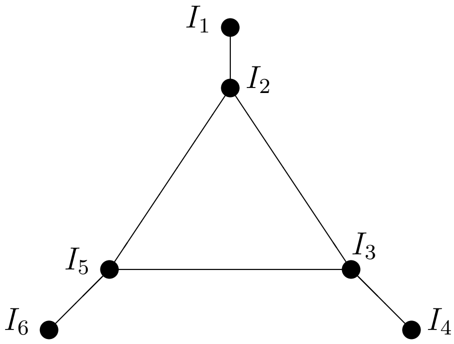

FIGURE 4. $A_{F_1 \times F_2 \times F_3}$

$$\begin{aligned} I_1 &= (0) \times (0) \times F_3, \\ I_2 &= F_1 \times F_2 \times (0), \\ I_3 &= F_1 \times (0) \times F_3, \\ I_4 &= (0) \times F_2 \times (0), \\ I_5 &= (0) \times F_2 \times F_3, \end{aligned}$$

$$I_6 = F_1 \times (0) \times (0).$$

(2) First suppose that $\overline{A_R}$ is chordal. If $|\mathrm{Max}(R)| \geq 4$, then we put

$$\begin{aligned} I_1 &= (0) \times R_2 \times R_3 \times (0) \times R_5 \times \cdots \times R_n, \\ I_2 &= (0) \times R_2 \times (0) \times R_4 \times R_5 \times \cdots \times R_n, \\ I_3 &= R_1 \times (0) \times (0) \times R_4 \times R_5 \times \cdots \times R_n, \\ I_4 &= R_1 \times (0) \times R_3 \times (0) \times R_5 \times \cdots \times R_n. \end{aligned}$$

Now, it is not hard to see that $I_1 - I_2 - I_3 - I_4 - I_1$ is a cycle of length 4, a contradiction. Thus $|\mathrm{Max}(R)| \leq 3$.

Conversely, suppose that $|\mathrm{Max}(R)| \leq 3$. We show that $\overline{A_R}$ is chordal. To see this, we consider the following cases.

Case 1. $|\mathrm{Max}(R)| = 1$. In this case, R is local and thus by Theorem 2.1, $\overline{A_R}$ is a complete graph. Hence $\overline{A_R}$ is chordal.

Case 2. $|\mathrm{Max}(R)| = 2$. By Theorem 2.2, $\overline{A_R} = K_{n_1} \bigvee (K_{n_2} + K_{n_3})$, where $n_1 = |A(\mathrm{Nil}(R))^*|$, $n_2 = |A(\mathfrak{m}_1)^*| - n_1$, $n_3 = |A(\mathfrak{m}_2)^*| - n_1$ and $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{Max}(R)$. Thus every cycle is a triangle, i.e, $\overline{A_R}$ is chordal.

Case 3. $|\mathrm{Max}(R)| = 3$. In this case, $R \cong R_1 \times R_2 \times R_3$. Let I_i be an ideal of R_i, for every $1 \leq i \leq 3$. Suppose that

$$\begin{aligned} A_1 &= \{I_1 \times I_2 \times I_3 \mid I_i \subseteq \mathrm{Nil}(R_i), \text{ for } i = 1, 2, 3\} \setminus \{(0) \times (0) \times (0)\}, \\ A_2 &= \{R_1 \times I_2 \times I_3 \mid I_i \subseteq \mathrm{Nil}(R_i), \text{ for } i = 2, 3\}, \\ A_3 &= \{I_1 \times R_2 \times I_3 \mid I_i \subseteq \mathrm{Nil}(R_i), \text{ for } i = 1, 3\}, \\ A_4 &= \{I_1 \times I_2 \times R_3 \mid I_i \subseteq \mathrm{Nil}(R_i), \text{ for } i = 1, 2\}, \\ B_1 &= \{R_1 \times R_2 \times I_3 \mid I_3 \subseteq \mathrm{Nil}(R_3)\}, \\ B_2 &= \{R_1 \times I_2 \times R_3 \mid I_2 \subseteq \mathrm{Nil}(R_2)\}, \\ B_3 &= \{I_1 \times R_2 \times R_3 \mid I_1 \subseteq \mathrm{Nil}(R_1)\}. \end{aligned}$$

Let $A = \cup_{i=1}^{4} A_i$ and $B = \cup_{i=1}^{3} B_i$. One may check that $A \cap B = \emptyset$ and $V(\overline{A_R}) = A \cup B$ and so $\{A, B\}$ is a partition of $V(\overline{A_R})$. We claim that $\overline{A_R}$ contains no induced cycle of length at least 4. Assume to the contrary, $a_1 - a_2 - \cdots - a_n - a_1$ is an induced cycle of length at least 4 in $\overline{A_R}$. We show that

$$\{a_1, a_2, \ldots, a_n\} \cap B_1 = \emptyset.$$

Suppose to the contrary (and with no loss of generality), $a_1 \in B_1$. Thus $a_1 = R_1 \times R_2 \times I_3$, where $I_3 \subseteq \mathrm{Nil}(R_3)$. Since a_2 and a_n are adjacent to a_1, we conclude that the third components of a_2 and a_n must be nilpotent ideals of R_3. This implies that a_2 and a_n are adjacent, a contradiction. Hence,

$$\{a_1, a_2, \ldots, a_n\} \cap B_1 = \emptyset.$$

Similarly,

$$\{a_1, a_2, \ldots, a_n\} \cap B_2 = \{a_1, a_2, \ldots, a_n\} \cap B_3 = \emptyset.$$

This means that

$$\{a_1, a_2, \ldots, a_n\} \subseteq A.$$

But this contradicts the fact that $\overline{A_R}[A]$ is a complete graph, and so $\overline{A_R}$ contains no induced cycle of length at least 4. Thus $\overline{A_R}$ is chordal. □

3. When A_R is Perfect?

In this section, we characterize all Artinian rings rings R whose A_R is Perfect. First, we need two celebrate results.

Theorem 3.1 (The Strong Perfect Graph Theorem [8]). *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length at least 5.*

In light of Theorem 3.1, we have the following corollary.

Corollary 3.1. *Let G be a graph. Then the following statements hold.*

(1) *G is a perfect graph if and only if $\overline{G}$ is a perfect graph.*

(2) *If G is a complete bipartite graph, then G is a perfect graph.*

Theorem 3.2. [9] *Every chordal graph is perfect.*

Lemma 3.1. *Let n be a positive integer and $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian ring for every $1 \leq i \leq n$. Let $I = I_1 \times \cdots \times I_n, J = J_1 \times \cdots \times J_n$ be two distinct ideals of R and $n \geq 2$. Then $I - J$ is an edge of A_R if and only if for every $1 \leq i \leq n$, $I_i \notin A(\mathrm{Nil}(R_i))$ or $J_i \notin A(\mathrm{Nil}(R_i))$.*

Proof. Let $I - J$ be an edge of A_R. If there exists $1 \leq i \leq n$ such that $I_i, J_i \in A(\mathrm{Nil}(R_i))$, then by Lemma 2.2, $\mathrm{Ann}(I_i) \cap \mathrm{Ann}(J_i) \neq (0)$. So if $0 \neq a_i \in \mathrm{Ann}(I_i) \cap \mathrm{Ann}(J_i)$, then $(0) \times \cdots \times (0) \times R_i a_i \times (0) \times \cdots \times (0) \subseteq \mathrm{Ann}(I) \cap \mathrm{Ann}(J)$ and thus $I - J$ is not an edge of A_R, a contradiction.

Conversely, suppose that $I_i \notin A(\mathrm{Nil}(R_i))$ or $J_i \notin A(\mathrm{Nil}(R_i))$, for every $1 \leq i \leq n$. Thus $I_i = R_i$ or $J_i = R_i$, for every $1 \leq i \leq n$. This implies that $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) = (0)$. Hence $I - J$ is an edge of A_R. □

We are now in a position to state our main result in this paper.

Theorem 3.3. *Let R be an Artinian rings. Then $\overline{A_R}$ is a perfect graph if and only if $|\mathrm{Max}(R)| \leq 4$.*

Proof. First suppose $\overline{A_R}$ is perfect. Since R is an Artinian ring, there exists a positive integer $n = |\mathrm{Max}(R)|$ such that $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$, by Lemma 2.1. If $n \geq 5$, then we put

$$\begin{aligned}
I_1 &= (0) \times R_2 \times R_3 \times (0) \times R_5 \times R_6 \times \cdots \times R_n, \\
I_2 &= (0) \times R_2 \times (0) \times R_4 \times R_5 \times R_6 \times \cdots \times R_n, \\
I_3 &= R_1 \times (0) \times (0) \times R_4 \times R_5 \times R_6 \times \cdots \times R_n, \\
I_4 &= R_1 \times (0) \times R_3 \times R_4 \times (0) \times R_6 \times \cdots \times R_n,
\end{aligned}$$

$$I_5 = R_1 \times R_2 \times R_3 \times (0) \times (0) \times R_6 \times \cdots \times R_n.$$

Then it is easily seen that

$$I_1 - I_2 - I_3 - I_4 - I_5 - I_1$$

is a cycle of length 5 in $\overline{A_R}$, a contradiction (by Theorem 3.1). So $n \leq 4$.

Conversely, suppose that $|\mathrm{Max}(R)| \leq 4$. We show that $\overline{A_R}$ is a perfect graph. If $|\mathrm{Max}(R)| \leq 3$, then by part (2) of Theorem 2.3, $\overline{A_R}$ is chordal and thus by Theorem 3.2, $\overline{A_R}$ is a perfect graph. Therefore, we need only to check the case $|\mathrm{Max}(R)| = 4$. Let $R \cong R_1 \times R_2 \times R_3 \times R_4$. We have the following claims.

Claim 1. $\overline{A_R}$ contains no induced odd cycle of length at least 5. We consider the following partition for $V(\overline{A_R})$:

$$\begin{aligned}
A &= \{I_1 \times I_2 \times I_3 \times I_4 \mid I_i \in A(R_i) \text{ for every } 1 \leq i \leq 4 \text{ and } I_4 \in A(\mathrm{Nil}(R_4))\},\\
B &= \{I_1 \times I_2 \times I_3 \times R_4 \mid I_i \in A(R_i) \text{ for every } 1 \leq i \leq 3 \text{ and } I_3 \in A(\mathrm{Nil}(R_3))\},\\
C &= \{I_1 \times I_2 \times R_3 \times R_4 \mid I_i \in A(R_i) \text{ for every } 1 \leq i \leq 2 \text{ and } I_2 \in A(\mathrm{Nil}(R_2))\},\\
D &= \{R_1 \times I_2 \times R_3 \times R_4, I_1 \times R_2 \times R_3 \times R_4 \mid \text{for every } 1 \leq i \leq 2 \; I_i \in A(\mathrm{Nil}(R_i))\}.
\end{aligned}$$

Now, assume to the contrary, $a_1 - a_2 - \cdots - a_n - a_1$ is an induced odd cycle of length at least 5 in $\overline{A_R}$. We consider the following cases.

Case 1. $\{a_1, a_2, \ldots, a_n\} \cap D = \emptyset$. Let $a_i \in \{a_1, a_2, \ldots, a_n\} \cap D$, for some $1 \leq i \leq n$. Then we can let $a_i = I_1 \times R_2 \times R_3 \times R_4$ or $a_i = R_1 \times I_2 \times R_3 \times R_4$. If $a_i = I_1 \times R_2 \times R_3 \times R_4$, then the first components of a_{i-1} and a_{i+1} must be in $A(\mathrm{Nil}(R_i))$ and $A(\mathrm{Nil}(R_i))$, respectively. So by Lemma 3.1, a_{i-1} is adjacent to a_{i+1}, a contradiction. Thus, $a_i \neq I_1 \times R_2 \times R_3 \times R_4$. Similarly, $a_i \neq R_1 \times I_2 \times R_3 \times R_4$. This means that $\{a_1, a_2, \ldots, a_n\} \cap D = \emptyset$.

Case 2. $\{a_1, a_2, \ldots, a_n\} \cap C = \emptyset$. First we show that $|\{a_1, a_2, \ldots, a_n\} \cap C| \leq 1$. Let $a, b \in \{a_1, a_2, \ldots, a_n\} \cap C$. Then we can easily check that if there exits $x \in V(\overline{A_R})$ such that $\mathrm{Ann}(x) \cap \mathrm{Ann}(a) \neq (0)$, then $\mathrm{Ann}(x) \cap \mathrm{Ann}(b) \neq (0)$. This means that if x is adjacent to a, then x is adjacent to b, a contradiction. So $|\{a_1, a_2, \ldots, a_n\} \cap C| \leq 1$. This together with the fact that $\overline{A_R}[A]$ and $\overline{A_R}[B]$ are complete subgraphs, imply that $n = 5$ and $|\{a_1, a_2, \ldots, a_n\} \cap B| = |\{a_1, a_2, \ldots, a_n\} \cap A| = 2$. Hence $|\{a_1, a_2, \ldots, a_n\} \cap C| = 1$, and thus we can let $a \in \{a_1, a_2, \ldots, a_n\} \cap C$. Since a is adjacent to all vertices of $B \setminus \{R_1 \times R_2 \times I_3 \times R_4 \mid I_3 \subseteq \mathrm{Nil}(R_3)\}$ and $\overline{A_R}[B]$ is a complete subgraph, $a_i \in \{a_1, a_2, \ldots, a_n\} \cap \{R_1 \times R_2 \times I_3 \times R_4 \mid I_3 \subseteq \mathrm{Nil}(R_3)\}$, for some $1 \leq i \leq n$. We can let $a_i = R_1 \times R_2 \times I_3 \times R_4$. Since only one of the components of a_i is a nilpotent ideal of R_i, by a similar argument to that of case 1, we get a contradiction. Hence, $\{a_1, a_2, \ldots, a_n\} \cap C = \emptyset$.

By the above cases, $\{a_1, a_2, \ldots, a_n\} \subseteq A \cup B$, but this contradicts the fact $\overline{A_R}[A]$ and $\overline{A_R}[B]$ are complete graphs, and thus $\overline{A_R}$ contains no induced odd cycle of length at least 5.

Claim 2. A_R contains no induced odd cycle of length at least 5. We consider the following partition for $V(A_R)$:

$$
\begin{aligned}
A_1 =&\{I_1 \times R_2 \times R_3 \times R_4 \mid I_1 \in A(\mathrm{Nil}(R_1))\},\\
A_2 =&\{R_1 \times I_2 \times R_3 \times R_4 \mid I_2 \in A(\mathrm{Nil}(R_2))\},\\
A_3 =&\{R_1 \times R_2 \times I_3 \times R_4 \mid I_3 \in A(\mathrm{Nil}(R_3))\},\\
A_4 =&\{R_1 \times R_2 \times R_3 \times I_4 \mid I_4 \in A(\mathrm{Nil}(R_4))\},\\
B_1 =&\{I_1 \times I_2 \times R_3 \times R_4 \mid I_1 \in A(\mathrm{Nil}(R_1)), I_2 \in A(\mathrm{Nil}(R_2))\},\\
B_2 =&\{R_1 \times R_2 \times I_3 \times I_4 \mid I_3 \in A(\mathrm{Nil}(R_3)), I_4 \in A(\mathrm{Nil}(R_4))\},\\
B_3 =&\{I_1 \times R_2 \times I_3 \times R_4 \mid I_1 \in A(\mathrm{Nil}(R_1)), I_3 \in A(\mathrm{Nil}(R_3))\},\\
B_4 =&\{R_1 \times I_2 \times R_3 \times I_4 \mid I_2 \in A(\mathrm{Nil}(R_2)), I_4 \in A(\mathrm{Nil}(R_4))\},\\
B_5 =&\{I_1 \times R_2 \times R_3 \times I_4 \mid I_1 \in A(\mathrm{Nil}(R_1)), I_4 \in A(\mathrm{Nil}(R_4))\},\\
B_6 =&\{R_1 \times I_2 \times I_3 \times R_4 \mid I_2 \in A(\mathrm{Nil}(R_2)), I_3 \in A(\mathrm{Nil}(R_3))\},\\
C_1 =&\{R_1 \times I_2 \times I_3 \times I_4 \mid I_2 \in A(\mathrm{Nil}(R_2)), I_3 \in A(\mathrm{Nil}(R_3)), I_4 \in A(\mathrm{Nil}(R_4))\},\\
C_2 =&\{I_1 \times R_2 \times I_3 \times I_4 \mid I_1 \in A(\mathrm{Nil}(R_1)), I_3 \in A(\mathrm{Nil}(R_3)), I_4 \in A(\mathrm{Nil}(R_4))\},\\
C_3 =&\{I_1 \times I_2 \times R_3 \times I_4 \mid I_1 \in A(\mathrm{Nil}(R_1)), I_2 \in A(\mathrm{Nil}(R_2)), I_4 \in A(\mathrm{Nil}(R_4))\},\\
C_4 =&\{I_1 \times I_2 \times I_3 \times R_4 \mid I_1 \in A(\mathrm{Nil}(R_1)), I_2 \in A(\mathrm{Nil}(R_2)), I_3 \in A(\mathrm{Nil}(R_3))\},\\
D =&\{I_1 \times I_2 \times I_3 \times I_4 \mid I_1 \in A(\mathrm{Nil}(R_1)), I_2 \in A(\mathrm{Nil}(R_2)), I_3 \in A(\mathrm{Nil}(R_3)),\\
&I_4 \in A(\mathrm{Nil}(R_4))\}.
\end{aligned}
$$

If we put $A = \cup_{i=1}^4 A_i$, $B = \cup_{i=1}^6 B_i$ and $C = \cup_{i=1}^4 C_i$, then one may check that $\{A, B, C, D\}$ is a partition of $V(A_R)$. We show that A_R contains no induced odd cycle of length at least 5. Assume to the contrary, $a_1 - a_2 - \cdots - a_n - a_1$ is a induced odd cycle of length at least 5 in A_R. By Lemma 2.2, every vertex in D is an isolated vertex in A_R and thus $\{a_1, a_2, \ldots, a_n\} \cap D = \emptyset$. Next, we show that

$$\{a_1, a_2, \ldots, a_n\} \cap C_1 = \emptyset.$$

To see this, if $a_i \in \{a_1, a_2, \ldots, a_n\} \cap C_1$, for some $1 \leq i \leq n$, then with no loss of generality, assume that $a_1 \in C_1$. Since every vertex of C_1 is adjacent only to vertices of A_1, $a_2, a_n \in A_1$. This is impossible, as every vertex of A_R is adjacent to a_2 if and only if it is adjacent to a_n. Therefore

$$\{a_1, a_2, \ldots, a_n\} \cap C_1 = \emptyset.$$

Similarly,

$$\{a_1, a_2, \ldots, a_n\} \cap C_2 = \{a_1, a_2, \ldots, a_n\} \cap C_3 = \{a_1, a_2, \ldots, a_n\} \cap C_4 = \emptyset.$$

Thus

$$\{a_1, a_2, \ldots, a_n\} \cap C = \emptyset.$$

Finally, we show that

$$\{a_1, a_2, \ldots, a_n\} \cap B_1 = \emptyset.$$

Assume to the contrary and with no loss of generality, $a_1 \in B_1$. As a_1 is adjacent only to vertices of $B_2 \cup A_3 \cup A_4$, $\{a_2, a_n\} \subseteq B_2 \cup A_3 \cup A_4$. If $a_2 \in B_2$, then a_3 is adjacent to a_n (since if a is adjacent to a_2 and b is adjacent to a_1, a is adjacent to b), a contradiction. Thus $a_2 \notin B_2$. Similarly, $a_n \notin B_2$ and so $\{a_2, a_n\} \subseteq A_3 \cup A_4$. Since $A_R[A_3 \cup A_4]$ is a complete bipartite graph, we conclude that $\{a_2, a_n\} \subseteq A_3$ or $\{a_2, a_n\} \subseteq A_4$. With no loss of generality, we may assume that $\{a_2, a_n\} \subseteq A_3$. This implies that a_3 is adjacent to a_2 and a_n (since a vertex is adjacent to a_2 if and only if it is adjacent to a_n), a contradiction. Hence,

$$\{a_1, a_2, \ldots, a_n\} \cap B_1 = \emptyset.$$

Similarly, for every $2 \leq i \leq 6$

$$\{a_1, a_2, \ldots, a_n\} \cap B_i = \emptyset.$$

This means that

$$\{a_1, a_2, \ldots, a_n\} \subseteq A.$$

But $A_R[A]$ is a complete 4-partite graph with parts A_i for $1 \leq i \leq 4$, a contradiction. Therefore, A_R contains no induced odd cycle of length at least 5 and thus by Claim 1, Claim 2 and Theorem 3.1, we have A_R is a perfect graph. $\square$

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