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PSEUDO-BCK ALGEBRAS DERIVED FROM DIRECTOIDS

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Abstract. The aim of this paper is to derive pseudo-BCK algebras from directoids and vice versa. We generalize some results proved by Ivan Chajda et al. in the case of BCK-algebras. We assign to an arbitrary pseudo-BCK algebra a semilattice-like structure and observe that this is the point where directoids are different from the semilattice-like structures. Finally, the relation between commutative deductive systems and derive directoids from a bounded pseudo-BCK(pDN) algebras and a characterization of commutative deductive systems of a bounded pseudo-BCK(pDN) algebra in terms of directoids is discussed.

1. Introduction

BCK-algebras were introduced by Y. Imai and K. Iséki in 1966 ([15, 19]) as algebras with a binary operation $*$ modeling the set-theoretical difference and with a constant element 0 that is a least element. S. Tanaka defined a special class of BCK-algebras called commutative BCK-algebras in 1975 (see [31]). In BCK-algebras, some lattices, as bounded commutative BCK-algebras, involutive BCK-lattices and bounded implicative BCK-algebras were defined and among the relationship between them were discussed [23]. Some recent researchers led to generalizations of the notion of pseudo structure on some types of algebras. G. Georgescu et al. [10] and independently J. Rachůnek [24], introduced pseudo-MV algebra which is a non-commutative generalization of MV-algebra. After a pseudo-MV algebra, the pseudo-BL algebra [11], the pseudo-BCK algebra [12] and as a generalization of BCI-algebra, the notion of pseudo-BCI algebra is introduced by W. A. Dudek et al. in [9]. A. Walendziak [32] introduced pseudo-BCH algebras as an extension of BCH-algebras. Further, he

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proved that every branchwise commutative pseudo-BCH algebra is a pseudo-BCI algebra [33]. Commutative pseudo-BCK algebras were originally defined by G. Georgescu et al. in [12] under the name of *semilattice-ordered pseudo-BCK algebras* and some properties of these structures were investigated by J. Kühr in [21, 22]. R. A. Borzooei et al. introduced in [1] (see also [2, 26, 27]) a pseudo-BE algebra as generalization of BE-algebra, and the commutative pseudo-BE algebra have recently been investigated by L. C. Ciungu. It was proved that the class of commutative pseudo-BE algebras is equivalent to the class of commutative pseudo-BCK algebras. Based on this result, all results holding for commutative pseudo-BCK algebras also hold for commutative pseudo-BE algebras [5]. Then she gave a characterization of commutative pseudo-BCK algebras and defined the commutative deductive systems of pseudo-BCK algebras and proved that a pseudo-BCK algebra $\mathfrak{X}$ is commutative if and only if all the deductive systems of $\mathfrak{X}$ are commutative. Also, she showed that the class of commutative pseudo-BCK algebras is a variety [6] (see also, [14]). A. Rezaei et al. introduced the notion of pseudo-CI algebras as an extension of pseudo-BE algebras and proved that the class of commutative pseudo-CI algebras coincide with the class of commutative pseudo-BCK algebras [28]. G. Georgescu et al. proved that every Wajsberg pseudo-hoop is a basic pseudo-hoop and every simple basic pseudo-hoop is a linearly ordered Wajsberg pseudo-hoop [13]. L. C. Ciungu in [7] showed that every pseudo-hoop is a pseudo-BCK-meet semilattice. The relation between FL_w-algebras, bounded pseudo-BCK(pP) algebras, pseudo-MTL algebras, pseudo-BL algebras and pseudo-MV algebras proved in [16]. Also, in [29, 30], the interrelationships between dual pseudo-Q/QC algebras and other pseudo algebras are visualized with a diagram and then they introduced the concepts of branchwise commutative pseudo-CI algebras and pointed pseudo-CI algebras and investigated some of properties. A. Iorgulescu for the first time introduced the notation of quasi-pseudo-M algebras as generalizations of pseudo-M algebras and (involutive) quasi-implicative-groups and the (strong involutive) (super) quasi-implicative-hoops, as generalizations of implicative-groups and implicative-hoops, respectively in [18]. I. Chajda et al. showed that one can be assign to an arbitrary BCK-algebra a semilattice-like structure every section of which possesses a certain antitone mappings [3], it arises a natural question of generalization of these concepts also for pseudo-BCK algebras. Since lattice theory has many applications in computer science and has an important and vital role in investigating the structure of a logical system, this motivated our investigations on directoids and pseudo-BCK-algebras to characterized several of its important properties. The main result of this paper establishes a bijective correspondence between pseudo-BCK algebras and some algebraic structures defined by two directoids. A characterization of commutative deductive systems of a bounded pseudo-BCK(pDN) algebra in terms of directoids is discussed and various results obtained mentioned in this paper can be transferred to the pseudo-BCK algebras. The core of the paper is based on by presenting a survey of some results of logic in the non-commutative case (see [3] for the commutative case) and extension of [25] (see also [4]).

2. Preliminaries

In this section we recall some basic notions and results regarding (commutative) pseudo-BCK algebras.

Definition 2.1 ([9, 17]). An algebra $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called a *pseudo-BCI algebra if it satisfies the following axioms for all $x, y, z \in X$:*

(psBCI$_1$) $(x \to y) \rightsquigarrow ((y \to z) \rightsquigarrow (x \to z)) = 1$ and
$(x \rightsquigarrow y) \to ((y \rightsquigarrow z) \to (x \rightsquigarrow z)) = 1$;

(psBCI$_2$) $x \to ((x \to y) \rightsquigarrow y) = 1$ and $x \rightsquigarrow ((x \rightsquigarrow y) \to y) = 1$;

(psBCI$_3$) $x \to x = x \rightsquigarrow x = 1$;

(psBCI$_4$) $x \to y = y \rightsquigarrow x = 1 \Rightarrow x = y$;

(psBCI$_5$) $x \preceq y$ if and only if $x \to y = 1$ if and only if $x \rightsquigarrow y = 1$.

A *pseudo-BCK algebra* [20] is a pseudo-BCI algebra $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ satisfying the condition (psBCK), for all $x \in X$:

(psBCK) $x \to 1 = 1$.

I. Chajda et al. proved that for every pseudo-BCI algebra $x \to y = 1$ if and only if $x \rightsquigarrow y = 1$ (see [4, Lemma 2.1]).

Remark 2.1. If $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ is a pseudo-BCI algebra satisfying $x \to y = x \rightsquigarrow y$, for all $x, y \in X$, then $\mathfrak{X} = (X; \to, 1)$ is a BCI-algebra. Hence, every BCI-algebra is a pseudo-BCI algebra in a natural way.

Remark 2.2. By definition (psBCI$_1$)-(psBCI$_5$), pseudo-BCK algebras are contained in the class of pseudo-BCI algebras. A pseudo-BCI algebra which is not a pseudo-BCK algebra will be called *proper*.

From now on, $\mathfrak{X}$ is a pseudo-BCK algebra, unless it is stated.

Proposition 2.1 ([12, 17]). *In any pseudo-BCK algebra $\mathfrak{X}$ the following conditions hold for all $x, y, z \in X$:*

(1) $x \preceq y$ *implies* $z \to x \preceq z \to y$ *and* $z \rightsquigarrow x \preceq z \rightsquigarrow y$;
(2) $x \preceq y$ *implies* $y \to z \preceq x \to z$ *and* $y \rightsquigarrow z \preceq x \rightsquigarrow z$;
(3) $x \to y \preceq (z \to x) \rightsquigarrow (z \to y)$ *and* $x \rightsquigarrow y \preceq (z \rightsquigarrow x) \to (z \rightsquigarrow y)$;
(4) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$ *and* $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$;
(5) $x \preceq y \to x$ *and* $x \preceq y \rightsquigarrow x$;
(6) $((x \to y) \rightsquigarrow y) \to y = x \to y$ *and* $((x \rightsquigarrow y) \to y) \rightsquigarrow y = x \rightsquigarrow y$.

Theorem 2.1 ([6]). *Let $\mathfrak{X}$ be a pseudo-BCK algebra. The following statements are equivalent for all $x, y \in X$:*

(1) $\mathfrak{X}$ *is commutative;*
(2) $x \to y = ((y \to x) \rightsquigarrow x) \to y$ *and* $x \rightsquigarrow y = ((y \rightsquigarrow x) \to x) \rightsquigarrow y$;
(3) $(x \to y) \rightsquigarrow y = (((x \to y) \rightsquigarrow y) \to x) \rightsquigarrow x$ *and*
$(x \rightsquigarrow y) \to y = (((x \rightsquigarrow y) \to y) \rightsquigarrow x) \to x$;
(4) $x \preceq y$ *implies* $y = (y \to x) \rightsquigarrow x = (y \rightsquigarrow x) \to x$.

Definition 2.2 ([16]). If there is an element 0 of a pseudo-BCK algebra $\mathfrak{X}$, such that $0 \preceq x$ (i.e., $0 \to x = 0 \rightsquigarrow x = 1$), for all $x \in X$, then 0 is called the *zero* of $\mathfrak{X}$. A pseudo-BCK algebra with zero is called *bounded pseudo-BCK algebra* and it is denoted by $\mathfrak{X} = (X; \to, \rightsquigarrow, 0, 1)$.

Definition 2.3 ([16]). A *pseudo-BCK(pP) algebra* is a pseudo-BCK algebra $\mathfrak{X}$ satisfying (pP) condition:

(pP) There exists $x \odot y = \min\{z : x \preceq y \to z\} = \min\{z : y \preceq x \rightsquigarrow z\}$ for all $x, y \in X$.

Definition 2.4 ([16, 20]).

(1) A *pseudo-BCK lattice* is a pseudo-BCK algebra $\mathfrak{X}$ such that $(X; \preceq)$ is a lattice.

(2) A *pseudo-BCK join-semilattice* is a pseudo-BCK algebra $\mathfrak{X}$ such that $(X; \vee)$ is a *join-semilattice*, and $x \to y = 1$ if and only if $x \vee y = y$.

(3) A *pseudo-BCK meet-semilattice* is a pseudo-BCK algebra $\mathfrak{X}$ such that $(X; \wedge)$ is a *meet-semilattice*, and $x \to y = 1$ if and only if $x \wedge y = x$.

Definition 2.5 ([16]). A *pseudo-BCK algebra(pDN)* is a bounded pseudo-BCK algebra $\mathfrak{X} = (X; \preceq, \to, \rightsquigarrow, 0, 1)$ satisfying the condition:

(pDN) $(x^{\to})^{\rightsquigarrow} = (x^{\rightsquigarrow})^{\to} = x$, where $x^{\to} = x \to 0$ and $x^{\rightsquigarrow} = x \rightsquigarrow 0$ for all $x \in X$.

Definition 2.6 ([12]). A pseudo-BCK algebra $\mathfrak{X}$ is called *commutative* if for all $x, y, z \in X$, it satisfies the following identities:

(C_1) $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x$;

(C_2) $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x$.

Proposition 2.2 ([6]). *Any commutative pseudo-BCK algebra is a join-semilattice with respect to* $\preceq$.

Theorem 2.2 ([8]). *Let $\mathfrak{X}$ be a pseudo-BCK(pDN) algebra. The following statements are equivalent:*

(1) $(X; \preceq)$ *is a meet-semilattice;*

(2) $(X; \preceq)$ *is a join-semilattice;*

(3) $(X; \preceq)$ *is a lattice.*

Definition 2.7 ([6]). A subset D of a pseudo-BCK algebra $\mathfrak{X}$ is called a *deductive system* of $\mathfrak{X}$ if it satisfies the following conditions:

(DS_1) $1 \in D$;

(DS_2) $x \in D$ and $x \to y \in D$ imply $y \in D$.

A subset D of $\mathfrak{X}$ is a deductive system if and only if it satisfies (DS_1) and the condition:

(DS_3) $x \in D$ and $x \rightsquigarrow y \in D$ imply $y \in D$.

We will denote by $\mathfrak{D}S(X)$ the set of all deductive systems of $\mathfrak{X}$.

Definition 2.8 ([6]). A deductive system D of a pseudo-BCK algebra $\mathfrak{X}$ is called *commutative* if it satisfies the following conditions for all $x, y \in X$:

(CDS_1) $y \to x \in D$ implies $((x \to y) \rightsquigarrow y) \to x \in D$;
(CDS_2) $y \rightsquigarrow x \in D$ implies $((x \rightsquigarrow y) \to y) \rightsquigarrow x \in D$.

We will denote by $\mathfrak{D}S_c(X)$ the set of all commutative deductive systems of a pseudo-BCK algebra $\mathfrak{X}$.

Definition 2.9 ([3])**.** A *directoid* is a groupoid $\mathfrak{G} = (G; \vee)$ satisfying the following identities for all $x, y, z \in G$:

(D_1) $x \vee x = x$;
(D_2) $(x \vee y) \vee x = x \vee y$;
(D_3) $y \vee (x \vee y) = x \vee y$;
(D_4) $x \vee ((x \vee y) \vee z) = (x \vee y) \vee z$.

The relation $\leq$ given by $x \leq y$ if and only if $x \vee y = y$ is a partial order. The binary operation $\vee$ assigns to a pair $\{x, y\}$ is a common upper bound of them.

3. Pseudo-BCK Algebras Derived from Directoids

Following the idea used by I. Chajda and J. Kühr [3] for BCK-algebras in what follows we give a generalization of this results for pseudo-BCK algebras. In this section, we assign a semilattice-like structure the sections of which have certain antitone mappings, and also conversely. We have the following results.

Let $\mathfrak{X}$ be a pseudo-BCK algebra. Define binary operations $\vee_1$ and $\vee_2$ by:

(A) $x \vee_1 y := (x \to y) \rightsquigarrow y$ and $x \vee_2 y := (x \rightsquigarrow y) \to y$ for all $x, y \in X$.

The following examples shows that these operations $\vee_1$ and $\vee_2$ need not coincide in general.

Example 3.1 ([8]). Consider the set $X = \{0, a, b, c, 1\}$, where $0 < a, b < c < 1$, a, b inomparable and the operations $\to$ and $\rightsquigarrow$ given by the following tables:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

,

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	0	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

.

Then $\mathfrak{X} = (X; \to, \rightsquigarrow, 0, 1)$ is a bounded pseudo-BCK algebra, but

$$a \vee_1 0 = (a \to 0) \rightsquigarrow 0 = 0 \rightsquigarrow 0 = 1 \neq a \vee_2 0 = (a \rightsquigarrow 0) \to 0 = b \to 0 = a.$$

Theorem 3.1. *Let $\mathfrak{X}$ be a pseudo-BCK algebra. For every $a \in X$, define unary operations ${}^{\to a}$ and ${}^{\rightsquigarrow a}$ by $x^{\to a} = x \to a$ and $x^{\rightsquigarrow a} = x \rightsquigarrow a$. Then the algebraic structure $\mathfrak{S}(\mathfrak{X}) = (X; \vee_1, \vee_2, ({}^{\to a})_{a \in X}, ({}^{\rightsquigarrow a})_{a \in X}, 1)$ satisfies the following quasi-identities:*

(1) $x \vee_1 1 = 1$ *and* $x \vee_2 1 = 1$;
(2) $x^{\to x} = 1$ *and* $x^{\rightsquigarrow x} = 1$;
(3) $1^{\to x} = x$ *and* $1^{\rightsquigarrow x} = x$;
(4) $x \vee_1 x = x$ *and* $x \vee_2 x = x$;

(5) $(x \vee_1 y)^{\to y} = x \to y$ *and* $(x \vee_2 y)^{\rightsquigarrow y} = x \rightsquigarrow y$*;*
(6) $x \preceq y$ *if and only if* $x \vee_1 y = y$ *and* $x \vee_2 y = y$*;*
(7) $x \vee_1 y = y$ *and* $y \vee_1 x = x$ *imply* $x = y$ *and*
$x \vee_2 y = y$ *and* $y \vee_2 x = x$ *imply* $x = y$*;*
(8) $x \vee_1 y = (x \vee_1 y) \vee_1 y = x \vee_1 (x \vee_1 y) = y \vee_1 (x \vee_1 y)$ *and*
$x \vee_2 y = (x \vee_2 y) \vee_2 y = x \vee_2 (x \vee_2 y) = y \vee_2 (x \vee_2 y)$*;*
(9) $(x \vee_1 z) \vee_1 ((x \vee_1 y) \vee_1 z) = (x \vee_1 y) \vee_1 z$ *and*
$(x \vee_2 z) \vee_2 ((x \vee_2 y) \vee_2 z) = (x \vee_2 y) \vee_2 z$*;*
(10) $x \vee_1 y = (x \vee_1 y)^{\to y \rightsquigarrow y} = ((x \vee_1 y)^{\to y} \vee_2 y)^{\rightsquigarrow y}$ *and*
$x \vee_2 y = (x \vee_2 y)^{\rightsquigarrow y \to y} = ((x \vee_2 y)^{\rightsquigarrow y} \vee_1 y)^{\to y}$*;*
(11) $(x \vee_1 (y \vee_2 z)^{\rightsquigarrow z})^{\to (y \vee_2 z)^{\rightsquigarrow z}} = (y \vee_2 (x \vee_1 z)^{\to z})^{\rightsquigarrow (x \vee_1 z)^{\to z}}$*;*
(12) $(x \vee_1 y)^{\to y} \vee_2 ((x \vee_1 z) \vee_1 (y \vee_1 z))^{\to (y \vee_1 z)} = ((x \vee_1 z) \vee_1 (y \vee_1 z))^{\to (y \vee_1 z)}$ *and*
$(x \vee_2 y)^{\rightsquigarrow y} \vee_1 ((x \vee_2 z) \vee_2 (y \vee_2 z))^{\rightsquigarrow (y \vee_2 z)} = ((x \vee_2 z) \vee_2 (y \vee_2 z))^{\rightsquigarrow (y \vee_2 z)}$*;*
(13) $((x \vee_2 z)^{\rightsquigarrow z} \vee_1 (y \vee_1 z))^{\to (y \vee_1 z)} = ((y \vee_1 z)^{\to z} \vee_2 (x \vee_2 z))^{\rightsquigarrow (x \vee_2 z)}$ *and*
$((x \vee_1 z)^{\to z} \vee_2 (y \vee_2 z))^{\rightsquigarrow (y \vee_2 z)} = ((y \vee_2 z)^{\rightsquigarrow z} \vee_1 (x \vee_1 z))^{\to (x \vee_1 z)}$*;*
(14) $((x \vee_1 y) \vee_1 x)^{\to x} = (x \vee_1 y)^{\to x}$ *and* $((x \vee_2 y) \vee_2 x)^{\rightsquigarrow x} = (x \vee_2 y)^{\rightsquigarrow x}$.

Proof. The proof of (1)-(6) is straightforward by the definition and properties of pseudo-BCK algebras. (6) Assume that $x \preceq y$. Then $x \vee_1 y = (x \to y) \rightsquigarrow y = 1 \rightsquigarrow y = y$. Also,

$$x \vee_2 y = (x \rightsquigarrow y) \to y = 1 \to y = y.$$

Conversely, suppose that $x \vee_1 y = y$ and $x \vee_2 y = y$. Since

$$x \to y = x \to ((x \to y) \rightsquigarrow y) = (x \to y) \rightsquigarrow (x \to y) = 1$$

and

$$x \rightsquigarrow y = x \rightsquigarrow ((x \rightsquigarrow y) \to y) = (x \rightsquigarrow y) \to (x \rightsquigarrow y) = 1,$$

then $x \preceq y$.

(7) Suppose that $(x \to y) \rightsquigarrow y = y$ and $(y \to x) \rightsquigarrow x = x$. Then by (psBCI$_3$) we have

$$x \to y = x \to ((x \to y) \rightsquigarrow y) = (x \to y) \rightsquigarrow (x \to y) = 1$$

and

$$y \rightsquigarrow x = y \rightsquigarrow ((y \rightsquigarrow x) \to x) = (y \rightsquigarrow x) \to (y \rightsquigarrow x) = 1.$$

Now, using (psBCI$_5$) $x = y$. By a similar argument the second part is valid.

(8) By Proposition 2.1 (6), we have

$$\begin{aligned}(x \vee_1 y) \vee_1 y =&((x \vee_1 y) \to y) \rightsquigarrow y\\ =&(((x \to y) \rightsquigarrow y) \to y) \rightsquigarrow y\\ =&(x \to y) \rightsquigarrow y\\ =&x \vee_1 y.\end{aligned}$$

Similarly, we see that $x \vee_1 (x \vee_1 y) = x \vee_1 y$ and $y \vee_1 (x \vee_1 y) = x \vee_1 y$.

(9) According to ($psBCI_2$), $x \preceq x \vee_1 y$. By Proposition 2.1 (1) and (2), we have $x\vee_1 z \preceq (x\vee_1 y)\vee_1 z$. Now, using (6) it follows that $(x\vee_1 z)\vee_1((x\vee_1 y)\vee_1 z) = (x\vee_1 y)\vee_1 z$. By a similar argument we can verify $x\vee_2 y = (x\vee_2 y)\vee_2 y = x\vee_2(x\vee_2 y) = y\vee_2(x\vee_2 y)$.

(10) $(x \vee_1 y)^{\to y \rightsquigarrow y} = ((x \vee_1 y) \to y) \rightsquigarrow y = x \vee_1 y$ and $(x \vee_2 y)^{\rightsquigarrow y \to y} = ((x \vee_2 y) \rightsquigarrow y) \to y = x \vee_2 y$. By Proposition 2.1 (6), we have

$$\begin{aligned}
((x \vee_1 y)^{\to y} \vee_2 y)^{\rightsquigarrow y} =&(((x \vee_1 y) \to y) \vee_2 y)^{\rightsquigarrow y}\\
=&((((x \to y) \rightsquigarrow y) \to y) \vee_2 y)^{\rightsquigarrow y}\\
=&((x \to y) \vee_2 y)^{\rightsquigarrow y}\\
=&(((x \to y) \rightsquigarrow y) \to y)^{\rightsquigarrow y}\\
=&(x \to y)^{\rightsquigarrow y}\\
=&(x \to y) \rightsquigarrow y\\
=&x \vee_1 y.
\end{aligned}$$

Also, the proof of the second part is similar.

(11) From (5) and Proposition 2.1 (4), we conclude

$$\begin{aligned}
(x \vee_1 (y \vee_2 z)^{\rightsquigarrow z})^{\to (y\vee_2 z)^{\rightsquigarrow z}} =&(x \vee_1 (y \rightsquigarrow z))^{\to (y\rightsquigarrow z)}\\
=&x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)\\
=&(y \vee_2 (x \to z))^{\rightsquigarrow (x\to z)}\\
=&(y \vee_2 (x \vee_1 z)^{\to z})^{\rightsquigarrow (x\vee_1 z)^{\to z}}.
\end{aligned}$$

(12) Using (5), we have

$$\begin{aligned}
((x \vee_1 z) \vee_1 (y \vee_1 z))^{\to (y\vee_1 z)} =&(x \vee_1 z) \to (y \vee_1 z)\\
=&((x \to z) \rightsquigarrow z) \to ((y \to z) \rightsquigarrow z)\\
=&(y \to z) \rightsquigarrow (((x \to z) \rightsquigarrow z) \to z)\\
=&(y \to z) \rightsquigarrow (x \to z).
\end{aligned}$$

We have $(x \to y) \vee_2 ((y \to z) \rightsquigarrow (x \to z)) = 1 \to ((y \to z) \rightsquigarrow (x \to z))$. From this and ($psBCI_1$) we conclude

$$\begin{aligned}
(x \vee_1 y)^{\to y} \vee_2 ((x \vee_1 z) \vee_1 (y \vee_1 z))^{\to (y\vee_1 z)} =&(x \vee_1 y)^{\to y} \vee_2 ((y \to z) \rightsquigarrow (x \to z))\\
=&(x \to y) \vee_2 ((y \to z) \rightsquigarrow (x \to z))\\
=&1 \to ((y \to z) \rightsquigarrow (x \to z))\\
=&(y \to z) \rightsquigarrow (x \to z)\\
=&((x \vee_1 z) \vee_1 (y \vee_1 z))^{\to (y\vee_1 z)}.
\end{aligned}$$

(13) Applying (5), we have

$$\begin{aligned}((x \vee_2 z)^{\rightsquigarrow z} \vee_1 (y \vee_1 z))^{\to (y \vee_1 z)} =&(x \rightsquigarrow z) \to (y \vee_1 z)\\ =&(x \rightsquigarrow z) \to ((y \to z) \rightsquigarrow z)\\ =&(y \to z) \rightsquigarrow ((x \rightsquigarrow z) \to z)\\ =&(y \to z) \rightsquigarrow (x \vee_2 z)\\ =&((y \vee_1 z)^{\to z} \vee_2 (x \vee_2 z))^{\rightsquigarrow (x \vee_2 z)}.\end{aligned}$$

By a similar argument we have

$$((x \vee_1 z)^{\to z} \vee_2 (y \vee_2 z))^{\rightsquigarrow (y \vee_2 z)} = ((y \vee_2 z)^{\rightsquigarrow z} \vee_1 (x \vee_1 z))^{\to (x \vee_1 z)}.$$

(14) Using Proposition 2.1 (6), we get

$$\begin{aligned}((x \vee_1 y) \vee_1 x)^{\to x} =&(((x \to y) \rightsquigarrow y) \vee_1 x)^{\to x}\\ =&((((x \to y) \rightsquigarrow y) \to x) \rightsquigarrow x)^{\to x}\\ =&((x \to y) \rightsquigarrow y)^{\to x}\\ =&((x \to y) \rightsquigarrow y) \to x\\ =&(x \vee_1 y) \to x\\ =&(x \vee_1 y)^{\to x}.\end{aligned}$$

Similarly, $((x \vee_2 y) \vee_2 x)^{\rightsquigarrow x} = (x \vee_2 y)^{\rightsquigarrow x}$. □

Lemma 3.1. *Let $\mathfrak{X} = (X; \vee_1, \vee_2)$ be an algebra of type $(2, 2)$ satisfying the quasi-identities* (4)*,* (7)*,* (8) *and* (9) *of Theorem* 3.1*. Then the binary relation $\preceq$ defined by*

(B) *$x \preceq y$ if and only if $x \vee_1 y = y$ and $x \vee_2 y = y$ is a partial order on X.*

Proof. By (4) and (7), $\preceq$ is reflexive and antisymmetric. For transitivity, assume that $x \preceq y$ and $y \preceq z$. Using (8) and (9), we get

$$\begin{aligned}x \vee_1 z =&(x \vee_1 z) \vee_1 z\\ =&(x \vee_1 z) \vee_1 (y \vee_1 z)\\ =&(x \vee_1 z) \vee_1 ((x \vee_1 y) \vee_1 z)\\ =&(x \vee_1 y) \vee_1 z\\ =&y \vee_1 z = z\end{aligned}$$

and if $x \vee_2 y = y$ and $y \vee_2 z = z$, then we have

$$\begin{aligned}x \vee_2 z =&(x \vee_2 z) \vee_2 z\\ =&(x \vee_2 z) \vee_2 (y \vee_2 z)\\ =&(x \vee_2 z) \vee_2 ((x \vee_2 y) \vee_2 z)\\ =&(x \vee_2 y) \vee_2 z\\ =&y \vee_2 z = z.\end{aligned}$$

Thus, $\preceq$ is a partial order on X. $\square$

The following example shows that for every pseudo-BCK algebra $\mathfrak{X}$, $(X;\vee_1)$ and $(X;\vee_2)$ are not directoids in general.

Example 3.2. Let $\mathfrak{X}$ be the algebra given in Example 3.1. Then $(X;\vee_1)$ and $(X;\vee_2)$ are not directoids, since

$$c\vee_1 0 = (c\to 0)\rightsquigarrow 0 = 0\rightsquigarrow 0 = 1 \neq (c\vee_1 0)\vee_1 c = 0\vee_1 c = (0\to c)\rightsquigarrow c = 1\rightsquigarrow c = c$$

and

$$c\vee_2 0 = (c\rightsquigarrow 0)\to 0 = 0\to 0 = 1 \neq (c\vee_2 0)\vee_2 c = 0\vee_2 c = (0\rightsquigarrow c)\to c = 1\to c = c.$$

Theorem 3.2. *Let $\mathfrak{X}$ be a pseudo-BCK algebra, $\vee_1$ and $\vee_2$ be the binary operations defined by* (A)*. Then the following conditions are equivalent:*

(1) $(X;\vee_1)$ *and* $(X;\vee_2)$ *are directoids;*
(2) $\mathfrak{X}$ *is a commutative pseudo-BCK algebra;*
(3) $(X;\preceq)$ *is a join-semilattice, where $\preceq$ is defined by* (B).

Proof. (1)⇒(2) Assume that $(X;\vee_1)$ is a directoid. Then $x\preceq y$ implies $y\vee_1 x = y$ and so $\mathfrak{X}$ satisfies the quasi-identity

$$x\preceq y \Rightarrow y = (y\to x)\rightsquigarrow x.$$

Similary, $x\preceq y$ implies $y = (y\rightsquigarrow x)\to x$. Therefore, $\mathfrak{X}$ is a commutative pseudo-BCK algebra by Theorem 2.1.

(2)⇒(3) It follows from Proposition 2.2.

(3)⇒(1) It is obvious that every join-semilattice is a directoid. $\square$

Corollary 3.1. *Let $\mathfrak{X}$ be a pseudo-BCK(pDN), $\vee_1$ and $\vee_2$ be the binary operations defined by* (A)*. Then the following conditions are equivalent:*

(1) $(X;\vee_1)$ *and* $(X;\vee_2)$ *are directoids;*
(2) $\mathfrak{X}$ *is a commutative pseudo-BCK algebra;*
(3) $(X;\preceq)$ *is a join-semilattice;*
(4) $(X;\preceq)$ *is a meet-semilattice;*
(5) $(X;\preceq)$ *is a lattice.*

Proof. It follows from Theorems 3.2 and 2.2. $\square$

Corollary 3.2. *Let $\mathfrak{X}$ be a pseudo-BCK(pDN), $\vee_1$ and $\vee_2$ be the binary operations defined by* (A)*. Then the following conditions are equivalent:*

(1) $(X;\vee_1)$ *and* $(X;\vee_2)$ *are directoids;*
(2) $\mathfrak{X}$ *is a commutative pseudo-BCK algebra;*
(3) $\{1\}\in\mathfrak{D}S_c(X)$*;*
(4) $\mathfrak{D}S(X)=\mathfrak{D}S_c(X)$.

Proof. It follows from Theorem 3.2 and [6, Corollary 4.6, Theorem 4.7 and Corollary 4.8]. $\square$

In [8], L. C. Ciungu proved that for every pseudo-BCK(pDN) lattice the following conditions are equivalent (see [8, Proposition 3.5]):

(P_1) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$ and $(x \wedge y) \rightsquigarrow z = (x \rightsquigarrow z) \vee (y \rightsquigarrow z)$;

(P_2) $z \to (x \vee y) = (z \to x) \vee (z \to y)$ and $z \rightsquigarrow (x \vee y) = (z \rightsquigarrow x) \vee (z \rightsquigarrow y)$.

Also, she showed that the class of pseudo-BCK(pDN) lattices satisfies the conditions (P_1) and (P_2) is not empty, since every pseudo-MV algebra satisfies these conditions. Further, It was proved that if a pseudo-BCK(pDN) lattice $\mathfrak{X}$ satisfying (P_1) or (P_2), then $(X; \preceq)$ is a distributive lattice (see [8, Theorem 3.4, Corollary 3.2]).

Theorem 3.3. *Let $\mathfrak{S} = (S; \vee_1, \vee_2, (^{\to a})_{a\in S}, (^{\rightsquigarrow a})_{a\in S}, 1)$ be a structure algebraic, where $\vee_1$ and $\vee_2$ are binary operations on S and for each $a \in S$, $^{\to a}$ and $^{\rightsquigarrow a}$ are unary operations on $\{x \in S : a \vee_1 x = 1$ and $a \vee_2 x = 1\}$ and 1 is a distinguished element of S, satisfying the quasi-identities* (1)-(12) *from Theorem* 3.1*. Define the new binary operations $\to$ and $\rightsquigarrow$ on S by*

(C) $x \to y = (x \vee_1 y)^{\to y}$ *and* $x \rightsquigarrow y = (x \vee_2 y)^{\rightsquigarrow y}$.

Then $\mathfrak{X}(\mathfrak{S}) = (S; \to, \rightsquigarrow, 1)$ is a pseudo-BCK algebra.

Proof. The definition of $\to$ and $\rightsquigarrow$ are well define from Theorem 3.1 (8). Furthermore, we note that

(D) $x \vee_1 y = y$ and $x \vee_2 y = y$ if and only if $x \to y = 1$ and $x \rightsquigarrow y = 1$.

Indeed, if $x \vee_1 y = y$, then $x \to y = (x \vee_1 y)^{\to y} = y^{\to y} = 1$, by Theorem 3.1 (2). Similarly, if $x \vee_2 y = y$, then $x \rightsquigarrow y = (x \vee_2 y)^{\rightsquigarrow y} = y^{\rightsquigarrow y} = 1$.

For conversely, $1 = x \to y = (x \vee_1 y)^{\to y}$ implies $y = 1^{\rightsquigarrow y} = (x \vee_1 y)^{\to y \rightsquigarrow y} = x \vee_1 y$. Also, $1 = x \rightsquigarrow y = (x \vee_2 y)^{\rightsquigarrow y}$ implies $y = 1^{\to y} = (x \vee_2 y)^{\rightsquigarrow y \to y} = x \vee_2 y$, by Theorem 3.1 (3) and (10). Now, we verify the axioms of pseudo-BCK algebras as follows.

($psBCI_1$) Using Theorem 3.1 (5) and (12), we obtain

$$\begin{aligned}(x \to y) \vee_2 ((x \vee_1 z) \to (y \vee_1 z)) &= (x \vee_1 y)^{\to y} \vee_2 ((x \vee_1 z) \vee_1 (y \vee_1 z))^{\to (y \vee_1 z)} \\ &= ((x \vee_1 z) \vee_1 (y \vee_1 z))^{\to (y \vee_1 z)} \\ &= (x \vee_1 z) \to (y \vee_1 z).\end{aligned}$$

Thus, $(x \to y) \rightsquigarrow ((x \vee_1 z) \to (y \vee_1 z)) = 1$.

Also, according to Theorem 3.1 (10) and (11), we get

$$(x \to y) \rightsquigarrow y = ((x \vee_1 y)^{\to y} \vee_2 y)^{\rightsquigarrow y} = x \vee_1 y$$

and

$$x \to (y \rightsquigarrow z) = (x \vee_1 (y \vee_2 z)^{\rightsquigarrow z})^{\to (y \vee_2 z)} = (y \vee_2 (x \vee_1 z)^{\to z})^{\rightsquigarrow (x \vee_1 z)} = y \rightsquigarrow (x \to z).$$

Then

$$\begin{aligned}(x \vee_1 z) \to (y \vee_1 z) &= ((x \to z) \rightsquigarrow z) \to ((y \to z) \rightsquigarrow z) \\ &= (y \to z) \rightsquigarrow (((x \to z) \rightsquigarrow z) \to z) \\ &= (y \to z) \rightsquigarrow (x \to z),\end{aligned}$$

since

$$\begin{aligned}((x \to z) \to z) \rightsquigarrow z =& (((x \vee_1 z)^{\to z} \vee_2 z)^{\to z} \vee_1 z)^{\to z}\\ =& ((x \vee_1 z) \vee_1 z)^{\to z}\\ =& (x \vee_1 z)^{\to z}\\ =& x \to z.\end{aligned}$$

Altogether, we have

$$(x \to y) \rightsquigarrow ((y \to z) \rightsquigarrow (x \to z)) = (x \to y) \rightsquigarrow ((x \vee_1 z) \to (y \vee_1 z)) = 1.$$

The second part of axiom (psBCI_1) follows by duality.

(psBCI_2) Using Theorem 3.1 (10), we get

$$(x \to y) \rightsquigarrow y = ((x \vee_1 y)^{\to y} \vee_2 y)^{\rightsquigarrow y} = x \vee_1 y.$$

Hence, $x \vee_1 ((x \to y) \rightsquigarrow y) = x \vee_1 (x \vee_1 y) = x \vee_1 y = (x \to y) \rightsquigarrow y$. Then $x \to ((x \to y) \rightsquigarrow y) = 1$.

By a similar argument we have $x \rightsquigarrow ((x \rightsquigarrow y) \to y) = 1$.

(psBCI_3) Applying (D), from $x \vee_1 x = x \vee_2 x = x$ it follows $x \to x = x \rightsquigarrow x = 1$.

(psBCI_4) If $x \to y = 1$ and $y \to x = 1$, then by Theorem 3.1 (6), we have $x \vee_1 y = y$ and $y \vee_1 x = x$. Now, using Theorem 3.1 (7), it follows $x = y$.

(psBCI_5) This follows from Theorem 3.1 (6) and (D).

(psBCK) By Theorem 3.1 (1), $x \vee_1 1 = x \vee_2 1 = 1$. From (D) we see that $x \to 1 = x \rightsquigarrow 1 = 1$. □

Theorem 3.4. *Let* $\mathfrak{S} = (S; \vee_1, \vee_2, ({}^{\to a})_{a \in S}, ({}^{\rightsquigarrow a})_{a \in S}, 1)$ *be an algebra as in Theorem* 3.3 *satisfying* (1)-(12) *of Theorem* 3.1 *and* $\mathfrak{X}$ *be a pseudo-BCK algebra. Then* $\mathfrak{X}(\mathfrak{S}(\mathfrak{X})) = \mathfrak{X}$ *and* $\mathfrak{S}(\mathfrak{X}(\mathfrak{S})) = \mathfrak{S}$.

Proof. By Theorem 3.1, $\mathfrak{S}(\mathfrak{X}) = (X; \vee_1, \vee_2, ({}^{\to a})_{a \in X}, ({}^{\rightsquigarrow a})_{a \in X}, 1)$ is the structure satisfying (1)-(12) which is assigned to a given pseudo-BCK algebra $\mathfrak{X}$. Then in $\mathfrak{X}(\mathfrak{S}(\mathfrak{X})) = (X; \to_1, \rightsquigarrow_1, 1)$ we have

$$x \to_1 y = (x \vee_1 y)^{\to y} = ((x \to y) \rightsquigarrow y) \to y = x \to y$$

and

$$x \rightsquigarrow_2 y = (x \vee_2 y)^{\rightsquigarrow y} = ((x \rightsquigarrow y) \to y) \rightsquigarrow y = x \rightsquigarrow y.$$

Therefore, $\mathfrak{X}(\mathfrak{S}(\mathfrak{X})) = \mathfrak{X}$.

Also, assume that $\mathfrak{S} = (S; \vee_1, \vee_2, ({}^{\to a})_{a \in S}, ({}^{\rightsquigarrow a})_{a \in S}, 1)$ is a structure that satisfies (1)-(12) of Theorem 3.1, $\mathfrak{X}(\mathfrak{S}) = (S; \to, \rightsquigarrow, 1)$ its corresponding pseudo-BCK algebra (cf. Theorem 3.3) and $\mathfrak{S}(\mathfrak{X}(\mathfrak{S})) = (S; \sqcup_1, \sqcup_2, (r_{1a})_{a \in S}, (r_{2a})_{a \in S}, 1)$. Then

$$x \sqcup_1 y = (x \to y) \rightsquigarrow y = ((x \vee_1 y)^{\to y} \vee_2 y)^{\rightsquigarrow y} = x \vee_1 y$$

and

$$x \sqcup_2 y = (x \rightsquigarrow y) \to y = ((x \vee_2 y)^{\rightsquigarrow y} \vee_1 y)^{\to y} = x \vee_2 y.$$

Further, for $x \in [a, 1]$, we have

$$r_{1a}(x) = x \to a = (x \vee_1 a)^{\to a} = ((a \vee_1 x) \vee_1 a)^{\to a} = (a \vee_1 x)^{\to a} = x^{\to a}$$

and

$$r_{2a}(x) = x \rightsquigarrow a = (x \vee_2 a)^{\rightsquigarrow a} = ((a \vee_2 x) \vee_2 a)^{\rightsquigarrow a} = (a \vee_2 x)^{\rightsquigarrow a} = x^{\rightsquigarrow a}.$$

Therefore, $\mathfrak{S}(\mathfrak{X}(\mathfrak{S})) = \mathfrak{S}$. □

Corollary 3.3. *Let* $\mathfrak{S} = (S; \vee_1, \vee_2, (^{\to a})_{a\in S}, (^{\rightsquigarrow a})_{a\in S}, 1)$ *be an algebraic structure satisfying* (1)-(13) *of Theorem* 3.1. *Then the relation defined by* (B) *is a partial order on* S, 1 *is the greatest element of* S *and for every* $x, y \in S$, $x, y \preceq x \vee y$, *where* $\vee = \vee_1 = \vee_2$. *Moreover, for each* $a \in S$, $^{\to a}$ *and* $^{\rightsquigarrow a}$ *are antitone mappings on* $[a, 1] = \{x \in S : a \preceq x\}$.

Conclusion

We consider that this paper could contribute to the study of algebraic structures and to the development of pseudo-BCK algebras. So, we hope it would be served as a foundation and another topic of research to define and investigate among algebraic structures derived from pseudo-BCK algebras. As another direction of research, one could investigate relationship between commutative pseudo-valuation on pseudo-BCK algebras with directoids.

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