Kragujevac Journal of Mathematics
Volume 50(9) (2026), Pages 1377–1392.

EXISTENCE RESULTS FOR DOUBLE PHASE PROBLEM INVOLVING FRACTIONAL OPERATORS AND SINGULAR NONLINEARITY

RYM CHAMMEM[1], ABDELJABBAR GHANMI[2], AND MAHFOUDH MECHERGUI[1]

Abstract. The purpose of this paper is to extend some existing results related to the fractional Laplacian operator to a more general fractional operator. More precisely, we combine the variational method with the min-max problem, to study the existence of solutions for some singular problems with double-phase nonlocal operators. To validate our main results an illustrative example is presented.

1. Introduction

One of the key operators in the field of harmonic analysis and partial differential equations is the fractional Laplace operator which may be defined as

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}dy, \tag{1.1}$$

where $N > 2s$ and $s \in (0,1)$. Indeed, the elliptic partial differential equations approach in a nonlocal framework has recently been used very often in Ambrosetti et al. [2], Ghanmi [19], Nezza et al. [28], Saoudi et al. [31], Servadei [32], Servadei and Valdinoci [33] and Servadei et al. [34]. Also, we note that the interest in this operator has increased considerably, given its importance and its rigorous applications in several fields including, in particular, the mechanics of continuous media, phase transition phenomena, population dynamics, for more applications, financial mathematics, see for example Autuori and Pucci [3], Barrios et al. [5], Cabre and Tan [7], Caffarelli and Silvestre [8], Capella [9], Dong and Kim [15], Servadei [32] and references therein.

Key words and phrases. Elliptic equation, variational method, min-max, nonlocal operator, singular problem.
2020 *Mathematics Subject Classification.* Primary: 31B30. Secondary: 35J35.
DOI
Received: October 04, 2024.
Accepted: December 23, 2024.

Many authors use several methods to study problems of the type:

$$\begin{cases} (-\Delta)^s u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.2}$$

where $N > 2s$, $0 < s < 1$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, we refer the interested readers to the papers of Teng [35] (Critical point theory for non-differentiable functionals), [36] (Extending three-critical-points theorem), Wu et al. [38](Stampacchia truncation method and the Leray-Schauder fixed point theorem).

Recently, several researchers have been concentrated on the generalization of the fractional Laplace operator Servadei and Valdinoci [33] use a more general non-local operator $\mathcal{L}_K$ to study the following elliptic problem

$$\begin{cases} \mathrm{L_K} u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.3}$$

where $K : \mathbb{R}^N \setminus \Omega \to (0, +\infty)$ is a function satisfying some appropriate assumptions, $f \in C^1(\overline{\Omega} \times \mathbb{R})$, and $\mathrm{L_K}$ is a nonlocal operator which generalizes the fractional Laplacian one and defined as follows:

$$\mathrm{L_K} u(x) = \frac{1}{2} \int_{\mathbb{R}^N} \left(u(x+y) + u(x-y) - 2u(x)\right) K(y) dy. \tag{1.4}$$

By using the Mountain Pass Theorem they showed the existence of a non-trivial solution.

We note that the operator $\mathrm{L_K}$ is a generalization of the operator $-(-\Delta)^s$ in the sense that if $K(x) = |x|^{-(N+2s)}$ then $\mathcal{L}_K = -(-\Delta)^s$.

Next, many authors developed some problems like (1.3), we mention the works of Caffarelli and Silvestre [8], Chammem et al. [10, 11], Saoudi et al. [31], Servadei [32].

Very recently, Ghanmi and Saoudi [21] studied the following singular problem

$$\begin{cases} \mathrm{L_K} u = \frac{a(x)}{u^\gamma} + \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.5}$$

where $0 < \gamma < 1$, $\lambda > 0$, $a \in C(\Omega)$ and $h \in C^1(\bar{\Omega} \times \mathbb{R})$. They showed (using the method of Nehari manifold) the existence of λ_0 such that for all $\lambda \in (0, \lambda_0)$, the problem (1.5) admits two nontrivial weak solutions. For the interested readers, other singular problems like (1.5) can be found in [12, 16, 17, 20–23, 27, 37].

In recent years, problems involving mixed mixed-fractional Laplacian (double phase problems) operators attracted the attention of several researchers. In particular, Huyuan et al. [25] considered the following problem

$$\begin{cases} (-\Delta)^{s_1} u = \lambda \left((-\Delta)^{s_2} u + \mu u\right), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where $0 < s_2 < s_1 < 1$ and $N > 2s_1$. They proved that the above problem admits a sequence of nontrivial solutions. More recently, Giacomoni et al. [24] developed the following interesting problem

$$(1.6)\qquad \begin{cases}(-\Delta)_p^{s_1}u+(-\Delta)_q^{s_2}u=f(x), & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\backslash\Omega,\end{cases}$$

where $1<q\le p<+\infty$, $0<s_2\le s_1<1$ and $f\in L^\infty_{loc}(\Omega)$. The fractional p-Laplace operator $(-\Delta)_p^s$ is defined as

$$(-\Delta)_p^s u(x)=2\lim_{\epsilon\to 0}\int_{\mathbb{R}^N\backslash B_\epsilon(x)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}dy.$$

They study the Hölder continuity results and maximum principle for weak solutions to the problems (1.6).

We note that in general, double phase differential operators and corresponding energy functionals appear in several physical applications for example in the elasticity theory see Zhikov [39] and in other applications which can be found in the works of Bahroumi et al. [4]. Also, we note that double-phase problems appear in rigorous journals, we cite for example the papers [14, 18, 29, 30, 40], and the references therein.

Inspired by the above-mentioned works, in this paper we study the following problem

$$(Q_\lambda)\begin{cases}\mathcal{L}_{K_1}u+\mathcal{L}_{K_2}u=\frac{b(x)}{u^\gamma}-\lambda h(x,u), & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\backslash\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^N$, $N>2$, is a bounded smooth domain, $h\in C^1(\bar{\Omega}\times\mathbb{R})$, $\lambda\in\mathbb{R}$ and $0<\gamma<1$.

In the rest of this paper, the letter i will denote the integers one or two and we suppose that the function $K_i:\mathbb{R}^N\setminus\{0\}\to(0,+\infty)$ satisfies the following hypotheses:

(N1) $\min\{|x|,1\}K_i\in L^1(\mathbb{R}^N)$;

(N2) there exists $\mu_i>0$ such that for all $x\in\mathbb{R}^N\backslash\{0\}$ we have

$$K_i(x)>\mu_i|x|^{-(N+2s_i)},\quad s_i\in(0,1);$$

(N3) $K_i(x)=K_i(-x)$, for any $x\in\mathbb{R}^N\backslash\{0\}$.

We mentioned that the function $K_i(x)=|x|^{-(N+2s_i)}$ satisfies the above conditions. Also, we mentioned that the problem (Q_λ) is studied by M. Chhetri et al. [13] in the case when $\mathcal{L}_{K_1}=\mathcal{L}_{K_2}=|x|^{-(N+2s)}$, $b=0$ and $\lambda<0$.

Hereafter, we put $s=\min\{s_1,s_2\}$ and we assume that the function b satisfies:

$$(1.7)\qquad 0<b\in L^{\frac{\beta}{\beta+\gamma-1}}(\Omega),\quad 1\le\beta<2^*:=\frac{2N}{N-2s}.$$

The double-phase problem (Q_λ) is inspired by various models in mathematical physics. For example, consider the fourth-order relativistic operator

$$\varphi\mapsto \operatorname{div}\left(\frac{|\nabla\varphi|^2}{(1-|\nabla\varphi|^4)^{\frac{3}{4}}}\nabla\varphi\right),$$

which characterizes a wide range of phenomena in relativistic quantum mechanics. If we apply Taylor's formula to the function $y^2(1-y^4)^{-\frac{3}{4}}$, then we can approximate the last operator by the following double-phase operator

$$\varphi \mapsto \Delta_4\varphi + \frac{3}{4}\Delta_8\varphi.$$

The main body of the paper is organized as follows. In Section 2, we recall some basic definitions and collect some variational settings. Next, we study the existence and the multiplicity of solutions of problem Q_λ, in Section 3 for the case $\lambda < 0$, and in Section 4 for the case $\lambda \geq 0$. To illustrate the validity of the main results, an example is given in Section 5.

2. Preliminaries and Variational Setting

In this section, we collect and introduce some results related to the variational setting for the problem (Q_λ). First, we introduce the fractional Sobolev space

$$\mathcal{H}^s(\Omega) = \left\{u \in L^2(\Omega) : [u]_{s,2} < +\infty\right\},$$

with the norm

$$||u||_{\mathcal{H}^s(\Omega)} = |u|_{L^2(\Omega)} + [u]_{s,2},$$

where

$$[u]_{s,2} = \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{\frac{1}{2}}$$

is the Gagliardo norm. For a detailed account on the properties of $\mathcal{H}^s$, we refer the reader to [1, 28].

Denote $\mathbf{T} = \mathbb{R}^{2N} \setminus \mathcal{D}$, with $\mathcal{D} = (\Omega^c) \times (\Omega^c) \subset \mathbb{R}^{2N}$, such as : $(\Omega^c) = \mathbb{R}^N \setminus \Omega$. We denote by $\mathbf{O} = \{u : \mathbb{R}^N \to \ \mathbb{R}$ Lebesgue measurable$\}$. We define the space

$$X_i = \left\{u \in \mathbf{O} : u_{\setminus\Omega} \in L^2(\Omega) \text{ and } (u(x)-u(y))\sqrt{K_i(x-y)} \in L^2(\mathbf{T})\right\},$$

with the norm

$$||u||_{X_i} = \ ||u||_{L^2(\Omega)} + \left(\int_{\mathbf{T}} |u(x)-u(y)|^2 K_i(x-y) dxdy\right)^{\frac{1}{2}}.$$

We consider the following space:

$$F_i = \{u \in X_i : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \{\Omega\}\},$$

with the norm

$$||u||_{F_i} = \left(\int_{\mathbf{T}} \ |u(x)-u(y)|^2 K_i(x-y) dxdy\right)^{\frac{1}{2}}.$$

We stress that $(F_i, ||\cdot||_{F_i})$ is a Hilbert space [33]. Also, $C_0^2(\Omega) \subset F_i$, $X_i \subset \mathcal{H}^s(\Omega)$ and $F_i \subset \mathcal{H}^s(\mathbb{R}^N)$.

Theorem 2.1. *For all $1 \leq \beta < 2^*$, the injection $F_i \hookrightarrow L^\beta(\Omega)$ is continuous and compact.*

Put $F = F_1 \cap F_2$. Then, we obviously have that $(F, ||\cdot||_F)$ is a Hilbert space, with

$$||u||_F = ||u||_{F_1} + ||u||_{F_2}.$$

Moreover, for $1 \leq \beta < 2^*$, the embedding

$$F \hookrightarrow L^\beta(\Omega) \text{ is continuous and compact.} \tag{2.1}$$

Remark 2.1. Let $u \in F$. Then, we have

(i) $||u||_{F_i} \leq ||u||_F$;
(ii) $\frac{1}{2}||u||_F^2 \leq ||u||_{F_1}^2 + ||u||_{F_2}^2 \leq ||u||_F^2$;
(iii) according to 2.1, if we note that S_β is the best Sobolev constant, then we have

$$\int_\Omega |u|^\beta dx \leq S_\beta ||u||_F^\beta.$$

To use the variational method to prove the existence of solutions in F, we define the energy functional $\phi_\lambda : F \to \mathbb{R}$ associated to problem (Q_λ) by:

$$\phi_\lambda(u) = \frac{1}{2}||u||_{F_1}^2 + \frac{1}{2}||u||_{F_2}^2 + \lambda \int_\Omega H(x,u)dx - \frac{1}{1-\gamma}\int_\Omega b(x)|u|^{1-\gamma}dx, \tag{2.2}$$

where

$$H(x,u) = \int_0^u h(x,s)ds. \tag{2.3}$$

We say that $u \in F$ is a weak solution of problem (Q_λ), for every $v \in F$, we have

$$\begin{aligned}&\int_{\mathbf{T}} (u(x)-u(y))(v(x)-v(y))(K_1(x-y)+K_2(x-y))dxdy\\ =&\int_\Omega b(x)u^{-\gamma}v(x)dx - \lambda\int_\Omega h(x,u(x))v(x)dx.\end{aligned}$$

Now, we aim to study the existence and the multiplicity of solutions to problem (Q_λ) by using variational methods [6, 26].

3. Existence Result for $\lambda < 0$

In this section, we will present and prove the first main result of this paper. For this aim, we shall consider the following hypotheses.

(M1) There exist $f \in L^r(\Omega)$ continuous function and $1 < q < 2 < \frac{N}{s} < r$ such that

$$h(x,t) \leq Cf(x)|t|^{q-2}t, \quad \text{a.e. } x \in \Omega, t \in \mathbb{R},$$

where C is a positive constant.

(M2) There exists $\Omega' \subset \Omega$ with $|\Omega'| > 0$ such that

$$h(x,t), f(x) > 0, \quad \text{in } \Omega'.$$

The first main result of this paper is the following theorem.

Theorem 3.1. *Assume that* (M1) *and* (M2) *hold. If* $\lambda < 0$*, then problem* (Q_λ) *has at least one nontrivial weak solution with negative energy.*

To prove Theorem 3.1, we need to prove several lemmas.

Lemma 3.1. *Let $\lambda < 0$ and under assumption* (M1)*, the functional ϕ_λ is coercive and bounded below on F.*

Proof. Let $u \in F$. Then, using (1.7) and (2.1) and Hölders inequality, we obtain

$$\int_\Omega b(x)u^{1-\gamma}dx \le ||b||_{\frac{\beta}{\beta+\gamma-1}}||u||_\beta^{1-\gamma} \le S_\beta^{1-\gamma}||b||_{\frac{\beta}{\beta+\gamma-1}}||u||_F^{1-\gamma}. \tag{3.1}$$

Consequently, from (3.1), we have

$$\begin{aligned}\phi_\lambda(u) &\ge \frac{1}{2}||u||_{F_1}^2 + \frac{1}{2}||u||_{F_2}^2 + \lambda\int_\Omega H(x,u)dx - S_\beta^{1-\gamma}||b||_{\frac{\beta}{\beta+\gamma-1}}||u||_F^{1-\gamma}\\ &\ge \frac{1}{4}||u||_F^2 + \lambda\int_\Omega H(x,u)dx - S_\beta^{1-\gamma}||b||_{\frac{\beta}{\beta+\gamma-1}}||u||_F^{1-\gamma}. \end{aligned}\tag{3.2}$$

On the other hand, from (M1) and Hölders inequality, we get

$$\int_\Omega H(x,u)dx \le \frac{C}{q}\int_\Omega f(x)|u|^q(x)dx \le \frac{C}{q}||f||_r||u||_{qr'}^q, \tag{3.3}$$

where r' is such that $\frac{1}{r} + \frac{1}{r'} = 1$.

Note that, from (M1), we have

$$2^* - qr' = \frac{Nr(2-q) + 2(qrs - N)}{(N-2s)(r-1)} > 0. \tag{3.4}$$

From (3.3), applying (2.1), we get

$$\int_\Omega H(x,u)dx \le \ C_1||f||_r||u||_F^q, \tag{3.5}$$

for some positive constant C_1.

Finally, combining (3.2) and (3.5) and since $\lambda < 0$, we get

$$\phi_\lambda(u) \ge \frac{1}{4}||u||_F^2 + \lambda C_1||f||_r||u||_F^q - S_\beta^{1-\gamma}||b||_{\frac{\beta}{\beta+\gamma-1}}||u||_F^{1-\gamma}.$$

Since, $2 > q > 1-\gamma$, we infer that $\phi_\lambda \to +\infty$ if $||u||_F \to +\infty$. So, the functional ϕ_λ is coercive and bounded from below on F. That gives the proof of Lemma 3.1. □

Lemma 3.2. *Let $\lambda < 0$ and assume that assumption* (M2) *holds, then there exists $e \in E$ with $e > 0$ such that $\phi_\lambda(te) < 0$ for all $t > 0$ small enough.*

Proof. Let $e \in C_0^\infty$ such that $\text{supp}(e) \subset \Omega' \subset \Omega$, $e = 1$ in a subset $\Omega'' \subset \text{supp}(e)$ and $0 \le e \le 1$ in Ω'. Using assumption (M2), Remark 2.1 and the fact that $\lambda < 0$, we get

$$\begin{aligned}\phi_\lambda(te) &= \frac{1}{2}||te||_{F_1}^2 + \frac{1}{2}||te||_{F_2}^2 + \lambda\int_\Omega H(x,te)dx - \frac{1}{1-\gamma}\int_\Omega b(x)|te|^{1-\gamma}dx\\ &\le \frac{1}{2}t^2||e||_F^2 + \lambda\int_\Omega H(x,te)dx - \frac{1}{1-\gamma}t^{1-\gamma}\int_\Omega b(x)|e|^{1-\gamma}dx\\ &\le \frac{1}{2}t^2||e||_F^2 + \lambda\int_{\Omega'} H(x,te)dx - \frac{1}{1-\gamma}t^{1-\gamma}\int_\Omega b(x)|e|^{1-\gamma}dx.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\phi_\lambda(te) &\leq t^2\frac{1}{2}||e||_F^2 - t^{1-\gamma}\frac{1}{1-\gamma}\int_\Omega b(x)|e|^{1-\gamma}dx \\ &\leq t^{1-\gamma}\left(t^{2+\gamma-1}\frac{1}{2}||e||_F^2 - \frac{1}{1-\gamma}\int_\Omega b(x)|e|^{1-\gamma}dx\right). \qquad (3.6)\end{aligned}$$

Since $||e||_F^2 > 0$ (if $||e||_F^2 = 0$, then $e = 0$ in Ω, which is a contradiction). Finally, if t is such that $0 < t < \min\{1, t_0\}$, then from equation (3.6), we get $\phi_\lambda(te) < 0$, where t_0 is given by

$$t_0 = \left(\frac{\frac{1}{1-\gamma}\int_\Omega b(x)|e|^{1-\gamma}dx}{\frac{1}{2}||e||_F^2}\right)^{\frac{1}{2+\gamma-1}}.$$

□

In the sequel, we put

$$m_\lambda = \inf_{u\in F}\phi_\lambda(u). \qquad (3.7)$$

Lemma 3.3. *Let $\lambda < 0$. Under assertions (1.7) and (M1)-(M2), the function ϕ_λ reaches its global minimizer in F, that is, there exists $v_\lambda \in F$ such that*

$$\phi_\lambda(v_\lambda) = m_\lambda < 0.$$

Proof. Let $\{v_n\}$ be a minimizing sequence, then $\phi_\lambda(v_n) \to m_\lambda$. We claim that $\{v_n\}$ is bounded in F.

Indeed, if $\{v_n\}$ is not bounded in F, so $||v_n||_F \to +\infty$ as $n \to +\infty$. Since ϕ_λ is coercive,

$$\phi_\lambda(v_n) \to +\infty, \quad \text{as } ||v_n||_F \to +\infty.$$

This contradicts the fact that $\{v_n\}$ is a minimizing sequence. Therefore, $\{v_n\}$ is bounded in the reflexive space F_i. Then, from Theorem 2.1, up to a subsequence, still denoted by $\{v_n\}$ there exists $v_\lambda \in F_i$, such that

$$\begin{cases} v_n \rightharpoonup v_\lambda \text{ weakly in } F_i, \\ v_n \to v_\lambda \text{ strongly in } L^\beta(\Omega), \\ v_n \to v_\lambda \text{ a.e in } \Omega. \end{cases}$$

We begin by proving that ϕ_λ is weakly lower semi-continuous on F. So, we begin by remarking that from the fact that $v_n \rightharpoonup v_\lambda$ weakly in F_i, then we have

$$||v_\lambda||_{F_i} \leq \liminf_{n\to+\infty} ||v_n||_{F_i}. \qquad (3.8)$$

Next, by Vitali's Theorem [39], we can claim that

$$\lim_{n\to+\infty}\int_\Omega b(x)|v_n|^{1-\gamma}dx = \int_\Omega b(x)|v_\lambda|^{1-\gamma}dx. \qquad (3.9)$$

Indeed, we only need to prove that

$$\left\{\int_\Omega b(x)|v_n|^{1-\gamma}dx : n \in \mathbb{N}\right\},$$

is equi-absolutely-continuous.

Let $\varepsilon > 0$. Then, by the absolutely-continuity of $\int_\Omega |b(x)|^{\frac{\beta}{\beta+\gamma-1}} dx$, there exists $\alpha > 0$ such that

$$\int_{\Omega_3} |b(x)|^{\frac{\beta}{\beta+\gamma-1}} dx < \varepsilon^{\frac{\beta}{\beta+\gamma-1}}, \quad \text{for every } \Omega_3 \subset \Omega, \text{ with } |\Omega_3| < \alpha.$$

Note that $\{v_n\}$ is bounded, by the Sobolev embedding theorem, there exists a constant $A > 0$, such that $||v_n||_F \leq A$.

Consequently, by Hölder's inequality, we have

$$\int_{\Omega_3} b(x)|v_n|^{1-\gamma} dx \leq ||v_n||_F^{1-\gamma} \left(\int_{\Omega_3} |b(x)|^{\frac{\beta}{\beta+\gamma-1}} dx \right)^{\frac{\beta+\gamma-1}{\beta}} \leq A^{1-\gamma}\varepsilon.$$

Finally, we will prove that

$$\lim_{n\to+\infty} \int_\Omega H(x, v_n) dx = \int_\Omega H(x, v_\lambda) dx. \tag{3.10}$$

Let $\epsilon > 0$. Then, from (M1), there exists $C_\epsilon > 0$ such that

$$H(x, v_n(x)) \leq \frac{C_\epsilon}{q} |f(x)| \cdot |v_n|^q.$$

Since $v_n \rightharpoonup v_\lambda$ in F and from (3.4), we obtain the strong convergence for a subsequence of $\{v_n\}$ in $L^{r'q}(\Omega)$, still denoted by $\{v_n\}$. Therefore, we get $v_n \to v_\lambda$ a.e. in Ω, from [6, Theorem IV-9] there exists $l \in L^{r'q}(\Omega)$ such that

$$|v_n(x)| \leq l(x). \tag{3.11}$$

Then, assumption (M1), equations (2.3), (3.11) and Hölder's inequality imply that

$$\begin{aligned} \int_\Omega H(x, v_n) dx &\leq \frac{C_\epsilon}{q} \int_\Omega f(x)|v_n|^q dx \leq \frac{C_\epsilon}{q} \int_\Omega f(x) l^q(x) dx \\ &\leq \frac{C_\epsilon}{q} ||f||_r ||l||_{qr'}^q \leq C_3 ||f||_r ||l||_F^q. \end{aligned}$$

Hence, by the dominated convergence theorem, (3.10) holds.

Finally, by combining (2.2), (3.9), (3.8) with (3.10), we get ϕ_λ is weakly lower semi-continuous on F. Therefore,

$$\phi_\lambda(v_\lambda) \leq \liminf_{n\to+\infty} \phi_\lambda(v_n) = m_\lambda. \tag{3.12}$$

On the other hand, from (3.7), we have

$$\phi_\lambda(v_\lambda) \geq m_\lambda. \tag{3.13}$$

Combining (3.12) and (3.13), we conclude that $\phi_\lambda(v_\lambda) = m_\lambda$. □

Proof of Theorem 3.1. From Lemma 3.3, v_λ is a global minimum for ϕ_λ. Let $\varphi \in F$, then, for sufficiently small $t > 0$, we have

$$\begin{aligned}0 \leq &\phi_\lambda(v_\lambda + t\varphi) - \phi_\lambda(v_\lambda)\\ =&\frac{1}{2}\left(||v_\lambda + t\varphi||_{F_1}^2 - ||v_\lambda||_{F_1}^2 + ||v_\lambda + t\varphi||_{F_2}^2 - ||v_\lambda||_{F_2}^2\right)\\ &+ \lambda \int_\Omega H(x, v_\lambda + t\varphi) - H(x, v_\lambda) dx - \frac{1}{1-\gamma}\int_\Omega b(x)\left(|v_\lambda + t\varphi|^{1-\gamma} - |v_\lambda|^{1-\gamma}\right) dx,\end{aligned}$$

dividing this by $t > 0$ and letting $t \to 0^+$, we obtain

$$\begin{aligned}0 \leq &\int_\Omega (v_\lambda(x) - v_\lambda(y))(\varphi(x) - \varphi(y))(K_1(x-y) + K_2(x-y)) dx dy\\ &- \int_\Omega b(x)|v_\lambda|^{-\gamma}\varphi(x)dx + \lambda\int_\Omega h(x, v_\lambda(x))\varphi(x)dx.\end{aligned}$$

Since φ is arbitrary in F, we can replace φ by $-\varphi$. Thus, the equality holds for all $\varphi \in F$, which means that v_λ is a weak solution of problem (Q_λ). Moreover, the fact that $\phi_\lambda(v_\lambda) < 0$ implies that v_λ is nontrivial. □

4. Existence Result for $\lambda > 0$

In this section, we will present and prove the second main result of this paper. For this aim, we shall consider the following hypothesis.

(M3) There exist $\Omega_2 \subset \Omega$ with $|\Omega_2| > 0$ and a nonnegative function f_1 on Ω_2 such that $f_1 \in L^{r_1}(\Omega)$ with

$$\lim_{|t|\to 0} \frac{h(x,t)}{f_1(x)|t|^{p-1}} = 0, \quad \text{for } x \in \Omega \text{ uniformly.}$$

(M4) There exists a nonnegative function f_2 on Ω such that $f_2 \in L^{r_2}(\Omega)$ and

$$\lim_{|t|\to +\infty} \frac{h(x,t)}{f_2(x)|t|^{p'-1}} = 0, \quad \text{for } x \in \Omega \text{ uniformly,}$$

where

$$1 < \max\{p, p'\} < 2 < \frac{N}{s} < \min\{r_1, r_2\}. \tag{4.1}$$

(M5) There exists $B > 0$ such that

$$\int_\Omega H(x,t)dx \geq 0, \quad \text{for any } t > B.$$

The second main result of this paper is the following theorem.

Theorem 4.1. *Assume that* (M3), (M4) *and* (M5) *hold. If $\lambda > 0$, then problem* (Q_λ) *has at least one nontrivial weak solution with negative energy.*

To prove Theorem 4.1, we need to prove several lemmas.

Lemma 4.1. *Under assumption* (M5) *and the fact that $\lambda > 0$, the functional ϕ_λ is coercive and bounded below on F.*

Proof. Let $u \in F$ and $\lambda > 0$. Then, using (M5), (2.1) and (3.1) for any $||u|| > \max\{1, B\}$, we get

$$\begin{aligned}\phi_\lambda(u) &= \frac{1}{2}||u||_{F_1}^2 + \frac{1}{2}||u||_{F_2}^2 + \lambda\int_\Omega H(x,u)dx - \frac{1}{1-\gamma}\int_\Omega b(x)|u|^{1-\gamma}dx \\ &\geq \frac{1}{4}||u||_F^2 + \lambda\int_\Omega H(x,u)dx - S_\beta^{1-\gamma}||b||_{\frac{\beta}{\beta+\gamma-1}}||u||_F^{1-\gamma} \\ &\geq \frac{1}{4}||u||_F^2 - S_\beta^{1-\gamma}||b||_{\frac{\beta}{\beta+\gamma-1}}||u||_F^{1-\gamma}.\end{aligned}$$

Now, since $2 > 1-\gamma$, we have

$$\lim_{||u||_F\to+\infty} \frac{1}{4}||u||_F^2 - S_\beta^{1-\gamma}||b||_{\frac{\beta}{\beta+\gamma-1}}||u||_F^{1-\gamma} = +\infty.$$

So, the functional ϕ_λ is coercive and bounded from below on F which gives the proof of Lemma 4.1. □

Lemma 4.2. *Under assumption* $(M3)$ *and for* $\lambda > 0$*, there exists* $\zeta \in F$ *with* $\zeta > 0$ *such that* $\phi_\lambda(t\zeta) < 0$ *for all sufficiently small* $t > 0$*.*

Proof. Let $\zeta \in C_0^\infty(\Omega)$ such that $supp(\zeta) \subset \Omega_2 \subset \Omega$, $\zeta = 1$ in subset $\Omega_2' \subset supp(\zeta)$ and $0 < \zeta \leq 1$ in Ω_2. Using (2.3) and assumption $(M3)$, there exists $C_2 > 0$ such that

$$\int_\Omega H(x,t\zeta)dx \leq C_2|t|^p\int_{\Omega_2} f_1(x)|\zeta(x)|^p dx. \tag{4.2}$$

Assume that $0 < t < 1$, then by (4.2) and Remark 2.1, we get

$$\begin{aligned}\phi_\lambda(t\zeta) &= \frac{1}{2}||t\zeta||_{F_1}^2 + \frac{1}{2}||t\zeta||_{F_2}^2 + \lambda\int_\Omega H(x,t\zeta)dx - \frac{1}{1-\gamma}\int_\Omega b(x)|t\zeta|^{1-\gamma}dx \\ &\leq \frac{1}{2}t^2||\zeta||_F^2 + \lambda C_2|t|^p\int_{\Omega_2} f_1(x)|\zeta(x)|^p dx - \frac{1}{1-\gamma}t^{1-\gamma}\int_\Omega b(x)|\zeta|^{1-\gamma}dx \\ &\leq t^{\min\{2,p\}}\left[\frac{1}{2}||\zeta||_F^2 + \lambda C_2\int_{\Omega_2} f_1(x)|\zeta|^p dx\right] - t^{1-\gamma}\frac{1}{1-\gamma}\int_\Omega b(x)|\zeta|^{1-\gamma}dx \\ &\leq t^{1-\gamma}\left(\frac{1}{2}||\zeta||_F^2 + \lambda C_2\int_{\Omega_2} f_1(x)|\zeta|^p dx\right)\left(t^{\min\{2,p\}+\gamma-1} - t_0^{\min\{2,p\}+\gamma-1}dx\right), \qquad (4.3)\end{aligned}$$

where

$$t_0 = \left(\frac{\frac{1}{1-\gamma}\int_\Omega b(x)|\zeta|^{1-\gamma}dx}{\frac{1}{2}||\zeta||_F^2 + \lambda C_2\int_{\Omega_2} f_1(x)|\zeta|^p dx}\right)^{\frac{1}{\min\{2,p\}+\gamma-1}}.$$

Since $\lambda > 0$, $\zeta \neq 0$ and $f_1(x) \geq 0$, then

$$\frac{1}{2}||\zeta||_F^2 + \lambda C_2\int_{\Omega_2} f_1(x)\zeta^p dx > 0.$$

We see from (4.3), that if $0 < t < \min\{1, t_0\}$, then $\phi_\lambda(t\zeta) < 0$, which gives the proof of Lemma 4.2. □

In the sequel, we put

$$m'_\lambda = \inf_{u \in F} \phi_\lambda(u). \tag{4.4}$$

Lemma 4.3. *Let $\lambda > 0$. Under assertions* (1.7) *and* (M3)-(M5)*, the function ϕ_λ reaches its global minimizer in F, that is, there exists $\nu_\lambda \in F$ such that*

$$\phi_\lambda(\nu_\lambda) = m'_\lambda < 0.$$

Proof. Let $\{\nu_n\}$ be a minimizing sequence, then, $\phi_\lambda(\nu_n) \to m'_\lambda$. $\{\nu_n\}$ is bounded in F.

Indeed, if $\{\nu_n\}$ is not bounded in F, so $||\nu_n||_F \to +\infty$ as $n \to +\infty$. Since ϕ_λ is coercive

$$\phi_\lambda(\nu_n) \to +\infty, \quad \text{as } ||\nu_n||_F \to +\infty.$$

This contradicts the fact that $\{\nu_n\}$ is a minimizing sequence. Therefore, $\{\nu_n\}$ is bounded in the reflexive space F_i. Then from Theorem 2.1, up to a subsequence, still denoted by $\{\nu_n\}$ there exists $\nu_\lambda \in F_i$ such that

$$\begin{cases} \nu_n \rightharpoonup \nu_\lambda \text{ weakly in } F_i, \\ \nu_n \to \nu_\lambda \text{ strongly in } L^\beta(\Omega), \\ \nu_n \to \nu_\lambda \text{ a.e in } \Omega. \end{cases}$$

We will prove that

$$\lim_{n \to +\infty} \int_\Omega H(x, \nu_n) dx = \int_\Omega H(x, \nu_\lambda) dx. \tag{4.5}$$

Assume that $(M3)$, $(M4)$ and (2.3), that for all $\delta > 0$ there exists A_δ such that

$$|H(x, \nu_n)| \le \delta \frac{C_3}{p} \int_\Omega f_1(x) |\nu_n|^p dx + A_\delta \frac{C'_3}{p'} \int_\Omega f_2(x) |\nu_n|^{p'} dx.$$

Since $\nu_n \to \nu_\lambda$ strongly in $L^\beta(\Omega)$, $1 \le \beta < 2^*$, from [6, Theorem IV-9] there exists $l_1 \in L^\beta(\Omega)$ such that

$$|\nu_n(x)| \le l_1(x), \quad \text{a.e in } \Omega. \tag{4.6}$$

Then, by Hölder's inequality, we obtain

$$\begin{aligned}
|H(x,\nu_n)| \leq & \delta\frac{C_3}{p}\int_\Omega f_1(x)|\nu_n|^p dx + A_\delta\frac{C_3'}{p'}\int_\Omega f_2(x)|\nu_n|^{p'}dx \\
\leq & \delta\frac{C_3}{p}\left(\int_\Omega |f_1(x)|^{r_1}dx\right)^{\frac{1}{r_1}}\left(\int_\Omega |\nu_n|^{pr_1'}dx\right)^{\frac{1}{r_1}} \\
& + A_\delta\frac{C_3'}{p'}\left(\int_\Omega |f_2(x)|^{r_2}dx\right)^{\frac{1}{r_2}}\left(\int_\Omega |\nu_n|^{p'r_2'}dx\right)^{\frac{1}{r_2'}} \\
\leq & \delta\frac{C_3}{p}\left(\int_\Omega |f_1(x)|^{r_1}dx\right)^{\frac{1}{r_1}}\left(\int_\Omega |l_1|^{pr_1'}dx\right)^{\frac{1}{r_1}} \\
& + A_\delta\frac{C_3'}{p'}\left(\int_\Omega |f_2(x)|^{r_2}dx\right)^{\frac{1}{r_2}}\left(\int_\Omega |l_1|^{p'r_2'}dx\right)^{\frac{1}{r_2'}} \\
\leq & \delta\frac{C_3}{p}||f_1||_{r_1}||l_1||^p_{pr_1'} + A_\delta\frac{C_3'}{p'}||f_2||_{r_2}||l_1||^{p'}_{p'r_2'}, \qquad (4.7)
\end{aligned}$$

where $\frac{1}{r_1}+\frac{1}{r_1'}=1$ and $\frac{1}{r_2}+\frac{1}{r_2'}=1$.

Note that, from (4.1), we get

$$2^* - pr_1' = \frac{Nr_1(2-p)+2(pr_1s-N)}{(N-2s)(r_1-1)} > 0 \qquad (4.8)$$

and

$$2^* - p'r_2' = \frac{Nr_2(2-p')+2(p'r_2s-N)}{(N-2s)(r_2-1)} > 0.$$

From (4.7) applying (2.1), we get

$$|H(x,\nu_n)| \leq \delta\frac{C_3}{p}||f_1||_{r_1}||l_1||^p_F + A_\delta\frac{C_3'}{p'}||f_2||_{r_2}||l_1||^{p'}_F.$$

Hence, by the dominated convergence theorem, (4.5) holds.

By combining (2.2), (3.9), (3.8) and (4.5), we get that ϕ_λ is weakly lower semicontinuous. Therefore,

$$\phi_\lambda(\nu_\lambda) \leq \liminf_{n\to+\infty}\phi_\lambda(\nu_n) = m_\lambda'. \qquad (4.9)$$

On the other hand, from the definition of m_λ', we have

$$I_\lambda(\nu_\lambda) \geq m_\lambda'. \qquad (4.10)$$

Combining (4.9) and (4.10), we conclude that $\phi_\lambda(\nu_\lambda) = m_\lambda$. □

Proof of Theorem 4.1. Let $\lambda > 0$ and from Lemma 4.3, ν_λ is a global minimum for ϕ_λ. Let $\varphi \in F$, then, for sufficiently small $t > 0$,

$$\begin{aligned}0 \leq& \phi_\lambda(\nu_\lambda + t\varphi) - \phi_\lambda(\nu_\lambda)\\ =& \frac{1}{2}\left(||\nu_\lambda + t\varphi||_{F_1}^2 - ||\nu_\lambda||_{F_1}^2 + ||\nu_\lambda + t\varphi||_{F_2}^2 - ||\nu_\lambda||_{F_2}^2\right)\\ &+ \lambda \int_\Omega (H(x, \nu_\lambda + t\varphi) - H(x, \nu_\lambda))dx - \frac{1}{1-\gamma}\int_\Omega b(x)\left(|\nu_\lambda + t\varphi|^{1-\gamma} - |\nu_\lambda|^{1-\gamma}\right)dx,\end{aligned}$$

dividing this by $t > 0$ and letting $t \to 0^+$, we obtain

$$\begin{aligned}0 \leq& \int_{\mathbf{T}} (\nu_\lambda(x) - \nu_\lambda(y))(\varphi(x) - \varphi(y))(K_1(x-y) + K_2(x-y))dxdy\\ &- \int_\Omega b(x)|\nu_\lambda|^{-\gamma}\varphi(x)dx + \lambda\int_\Omega h(x, \nu_\lambda(x))\varphi(x)dx.\end{aligned}$$

Since φ is arbitrary in F, we can replace φ by $-\varphi$. Thus, the equality holds for all $\varphi \in F$, which means that ν_λ is a weak solution of problem (Q_λ). Moreover, the fact that $\phi_\lambda < 0$, implies that ν_λ is nontrivial. □

Remark 4.1. Let $\lambda = 0$, Lemmas 3.1 and 3.2 are true, then by Lemma 3.3 the function ϕ_0 reaches its global minimizer in F, that is, there exists $v_0 \in F$ such that

$$\phi_0(v_\lambda) = m_0 < 0.$$

We follow the same proof of Theorem 3.1 for $\lambda = 0$, we conclude that v_0 is a weak solution of problem (Q_λ). Moreover, the fact that $\phi_0(v_0) < 0$ implies that v_0 is nontrivial.

5. An Example

In this section, we give an example to illustrate our main result.

Example 5.1. Let $\Omega \subset \mathbb{R}^N$, $N > 2$, be a bounded domain. We consider the following problem:

$$(Q_\lambda)\begin{cases}\mathrm{L}_1 u + \mathrm{L}_2 u = \frac{b(x)}{u^\gamma} - \lambda h(x,u), & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N\backslash\Omega,\end{cases}$$

where λ is a parameter, b is a function that is almost everywhere positive in Ω provided that

$$b \in L^{\frac{\beta}{\beta+\gamma-1}}(\Omega), \quad 1 \leq \beta < 2^*, \text{ with } 2^* = \frac{2N}{N-2s}.$$

Case 1. Let $\lambda < 0$ and let h be the function defined respectively by:

$$h(x,t) = Cf(x)|t|^{q-2}t, \quad \text{for all } x \in \Omega,\ t \in \mathbb{R},$$

where, $f \in L^r(\Omega)$ is a continuous function and $1 < q < 2 < \frac{N}{s} < r$.

Then, assumptions (M1) and (M2) hold. So, for any $\lambda < 0$ problem (Q_λ) has at least one nontrivial weak solution in F with negative energy.

Case 2. Let $\lambda > 0$ and let h be the function defined respectively by:

$$h(x,t) = \begin{cases} f_1(x)|t|^{p-1}, & |t| \leq 1, \\ f_2(x)|t|^{p'-1}, & |t| > 1, \end{cases}$$

where $f_1 \in L^{r_1}(\Omega)$ and $f_2 \in L^{r_2}(\Omega)$ are nonnegative functions with

$$1 < \max\{p, p'\} < 2 < \frac{N}{s} < \min\{r_1, r_2\}.$$

Then, assumptions (M3), (M4) and (M5) hold. So for all $\lambda > 0$, problem (Q_λ) has at least one nontrivial weak solution in F with negative energy.

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122**(2) (1994), 519–543. https://doi.org/10.1006/jfan.1994.1078

[3] G. Autuori and P. Pucci, *Elliptic problems involving the fractional Laplacian in* $\mathbb{R}^N$, J. Differ. Equ. **255**(8) (2013), 2340–2362. https://doi.org/10.1016/j.jde.2013.06.016

[4] A. Bahrouni, V. D. Rădulescu and D. D. Repovš, *Double phase transonic flow problems with variable growth, nonlinear patterns and stationary waves*, Nonlinearity **32**(7) (2019), 2481–2495. https://doi.org/10.1088/1361-6544/ab0b03

[5] B. Barrios, E. Colorado, A. De Pablo and U. Sanchez, *On some critical problems for the fractional Laplacian operator*, J. Diffe. Equ. **252** (2012), 6133–6162. https://doi.org/10.1016/j.jde.2012.02.023

[6] H. Brezis, *Analyse Fonctionnelle. Théorie et Applications*, Masson, Paris, 1983.

[7] X. Cabré and J. Tan, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math. **224**(5) (2010), 2052–2093. https://doi.org/10.1016/j.aim.2010.01.025

[8] L. Caffarelli and L. Silvestre, *Regularity theory for fully nonlinear integrodifferential equations*, Commun. Pure Appl. Math. **62**(5) (2009), 597–638. https://doi.org/10.1002/cpa.20274

[9] A. Capella, *Solutions of a pure critical exponent problem involving the half-Laplacian in annular-shaped domains*, Commun. Pure Appl. Anal. **10**(6) (2011), 1645–1662. https://doi.org/10.3934/cpaa.2011.10.1645

[10] R. Chammem, A. Ghanmi and M. Mechergui, *Combined effects in nonlinear elliptic equations involving fractional operators*, J. Pseudo-Differ. Oper. Appl. **14**(3) (2023), Article ID 35. https://doi.org/10.1007/s11868-023-00530-w

[11] R. Chammem, A. Ghanmi and M. Mechergui, *Existence of solutions for a singular double phase Kirchhoff type problems involving the fractional* $q(x,\cdot)$*-Laplacian operator*, Complex Anal. Oper. Theory **18**, (2024), Article ID 25. https://doi.org/10.1007/s11785-023-01470-5

[12] R. Chammem, A. Ghanmi and A. Sahbani, *Existence of solution for a singular fractional Laplacian problem with variable exponents and indefinite weights*, Complex Var. Elliptic Equ. **66**(8) (2020), 1320–1332. https://doi.org/10.1080/17476933.2020.1756270

[13] M. Chhetri, P. Girg and E. Hollifield, *Existence of positive solutions for fractional Laplacian equations: Theory and numerical experiments*, Elect. J. Differ. Equ. **2020**(81) (2020), 81–31. https://doi.org/10.58997/ejde.2020.81

[14] A. Crespo-Blanco, L. Gasiński, P. Harjulejto and P. Winkert, *A new class of double phase variable exponent problems: Existence and uniqueness*, J. Differ. Equ. **323** (2022), 182–228. https://doi.org/10.1016/j.jde.2022.03.029

[15] H. Dong and D. Kim, *On L^p-estimates for a class of non-local elliptic equations*, J. Funct. Anal. **262**(3) (2012), 1166–1199. https://doi.org/10.1016/j.jfa.2011.11.002
[16] A. Fiscella, *A fractional Kirchhoff problem involving a singular term and a critical nonlinearity*, Adv. Nonlinear Anal. **8**(1) (2019), 645–660. https://doi.org/10.1515/anona-2017-0075
[17] A. Fiscella and P. K. Mishra, *The Nehari manifold for fractional Kirchhoff problems involving singular and critical terms*, Nonlinear Anal. **186**(2019), 6–32. https://doi.org/10.1016/j.na.2018.09.006
[18] L. Gasiński and N. S. Papageorgiou, *Double phase logistic equations with superdiffusive reaction*, Nonlinear Anal. Real World Appl. **70** (2023), Article ID 103782. https://doi.org/10.1016/j.nonrwa.2022.103782
[19] A. Ghanmi, *Existence of nonnegative solutions for a class of fractional p-Laplacian problems*, Nonlinear Stud. **22**(3) (2015), 373–379.
[20] A. Ghanmi and K. Saoudi, *A multiplicity results for a singular problem involving the fractional p-Laplacian operator*, Complex Var. Elliptic Equ. **61**(9) (2016), 1199–1216. https://doi.org/10.1080/17476933.2016.1154548
[21] A. Ghanmi and K. Saoudi, *The Nehari manifold for a singular elliptic equation involving the fractional Laplace operator*, Fract. Differ. Calc. **6**(2) (2016), 201–217. http://dx.doi.org/10.7153/fdc-06-13
[22] A. Ghanmi and K. Saoudi, *A multiplicity results for a singular equation involving the $p(x)$-Laplace operator*, Complex Var. Elliptic Equ. **62**(5) (2017), 695–725. https://doi.org/10.1080/17476933.2016.1238466
[23] A. Ghanmi, *Multiplicity of nontrivial solutions of a class of fractional p-Laplacian oroblem*, Z. Anal. Anwend. **34**(3) (2015), 309–319. https://doi.org/10.4171/ZAA/1541
[24] J. Giacomoni, D. Kumar and K. Sreenadh, *Interior and boundary regularity results for strongly nonhomogeneous p,q-fractional problems*, Adv. Calc. Var. **16**(2) (2023), 467–501. https://doi.org/10.1515/acv-2021-0040
[25] C. Huyuan, B. Mousomi and H. Hichem, *On the bounds of the sum of eigenvalues for a Dirichlet problem involving mixed fractional Laplacians*, J. Diff. Equ. **317** (2022), 31 pages. https://doi.org/10.1016/j.jde.2022.02.004
[26] O. Kavian, *Introduction à la Théorie des Points Critiques et Applications aux Problèmes Elliptiques*, Springer-Verlag, Paris, New York, 1993.
[27] J. F. Liao, X. F. Ke, C. Y. Lei and C. L. Tang, *A uniqueness result for Kirchhoff type problems with singularity*, Appl. Math. Lett. **59** (2016), 24–30. https://doi.org/10.1016/j.aml.2016.03.001
[28] E. D. Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136**(5) (2012), 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004
[29] N. S. Papageorgiou, V. D. Rădulescu and D. D. Repovš, *Existence and multiplicity of solutions for double phase Robin problems*, Bull. Lond. Math. Soc. **52**(3) (2020), 546–560. https://doi.org/10.1112/blms.12347
[30] K. Perera and M. Squassina, *Existence results for double-phase problems via Morse theory*, Commun. Contemp. Math. **20**(2) (2018), Article ID 17500023, https://doi.org/10.1142/S0219199717500237
[31] K. Saoudi, A. Ghanmi and S. Horrigue, *Multiplicity of solutions for elliptic equations involving fractional operator and sign-changing nonlinearity*, J. Pseudo-Differ. Oper. Appl. **11**(4) (2020), 1743–1756. https://doi.org/10.1007/s11868-020-00357-9
[32] R. Servadei, *The Yamabe equation in a non-local setting*, Adv. Nonlinear Anal. **2**(3) (2013), 235–270. https://doi.org/10.1515/anona-2013-0008
[33] R. Servadei and E. Valdinoci, *Mountain pass solutions for non-local elliptic operators*, J. Math. Anal. Appl. **389**(2), 887–898. https://doi.org/10.1016/j.jmaa.2011.12.032

[34] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. **33**(5) (2013), 2105–2137. https://doi.org/10.3934/dcds.2013.33.2105

[35] K. Teng, *Two nontrivial solutions for hemivariational inequalities driven by nonlocal elliptic operators*, Nonlinear Anal. Real World Appl. **14**(1) (2013), 867–874. https://doi.org/10.1016/j.nonrwa.2012.08.008

[36] K. Teng, *Multiplicity results for hemivariational inequalities driven by nonlocal elliptic operators*, J. Math. Anal. Appl. **396** (2012) 386–395. https://doi.org/10.1016/j.jmaa.2012.06.04

[37] L. Wang, K. Cheng and B. Zhang, *A uniqueness result for strong singular Kirchhoff-type fractional Laplacian problems*, Appl. Math. Opt. **83** (2021), 1859–1875. https://doi.org/10.1007/s00245-019-09612-y

[38] P. Wu, Y, Huang and Y. Zhou, *Existence and regularity of solutions for a class of fractional Laplacian problems*, J. Differ. Equ. **318**(5) (2022), 480–501. https://doi.org/10.1016/j.jde.2022.02.041

[39] V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR-Izv. **29**(1) (1987), 33–66. https://doi.org/10.1070/IM1987v029n01ABEH000958

[40] S. Leonardi and N. S. Papageorgiou, *Anisotropic Dirichlet double phase problems with competing nonlinearities*, Rev. Mat. Complut. **36**(2) (2023), 469–490. https://doi.org/10.1007/s13163-022-00432-3

[1]Department of Mathematics,
Tunis El Manar University,
Faculty of Sciences, Tunis 2092, Tunisia.
Email address: rym.chemmam@fst.utm.tn.
Email address: mecherguimahfoud78@gmail.com.

[2]Department of Mathematics,
Tunis El Manar University,
ENIT-LAMSIN, BP. 37, 1002 Tunis-Belvedère, Tunisia
Email address: abdeljabbar.Ghanmi@lamsin.rnu.tn
ORCID iD: https://orcid.org/0000-0001-8121-0496