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WELL-POSEDNESS AND ENERGY DECAY FOR A p-LAPLACIAN WAVE EQUATION WITH STRONG DAMPING AND STRONG FRACTIONAL DELAY FEEDBACK

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Abstract. We study a nonlinear wave equation involving the p-Laplacian operator posed in a bounded spatial domain, featuring a time-delay term within a strong fractional internal feedback mechanism of the form

$$w_{ss} - \operatorname{div}(|\nabla w|^{p-2}\nabla w) - \gamma_1 \Delta w_s - \gamma_2 \Delta(\partial_s^{\alpha,\upsilon} w)(s-\varsigma) = 0.$$

Under a suitable condition linking the coefficient of the delayed feedback to that of the non-delayed component, we establish the well-posedness of weak solutions in appropriate Sobolev spaces. This is achieved by combining the energy method with the Faedo-Galerkin approximation technique. Furthermore, we prove that the total energy of the system decays polynomially over time, using the multiplier method together with a general weighted integral inequality of Komornik type.

1. Introduction

In this paper, we study the well-posedness and asymptotic behavior of solutions to an initial-boundary value problem for a nonlinear p-Laplacian wave equation with strong damping and a delayed internal feedback governed by a generalized fractional time derivative. Specifically, we consider the following system:

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$$(\mathcal{P})\qquad \begin{cases} w_{ss}(z,s) - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial z_i}\left(\left|\frac{\partial w}{\partial z_i}\right|^{p-2}\frac{\partial w}{\partial z_i}\right) - \gamma_1 \Delta w_s(z,s) & \\ -\gamma_2 \Delta(\partial_s^{\alpha,\upsilon} w)(z,s-\varsigma) = 0, & \text{in } \Omega\times(0,+\infty),\\ w(z,s) = 0, & \text{on } \partial\Omega\times(0,+\infty),\\ w(z,0) = w_0(z), \quad w_s(z,0) = w_1(z), & \text{in } \Omega,\\ w_s(z,s-\varsigma) = f_0(z,s-\varsigma), & \text{in } \Omega\times(0,\varsigma). \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ denotes a bounded domain with a smooth boundary $\partial\Omega$. The parameter $\varsigma > 0$ represents a time delay, while $\gamma_1 > 0$ and $\gamma_2 \in \mathbb{R}$. The initial data (w_0, w_1, f_0) are assumed to belong to an appropriate function space.

The operator $\partial_s^{\alpha,\upsilon}$ represents a generalized Caputo-type fractional derivative of order α (with $0 < \alpha < 1$) with respect to the time, defined by

$$\partial_s^{\alpha,\upsilon}\hbar(s) = \frac{1}{\Gamma(1-\alpha)}\int_0^s (s-r)^{-\alpha} e^{-\upsilon(s-r)}\frac{d\hbar}{dr}(r)\,dr, \quad \upsilon \geq 0.$$

Physically, the equation can be interpreted as a Kelvin-Voigt model for viscoelastic materials incorporating delayed response. In this setting, the total stress consists of contributions from an elastic spring and two dashpots arranged in parallel, yielding the constitutive relation

$$\varphi(z,s) = \varepsilon(z,s) + \gamma_1\frac{d_\varepsilon(z,s)}{ds} + \gamma_2\frac{d_\varepsilon(z,s-\varsigma)}{ds},$$

where ε denotes the strain, $\varsigma > 0$ the delay time, and $\gamma_1 > 0, \gamma_2 \in \mathbb{R}$ are real parameters. For small deformations, the strain is assumed to be proportional to the deformation gradient, i.e. $\varepsilon = k\nabla w$. Substituting this relation into the equation of motion yields

$$w_{ss}(z,s) = \operatorname{div}\varphi(z,s),$$

recovers our governing equation. We refer to [10] for applications of the undelayed case $\gamma_2 = 0$ in biological tissue modeling. Indeed, although delay effects occur naturally in numerous physical and biological systems, they can also lead to instabilities.

The related research encompasses the investigation conducted by Nicaise and Pignotti [14], who examined a wave equation with internal damping $\gamma_1 w_s$ and a linear delayed feedback $\gamma_2 w_s(s-\varsigma)$. They proved that uniform stability holds if and only if $|\gamma_2| < \gamma_1$; otherwise, they constructed delay sequences for which solutions fail to decay. Subsequently, Messaoudi et al. [12] extended this to a strongly damped wave equation $-\gamma_1\Delta w_t$ with delayed strong feedback $-\gamma_2\Delta w_s(s-\varsigma)$, again establishing uniform decay under the condition $|\gamma_2| < \gamma_1$.

To our knowledge, the current issue involving both p-Laplacian nonlinearity and fractional delayed damping has not been discussed in the existing literature. Our main goals are to determine global solvability within Sobolev spaces and to obtain energy decay estimates for problem $(\mathcal{P})$. Global existence is obtained via a combined approach: the Faedo-Galerkin approximation scheme (see [9]) complemented by energy estimates. Notably, the semigroup framework employed in [14] appears unsuitable for handling the inherent nonlinearity of the p-Laplacian.

For the decay analysis, we utilize the multiplier method (see [7, 8]), which is based on meticulously designed weighted integral inequalities that were initially created to examine energy decay for equations of the type $w_{ss} - \Delta w + g(w_s) = 0$ in bounded domains.

2. Preliminaries and Main Results

We first state some lemmas that will be needed later.

Theorem 2.1 (see [11]). *Let β be the function:*

$$\beta(\zeta) = |\zeta|^{\frac{2\lambda-1}{2}}, \quad -\infty < \zeta < +\infty, 0 < \lambda < 1.$$

The relationship between the input $\mathcal{U}$ and the output $\mathcal{O}$ is given by the following system:

$$\partial_s \theta(\zeta, s) + (\zeta^2 + \upsilon)\theta(\zeta, s) - \mathcal{U}(s)\beta(\zeta) = 0, \quad -\infty < \zeta < +\infty, \upsilon > 0, s > 0,$$
$$\theta(\zeta, 0) = 0,$$
$$\mathcal{O}(s) = \frac{\sin(\lambda\pi)}{\pi} \int_{-\infty}^{+\infty} \beta(\zeta)\theta(\zeta, s)\, d\zeta$$

which implies that

$$\mathcal{O} = I^{1-\lambda,\upsilon} D\mathcal{U} = D^{\lambda,\upsilon}\mathcal{U},$$

where

$$[I^{\lambda,\upsilon} h](s) = \frac{1}{\Gamma(\lambda)} \int_0^s (s-r)^{\lambda-1} e^{-\upsilon(s-r)} h(r)\, dr.$$

Lemma 2.1 (see [2]). *If $\gamma \in D_\upsilon = \mathbb{C} \backslash (-\infty, -\upsilon]$, then*

$$\int_{-\infty}^{+\infty} \frac{\beta^2(\zeta)}{\gamma + \upsilon + \zeta^2}\, d\zeta = \frac{\pi}{\sin \lambda\pi} (\gamma + \upsilon)^{\lambda-1}.$$

Lemma 2.2 (Sobolev-Poincaré's inequality). *Let p satisfy*

$$2 \le p < +\infty, \quad \text{if } m = 1 \text{ or } 2, \quad \text{and} \quad 2 \le p \le \frac{2m}{m-2}, \quad \text{if } m \ge 3.$$

Then, there exists a constant $C_* = C_*(\Omega, p) > 0$ such that

$$\|w\|_{L^p(\Omega)} \le C_* \|\nabla w\|_{L^2(\Omega)}, \quad \text{for all } w \in H_0^1(\Omega).$$

Lemma 2.3 ([8]). *Let $\Xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function. Assume that there exist constants $\mu > -1$ and $\omega > 0$ such that*

$$\int_S^{+\infty} \Xi^{1+\mu}(s)\, ds \le \frac{1}{\omega} \Xi(0)^\mu \Xi(S), \quad 0 \le S < +\infty.$$

Then, the following estimates hold:

$$\Xi(s) = 0, \quad \text{for all } s \ge \frac{\Xi(0)^\mu}{\omega\, |\mu|}, \text{ if } -1 < \mu < 0,$$

$$\Xi(s) \leq \Xi(0)\left(\frac{1+\mu}{1+\omega\mu s}\right)^{\frac{1}{\mu}}, \quad \textit{for all } s \geq 0, \ \textit{if } \mu > 0,$$
$$\Xi(s) \leq \Xi(0)\Xi^{1-\omega s}, \quad \textit{for all } s \geq 0, \ \textit{if } \mu = 0.$$

We build the following assumption on the damping and the delay terms:

$$|\gamma_2|\upsilon^{\lambda-1} < \gamma_1.$$

As in [14], we define an auxiliary dependent variable

$$\chi(z,\rho,s) = w_s(z, s-\varsigma\rho), \quad \text{for } (z,\rho,s) \in \Omega \times (0,1) \times (0,+\infty).$$

It follows that

$$\varsigma\chi'(z,\rho,s) + \chi_\rho(z,\rho,s) = 0, \quad \text{in } \Omega \times (0,1) \times (0,+\infty).$$

Consequently, problem $(\mathcal{P})$ is equivalent to

$$\text{(2.1)} \quad \begin{cases} w_{ss}(z,s) - \operatorname{div}\Big(|\nabla w(z,s)|^{p-2}\nabla w\Big) - \gamma_1 \Delta w_s(z,s) & \\ -\kappa\displaystyle\int_{-\infty}^{+\infty} \beta(\zeta)\,\Delta\theta\,d\zeta = 0, & \text{in } \Omega \times (0,+\infty), \\ \partial_s\theta + (\zeta^2+\upsilon)\,\theta - \chi(z,1,s)\,\beta(\zeta) = 0, & \\ \varsigma\,\chi_s(z,\rho,s) + \chi_\rho(z,\rho,s) = 0, & \text{in } \Omega \times (0,1) \times \mathbb{R}_+^*, \\ w(z,s) = 0, & \text{on } \partial\Omega \times [0,+\infty), \\ \chi(z,0,s) = w_s(z,s), & \text{on } \Omega \times [0,+\infty), \\ w(z,0) = w_0(z), \quad w_s(z,0) = w_1(z), & \text{in } \Omega, \\ \chi(z,\rho,0) = f_0(z,-\rho\varsigma), & \text{in } \Omega \times (0,1). \end{cases}$$

with $\kappa = \frac{\sin(\lambda\pi)}{\pi}\gamma_2$. The total energy of the system governed by problem (2.1) is defined by:

$$\text{(2.2)} \quad \begin{aligned} \Xi(s) =& \frac{1}{2}\int_\Omega w_s^2(z,s)dz + \frac{1}{p}\int_\Omega |\nabla w(z,s)|^p dz + \nu\|\nabla\chi(z,\rho,s)\|^2_{L^2(\Omega\times(0,1))} \\ &+ |\kappa| \cdot \|\nabla\theta\|^2_{L^2(\Omega\times(-\infty,+\infty))}, \end{aligned}$$

where ν is a positive constant satisfying

$$\text{(2.3)} \quad \varsigma|\gamma_2|\frac{\upsilon^{\lambda-1}}{2\varepsilon} < \nu < \varsigma\left(2\gamma_1 - |\gamma_2|\frac{\upsilon^{\lambda-1}}{2\varepsilon}\right),$$

for all $0 < \varepsilon < \frac{1}{2}$.

Lemma 2.4. *Assume that (w,θ,χ) is a regular solution of system (2.1). Then, one can find a constant $K > 0$ for which the energy functional Ξ fulfills*

$$\frac{d\Xi(s)}{ds} \leq -K\left(\int_\Omega\int_{-\infty}^{+\infty} (\zeta^2+\upsilon)|\nabla\theta(z,\zeta,s)|^2\,dz\,d\zeta\right.$$

$$+\int_{\Omega}\left(|\nabla w_s(z,s)|^2+|\nabla\chi(z,1,s)|^2\right)dz\Big). \tag{2.4}$$

Proof. By multiplying the first equation in (2.1) by $\overline{w}_s$, integrating over the domain Ω, and invoking integration by parts in conjunction with the prescribed boundary conditions, we arrive at

$$\frac{1}{2}\cdot\frac{d}{ds}\left\{\|w_s\|^2_{L^2(\Omega)}+\frac{2}{p}\|\nabla w\|^p_{L^p(\Omega)}\right\}+\kappa\mathrm{Re}\int_{\Omega}\overline{w}_s\int_{-\infty}^{+\infty}\beta(\zeta)\Delta\theta(z,\zeta,s)\,d\zeta\,dz+\gamma_1. \tag{2.5}$$

A simple multiplication of $(2.1)_2$ by $-|\kappa|\Delta\overline{\theta}$, and integration over $\Omega\times(-\infty,+\infty)$, yield

$$\begin{aligned}&\frac{1}{2}\cdot\frac{d}{ds}\left\{|\kappa|\cdot\|\nabla\theta\|^2_{L^2(\Omega\times(-\infty,+\infty))}\right\}+|\kappa|\int_{\Omega}\int_{-\infty}^{+\infty}(\zeta^2+\upsilon)|\nabla\theta(z,\zeta,s)|^2\,d\zeta\,dz\\&-|\kappa|\mathrm{Re}\int_{\Omega}\nabla\chi(z,1,s)\int_{-\infty}^{+\infty}\beta(\zeta)\nabla\overline{\theta}(z,\zeta,s)\,d\zeta\,dz.\end{aligned} \tag{2.6}$$

Multiplying $(2.1)_3$ by $-\nu\Delta\overline{\chi}$, and integrating the resulting expression over $\Omega\times(0,1)$, we obtain

$$\frac{1}{2}\cdot\frac{d}{ds}\left\{\nu\|\nabla\chi\|^2_{L^2((\Omega)\times(0,1))}\right\}+\frac{\nu\varsigma^{-1}}{2}\int_{\Omega}\left(|\nabla\chi(z,1,s)|^2-|\nabla w_s(z,s)|^2\right)=0. \tag{2.7}$$

By adding (2.5), (2.6) and (2.7), we obtain

$$\begin{aligned}\frac{d\Xi(s)}{ds}=&-\gamma_1\|\nabla w_s\|^2_{L^2}-|\kappa|\int_{\Omega}\int_{-\infty}^{+\infty}(\zeta^2+\upsilon)|\nabla\theta(z,\zeta,s)|^2\,d\zeta\,dz\\&-\kappa\mathrm{Re}\int_{\Omega}\nabla\overline{w}_s\int_{-\infty}^{+\infty}\beta(\zeta)\nabla\theta(z,\zeta,s)\,d\zeta\,dz\\&+|\kappa|\mathrm{Re}\int_{\Omega}\nabla\chi(z,1,s)\int_{-\infty}^{+\infty}\beta(\zeta)\nabla\overline{\theta}(z,\zeta,s)\,d\zeta\,dz\\&+\frac{\nu\varsigma^{-1}}{2}\int_{\Omega}|\nabla w_s(z,s)|^2\,dz-\frac{\nu\varsigma^{-1}}{2}\int_{\Omega}|\nabla\chi(z,1,s)|^2\,dz.\end{aligned} \tag{2.8}$$

Hence, by the Cauchy-Schwarz inequality,

$$\begin{aligned}&\left|\int_{\Omega}\nabla\chi(z,1,s)\int_{-\infty}^{+\infty}\beta(\zeta)\nabla\overline{\theta}(z,\zeta,s)\,d\zeta\,dz\right|\\&\le\left(\int_{-\infty}^{+\infty}\frac{\beta^2(\zeta)}{\zeta^2+\upsilon}\,d\zeta\right)^{\frac{1}{2}}\|\nabla\chi(z,1,s)\|_{L^2(\Omega)}\left(\int_{\Omega}\int_{-\infty}^{+\infty}(\zeta^2+\upsilon)|\nabla\theta(z,\zeta,s)|^2\,dz\,d\zeta\right)^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{\Omega}\nabla\overline{w}_s(z,s)\int_{-\infty}^{+\infty}\beta(\zeta)\nabla\theta(z,\zeta,s)\,d\zeta\,dz\right|\\&\le\left(\int_{-\infty}^{+\infty}\frac{\beta^2(\zeta)}{\zeta^2+\upsilon}\,d\zeta\right)^{\frac{1}{2}}\|\nabla w_s(z,s)\|_{L^2(\Omega)}\left(\int_{\Omega}\int_{-\infty}^{+\infty}(\zeta^2+\upsilon)|\nabla\theta(z,\zeta,s)|^2\,dz\,d\zeta\right)^{\frac{1}{2}}.\end{aligned}$$

Exploiting Young's inequalities, (2.8) can be rewritten as

$$\frac{d\Xi(s)}{ds} \leq \left(-\gamma_1 + \frac{|\kappa|\mathcal{I}}{4\varepsilon} + \frac{\nu\varsigma^{-1}}{2}\right)\int_\Omega |\nabla w_s|^2\,dz + \left(\frac{|\kappa|\mathcal{I}}{4\varepsilon} - \frac{\nu\varsigma^{-1}}{2}\right)\int_\Omega |\nabla\chi(z,1,s)|^2\,dz$$
$$-(1-2\varepsilon)\int_\Omega\int_{-\infty}^{+\infty}(\zeta^2+\upsilon)|\nabla\theta(z,\zeta,s)|^2\,d\zeta\,dz,$$

where $0<\varepsilon<1/2$ and $\mathcal{I} = \int_{-\infty}^{+\infty}\frac{\beta^2(\zeta)}{\zeta^2+\upsilon}\,d\zeta$, which implies

$$\frac{d\Xi(s)}{ds} \leq -K\left(\int_\Omega\left(\int_{-\infty}^{+\infty}(\zeta^2+\upsilon)|\nabla\theta(z,\zeta,s)|^2\,d\zeta + |\nabla w_s(z,s)|^2 + |\nabla\chi(z,1,s)|^2\right)dz\right),$$

with

$$K = \min\left\{(1-2\varepsilon), \left(\gamma_2 - \frac{|\kappa|\mathcal{I}}{4\varepsilon} - \frac{\nu\varsigma^{-1}}{2}\right), \left(-\frac{|\kappa|\mathcal{I}}{4\varepsilon} + \frac{\nu\varsigma^{-1}}{2}\right)\right\}.$$

Choosing ν in accordance with assumption (2.3), the constant K is strictly positive. This finishes the proof. □

The main result of this paper is as follows.

Theorem 2.2. *Let conditions* $(H1)$*-*$(H2)$ *hold. Then, for any initial data* $(w_0,w_1,f_0)\in W_0^{1,p}\times L^2(\Omega)\times L^2(\Omega;L^2(0,1))$ *given problem* $(\mathcal{P})$ *has a unique solution* $w\in L^\infty([0,+\infty);W_0^{1,p}(\Omega))$, $w'\in L^2([0,+\infty);H_0^1(\Omega))$, $w''\in L^2_{loc}([0,+\infty);W_0^{1,-p'}(\Omega))$, *and there exists a constant* $K(\Xi(0))>0$, *depending continuously on the initial energy* $\Xi(0)$, *such that*

$$\Xi(s) \leq \left(\frac{K(\Xi(0))}{s}\right)^{\frac{p}{p-2}}, \quad \textit{for all } s>0.$$

3. Global Existence

We are now prepared to demonstrate Theorem 2.2, which will be conducted in the following two sections. Throughout this section, we assume the initial data satisfies

$$w_0\in H^2\cap H_0^1(\Omega), \quad w_1\in H_0^1(\Omega) \quad \text{and} \quad f_0\in H_0^1(\Omega;H^1(0,1)).$$

Let $\{\varpi_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $H^2(\Omega)\cap H_0^1(\Omega)$, and denote by V_k the finite-dimensional subspace spanned by $\{\varpi_1,\dots,\varpi_k\}$. Similarly, fix $T>0$ and let $\{\vartheta_k\}_{k\in\mathbb{N}}$ be a basis of $L^2\big(\Omega\times\mathbb{R}\big)$. The space generated by $\{\vartheta_1,\dots,\vartheta_k\}$ will be denoted by W_k.

For each $j=1,\dots,k$, define a function $\psi_j(z,\rho)$ on $\Omega\times[0,1]$ by prescribing its trace at $\rho=0$ as $\psi_j(z,0)=\varpi_j(z)$,

The function $\psi_j(z,0)$ is extended to $\psi_j(z,\rho)\in L^2\big(\Omega\times[0,1]\big)$, and let Z_k be the subspace generated by $\{\psi_1,\dots,\psi_k\}$.

We now construct approximate solutions (w_k,θ_k,χ_k), $k=1,2,3,\dots$, in the form

$$w_k(s)=\sum_{j=1}^k g_{jk}\varpi_j, \quad \chi_k(s)=\sum_{j=1}^k h_{jk}\psi_j, \quad \theta_k(s)=\sum_{j=1}^k l_{jk}\vartheta_j,$$

where the coefficient functions $g_{jk}, h_{jk}, l_{jk} \in C^1([0,T])$, $j = 1, 2, \dots, m$, are determined by the following system of equations:

$$\begin{cases} (w_k''(s), \varpi_j) - (\Delta_p w_k(s), \varpi_j) - \gamma_1(\Delta w_k'(s), \varpi_j) \\ - \kappa \int_{-\infty}^{+\infty} \beta(\zeta)\,(\Delta\theta_k(z,\zeta,s), \varpi_j)\, d\zeta = 0, \qquad 1 \le j \le k, \\ \chi_k(z,0,s) = w_k'(z,s), \end{cases} \tag{3.1}$$

$$w_k(0) = w_{0k} = \sum_{j=1}^{k} (w_0, \varpi_j)\varpi_j \to w_0, \quad \text{in } H^2 \cap H_0^1 \text{ as } m \to +\infty, \tag{3.2}$$

$$w_k'(0) = w_{1k} = \sum_{j=1}^{k} (w_1, \varpi_j)\varpi_j \to w_1, \quad \text{in } H_0^1 \text{ as } m \to +\infty \tag{3.3}$$

and

$$(\varsigma\, \chi_{ks} + \chi_{k\rho},\, \psi_j) = 0, \quad 1 \le j \le k, \tag{3.4}$$

$$\chi_k(\rho, 0) = \chi_{0k} = \sum_{j=1}^{k} (f_0, \psi_j)\psi_j \to f_0, \quad \text{in } H_0^1(\Omega; H^1(0,1)) \text{ as } k \to +\infty, \tag{3.5}$$

$$\begin{cases} \Big(\partial_s\theta_k(z,\zeta,s) + (\zeta^2 + \upsilon)\,\theta_k(z,\zeta,s) - \chi_k(z,1,s)\,\beta(\zeta),\, \vartheta_j\Big) = 0, \quad 1 \le j \le k, \\ \theta_k(z,\zeta,0) = 0. \end{cases} \tag{3.6}$$

For a fixed integer k, the system (3.1)–(3.6) constitutes a set of ordinary functional differential equations for the unknowns w_k, χ_k, and θ_k. By the standard theory of ordinary differential equations extended to functional settings the local existence and uniqueness of a solution follow from the local Lipschitz continuity of the linear terms in (3.1). Consequently, there exists a maximal interval of existence $[0, T_k)$ with $0 < T_k \le +\infty$, obtained via Zorn's lemma. Moreover, the approximation $w_k(s)$ is of class C^2.

To extend the solution globally to all $s > 0$, we now derive a priori estimates for w_k, χ_k, and θ_k, which will allow us to rule out finite-time blow-up and thus prove that $T_k = +\infty$.

Finally, a standard compactness argument (e.g. Aubin-Lions lemma or weak/weak-* convergence) can be applied in the limiting process $k \to +\infty$. Therefore, it suffices to establish uniform a priori bounds for the pair (w_k, χ_k).

3.1. **First energy estimate.**

The convergence of the sequences w_{0k}, w_{1k} and χ_{0k} implies, via (2.4), the existence of a positive constant K, independent on the approximation index k, such that

$$\Xi_k(s) = \frac{1}{2}\|w_k'(s)\|_2^2 + \frac{1}{p}\|\nabla_z w_k(s)\|_p^p + \nu\|\nabla\chi_k(z,\rho,s)\|_{L^2(\Omega\times(0,1))}^2 \tag{3.7}$$

$$+ |\kappa| \cdot \|\nabla\theta_k\|^2_{L^2(\Omega\times(-\infty,+\infty))} \leq K.$$

These estimates guarantee that the approximate solution (w_k, χ_k, θ_k) is defined globally on $[0, +\infty)$.

Moreover, inequality (3.7) yields

(3.8) w_k is uniformly bounded in the space $L^\infty_{loc}(0, +\infty; W^{1,p}_0(\Omega))$,

(3.9) w'_k is uniformly bounded in the space $L^\infty_{loc}(0, +\infty; L^2(\Omega))$,

(3.10) w'_k is uniformly bounded in the space $L^2(0, +\infty; W^{1,2}_0(\Omega))$,

(3.11) θ_k is uniformly bounded in the space $L^\infty_{loc}(0, +\infty; W^{1,2}(\Omega \times \mathbb{R}))$,

(3.12) $\zeta\theta_k$ is uniformly bounded in the space $L^2(0, +\infty; L^2(\Omega \times \mathbb{R}))$,

(3.13) $\chi_k(z, \rho, s)$ is uniformly bounded in the space $L^\infty_{loc}(0, +\infty; W^{1,2}(\Omega \times (0, 1)))$,

(3.14) $\chi^2_k(z, 1, s)$ is uniformly bounded in the space $L^1(\Omega \times (0, S))$.

3.2. **Second estimate.** Recall that the p-Laplacian operator $-\Delta_p w = -\text{div}\Big(|\nabla w|^{p-2}\nabla w\Big)$, for $p \geq 2$, is a uniformly bounded, monotone, and hemicontinuous mapping from $W^{1,p}_0(\Omega)$ into its dual $W^{-1,p'}_0(\Omega)$. Consequently, one readily verifies that

(3.15) $\{-\Delta_p w_k(s)\}$ is uniformly bounded in the space $L^\infty_{loc}(0, +\infty; W^{-1,p'}_0(\Omega))$.

Moreover, by means of a standard Galerkin projection argument applied to the approximate equation (3.1), together with the estimates (3.8) and (3.10), we deduce that

(3.16) $\{w''_k(s)\}$ is uniformly bounded in the space $L^2_{loc}(0, +\infty; H^{-r}(\Omega))$,

for some $r > 0$.

From (3.8)–(3.9), there exists a subsequence (still denoted by w_k) such that

$$w_k \rightharpoonup w \text{ weakly star in } L^\infty_{loc}(0, +\infty; W^{1,p}_0(\Omega)),$$
$$w'_k \rightharpoonup w' \text{ weakly star in } L^\infty_{loc}(0, +\infty; L^2(\Omega)),$$
$$w'_k \rightharpoonup w' \text{ weakly } L^2(0, +\infty; W^{1,2}_0(\Omega)).$$

Moreover, by virtue of estimate (3.16), there exist a function η such that

(3.17) $\{-\Delta_p w_k(s)\} \rightharpoonup \eta$ weakly star in $L^\infty_{loc}(0, +\infty; W^{-1,p'}_0(\Omega))$.

It follows from the Aubin-Lions compactness lemma in [9], we obtain, from (3.8) and (3.9), that

$$w_k \to w \text{ strongly in } L^2_{loc}(0, +\infty; L^2(\Omega))$$

and $w_k \to w$ in $\Omega \times (0, +\infty)$.

Since the embedding $W^{1,2}_0 \hookrightarrow L^2$ is compact, we obtain, from (3.15) and (3.16), that

$$w'_k \to w' \text{ strongly in } L^2_{loc}(0, +\infty; L^2(\Omega)).$$

By differentiating equation (3.4) with respect to the time variable s, we arrive at

$$\left(\varsigma\chi_k''(s) + \frac{\partial}{\partial\rho}\chi_k', \vartheta_j\right) = 0.$$

Multiplying this identity by $h_{jk}'(s)$ and summing over $j = 1, \ldots, k$, we deduce

$$\frac{1}{2}\varsigma\frac{d}{ds}\|\chi_k'(s)\|_2^2 + \frac{1}{2}\cdot\frac{d}{d\rho}\|\chi_k'(s)\|_2^2 = 0. \tag{3.18}$$

Integrating (3.18) over $(0, s)$ and applying Gronwall's lemma yields uniform bound

$$\|\chi_k'(z, \rho, s)\|^2_{L^2(\Omega\times(0,1))} \leq K,$$

for all $s \in \mathbb{R}_+$, hence, we deduce that

$$\chi_k \to \chi \ \text{ weak-star in } \ L^\infty_{loc}(0, +\infty; H_0^1(\Omega; L^2(0, 1))),$$

$$\chi_k' \to \chi' \ \text{ weak-star in } \ L^\infty_{loc}(0, +\infty; L^2(\Omega \times (0, 1))), \tag{3.19}$$

$$\theta_k \to \theta \ \text{ weak-star in } L^\infty_{loc}(0, +\infty; W^{1,2}(\Omega \times \mathbb{R})), \tag{3.20}$$

$$\zeta\theta_k \to \zeta\theta \ \text{ weak-star in } L^2(0, +\infty; L^2(\Omega \times \mathbb{R})). \tag{3.21}$$

Then, it follows immediately from (3.16), (3.17), (3.19) and (3.20), that, for any fixed $v \in L^2(0, T; H_0^1)$ and $u \in L^2(0, T; H_0^1(\Omega \times (0, 1)))$

$$\begin{aligned}
&\int_0^T \int_\Omega (w_k'' - \Delta_p w_k - \gamma_1 \Delta w_k' - \kappa \int_{-\infty}^{+\infty} \beta(\zeta)\Delta\theta_k(z, \zeta, s))v\,dz\,ds \\
&\to \int_0^T \int_\Omega \left(w'' - \eta - \gamma_1\Delta w' - \kappa\int_{-\infty}^{+\infty}\beta(\zeta)\Delta\theta(z, \zeta, s)\right) v\,dz\,ds, \\
&\int_0^T\int_0^1\int_\Omega(\varsigma\chi_k' + \frac{\partial}{\partial\rho}\chi_k)u\,dz\,d\rho\,ds \to \int_0^T\int_0^1\int_\Omega\left(\varsigma\chi' + \frac{\partial}{\partial\rho}\chi\right)u\,dz\,d\rho\,ds, \\
&\int_\Omega\int_{\mathbb{R}}(\partial_s\theta_k(z, \zeta, s) + (\zeta^2 + \upsilon)\theta_k(\zeta, s) - \chi_k(z, 1, s)\beta(\zeta))\vartheta dz\,d\zeta \\
&\to \int_\Omega\int_{\mathbb{R}}(\partial_s\theta(z, \zeta, s) + (\zeta^2 + \upsilon)\theta(\zeta, s) - \chi(z, 1, s)\beta(\zeta))\vartheta\,dz\,d\zeta,
\end{aligned}$$

as $k \to +\infty$.

Finally, using the standard monotonicity argument, we get that $-\Delta_p w = \eta$. Consequently, there exists a global weak solution w to the problem $(\mathcal{P})$.

3.3. **Uniqueness.** Let (w_1, θ_1, χ_1) and (w_2, θ_2, χ_2) be two solutions of the problem (2.1). Then $(\tilde{w}, \tilde{\theta}, \tilde{\chi}) = (w_1, \theta_1, \chi_1) - (w_2, \theta_2, \chi_2)$ satisfies

$$(3.22)\quad \begin{cases} \tilde{w}_{ss}(z,s) - \operatorname{div}\Big(|\nabla_z w_1|^{p-2}\nabla_z w_1 - |\nabla_z w_2|^{p-2}\nabla_z w_2\Big) \\ \gamma_1 \Delta \tilde{w}_s(z,s) - \kappa \displaystyle\int_{-\infty}^{+\infty} \beta(\zeta)\, \Delta\tilde{\theta}\, d\zeta = 0, & \text{in } \Omega\times\mathbb{R}_+^*, \\ \partial_s\tilde{\theta} + (\zeta^2+\upsilon)\,\tilde{\theta} - \tilde{\chi}(z,1,s)\,\beta(\zeta) \ = 0, & \text{in } \Omega\times\mathbb{R}\times\mathbb{R}_+^*, \\ \varsigma\,\tilde{\chi}_s(z,\rho,s) + \tilde{\chi}_\rho(z,\rho,s) = 0 & \text{in } \Omega\times(0,1)\times\mathbb{R}_+^*, \\ \tilde{w}(z,s) = 0, & \text{on } \partial\Omega\times\mathbb{R}_+, \\ \tilde{\chi}(z,0,s) = \tilde{w}_s(z,s), & \text{on } \Omega\times\mathbb{R}_+, \\ \tilde{w}(z,0) = 0, \quad \tilde{w}_s(z,0) = 0, & \text{in } \Omega, \\ \tilde{\chi}(z,\rho,0) = 0, & \text{in } \Omega\times(0,1). \end{cases}$$

Multiplying the first equation in (3.22) by $\tilde{w}_s$, and integrating over Ω we obtain

$$(3.23)\quad \frac{1}{2}\cdot\frac{d}{ds}\|\tilde{w}'\|_2^2 + \gamma_1\|\nabla_z\tilde{w}'\|_2^2 + \int_\Omega (|\nabla_z w_1)|^{p-2}\nabla_z w_1 - |\nabla_z w_2|^{p-2}\nabla_z w_2|\nabla\tilde{w}'\, dz \\ + \kappa\int_\Omega \nabla\tilde{w}' \int_{-\infty}^{+\infty} \beta(\zeta)\nabla\tilde{\theta}(z,\zeta,s)\, d\zeta\, dz = 0.$$

A simple multiplication of $(3.22)_2$by $-|\kappa|\Delta\overline{\tilde{\theta}}$, and integration over $\Omega\times(-\infty,+\infty)$, gives

$$(3.24)\quad \frac{1}{2}\cdot\frac{d}{ds}\Big\{|\kappa|\|\nabla\tilde{\theta}\|^2_{L^2(\Omega\times(-\infty,+\infty))}\Big\} + |\kappa|\int_\Omega\int_{-\infty}^{+\infty}(\zeta^2+\upsilon)|\nabla\tilde{\theta}(z,\zeta,s)|^2\, d\zeta \\ - |\kappa|\mathrm{Re}\int_\Omega \nabla\tilde{\chi}(z,1,s)\int_{-\infty}^{+\infty}\beta(\zeta)\nabla\overline{\tilde{\theta}}(z,\zeta,s)\, d\zeta\, dz.$$

Now, multiplying $(3.22)_3$ by $-\nu\Delta\overline{\tilde{\chi}}$, and integration over $\Omega\times(0,1)$, we get

$$(3.25)\quad \frac{1}{2}\cdot\frac{d}{ds}\Big\{\nu\|\nabla\tilde{\chi}\|^2_{L^2((\Omega)\times(0,1))}\Big\} + \frac{\nu\varsigma^{-1}}{2}\int_\Omega\Big(|\nabla\tilde{\chi}(z,1,s)|^2 - |\nabla\tilde{w}_s(z,s)|^2\Big) = 0.$$

From (3.24), (3.25) and by Cauchy-Schwartz and Sobolev-Poincaré inequalities we have

$$\frac{1}{2}\cdot\frac{d}{ds}\Big(\|\tilde{w}'\|_2^2 + |\kappa|\|\nabla\tilde{\theta}\|^2_{L^2(\Omega\times(-\infty,+\infty))} + \nu\|\nabla\tilde{\chi}\|^2_{L^2((\Omega)\times(0,1))}\Big) + c\|\nabla_z\tilde{w}'\|_2^2 \\ + \int_\Omega\Big(|\nabla_z w_1|^{p-2}\nabla_z w_1 - |\nabla_z w_2|^{p-2}\nabla_z w_2\Big)\nabla\tilde{w}'\, dz \le 0.$$

By integrating the previous identity over $(0,s)$, we obtain

$$(3.26)\quad \|\tilde{w}'\|_2^2 + |\kappa|\cdot\|\nabla\tilde{\theta}\|^2_{L^2(\Omega\times(-\infty,+\infty))} + \nu\|\nabla\tilde{\chi}\|^2_{L^2((\Omega)\times(0,1))} + c\int_0^s\|\nabla_z\tilde{w}'\|_2^2\, dr \\ + \int_0^s\int_\Omega\Big(|\nabla_z w_1|^{p-2}\nabla_z w_1 - |\nabla_z w_2|^{p-2}\nabla_z w_2\Big)\nabla\tilde{w}'\, dz\, dr \le 0.$$

Now, we have

$$\left|\int_0^s\int_\Omega\Big(|\nabla_z w_1|^{p-2}\nabla_z w_1\ - |\nabla_z w_2|^{p-2}\nabla_z w_2\Big)\nabla\tilde{w}'\, dz\, dr\right|$$

$$
\begin{aligned}
&\le C_1 \int_0^s \int_\Omega \int_0^\varsigma \left(|\nabla w_1(\varsigma)|^{p-2} + |\nabla w_2(\varsigma)|^{p-2}\right) |\nabla \tilde{w}'(r)| \cdot |\nabla \tilde{w}'(\varsigma)| \, dz \, dr \, d\varsigma \\
&\le C_1 \int_0^s \int_0^\varsigma \left(\|\nabla w_1(\varsigma)\|_p^{p-2} + \|\nabla w_2(\varsigma)\|_p^{p-2}\right) \|\nabla \tilde{w}'(r)\|_2 \|\nabla \tilde{w}'(\varsigma)\|_2 \, dr \, d\varsigma \\
&\le C_2 \int_0^s \int_0^\varsigma \|\nabla \tilde{w}'(r)\|_2 \|\nabla \tilde{w}'(\varsigma)\|_2 \, ds \, d\varsigma \\
&\le C_2 s \int_0^s \|\nabla \tilde{w}'(r)\|_2^2 \, dr.
\end{aligned} \tag{3.27}
$$

Hence, it follows from (3.26) and (3.27) that

$$
\begin{aligned}
&\|\tilde{w}'\|_2^2 + |\kappa| \cdot \|\nabla \tilde{\theta}\|^2_{L^2(\Omega\times(-\infty,+\infty))} + \nu \|\nabla \tilde{\chi}\|^2_{L^2((\Omega)\times(0,1))} + c\int_0^s \|\nabla_z \tilde{w}'\|_2^2 \, dr \\
&\le C_2 s \int_0^s \|\nabla \tilde{w}'(r)\|_2^2
\end{aligned} \tag{3.28}
$$

The integral inequality (3.28), implies the existence of a constant $S_1 > 0$ such that

$$\tilde{w}' = 0, \quad 0 \le s \le S_1.$$

In consequence, $\tilde{w}(z,s) = \tilde{w}(z,0) = 0,\ 0 \le s \le S_1$. Repeating the above argument, we deduce that $w_1(s) = w_2(s)$ on $[S_1, 2S_1], [2S_1, 3S_1] \ldots$, and $w_1(s) = w_2(s)$ on $[0, +\infty)$. This completes the proof.

4. Asymptotic Behavior

In this section, we establish a polynomial decay rate for the solutions of problem $(\mathcal{P})$, using a multiplier method.

Specifically, we multiply the first equation in (2.1) by $\Xi(s)^q w$ and integrate over $\Omega \times [S, R]$, where $0 \le R \le S \le +\infty$. This yields

$$
\begin{aligned}
0 =& \int_R^S \Xi(s)^q \int_\Omega w \left[w_{ss} - \Delta_p w - \gamma_1 \Delta w_s - \kappa \int_{-\infty}^{+\infty} \beta(\zeta)\, \Delta\theta(z,\zeta,s)\, d\zeta\right] dz\, ds \\
=& \left[\Xi(s)^q \int_\Omega w w_s \, dz\right]_R^S - \int_R^S q\Xi'(s)\Xi(s)^{q-1} \int_\Omega w w_s \, dz\, ds \\
&- \left(1 + \frac{p}{2}\right) \int_R^S \Xi(s)^q \int_\Omega |w_s|^2 \, dz\, ds \\
&+ \frac{p}{2} \int_R^S \Xi(s)^q \int_\Omega \left(|w_s|^2 + \frac{2}{p}|\nabla w|^p\right) dz\, ds \\
&+ \gamma_1 \int_R^S \Xi(s)^q \int_\Omega \nabla w \nabla w_s \, dz\, ds \\
&+ \kappa \int_R^S \Xi(s)^q \int_\Omega \nabla w \left[\int_{-\infty}^{+\infty} \beta(\zeta) \nabla\theta(z,\zeta,s)\, d\zeta\right] dz\, ds.
\end{aligned} \tag{4.1}
$$

Cauchy-Schwartz inequality gives

$$\left|\int_R^S \Xi(s)^q \int_\Omega \nabla w \nabla w_s \, dz\, ds\right| \le \int_R^S \Xi(s)^q \|\nabla w\|_{L^2(\Omega)} \|\nabla w_s\|_{L^2(\Omega)}\, ds.$$

Set $\varepsilon > 0$. Then by Young's inequality, Lemma 2.4 and the energy identity (2.2), we obtain

$$\left|\int_R^S \Xi(s)^q \int_\Omega \nabla w \nabla w_s \, dz\, ds\right| \leq C \int_R^S \Xi(s)^{q+\frac{1}{p}} (-\Xi'(s))^{\frac{1}{2}} \, ds$$
$$\leq \varepsilon \int_R^S \Xi(s)^{2(q+\frac{1}{p})} \, ds + C(\varepsilon)\Xi(R).$$

Moreover, we have

$$\int_R^S \Xi(s)^q \int_\Omega |w_s|^2 \, dz\, ds \leq \int_R^S \Xi(s)^q \int_\Omega |\nabla w_s|^2 \, dz\, ds \leq \int_R^S \Xi(s)^q \left(-\Xi'(s)\right) \, ds$$
$$\leq C\Xi(R)^{q+1},$$

where we have also used Sobolev-Poincaré, Cauchy-Schwartz inequalities and Lemma 2.4. Also, we have

$$\left|q \int_R^S \Xi(s)^{q-1}\Xi'(s) \int_\Omega w w_s \, dz ds\right| \leq q \int_R^S \Xi(s)^{q-1} |\Xi'(s)| \|\nabla w\|_{L^p(\Omega)} \|w_s\|_{L^2(\Omega)} \, dz ds$$
$$\leq C\Xi(R)^{q+\frac{1}{2}+\frac{1}{p}}$$

and

$$\left|\left[\Xi(s)^q \int_\Omega w w_s \, dz\right]_R^S\right| \leq C\Xi(S)^{q+\frac{1}{2}+\frac{1}{p}}.$$

The last term in the left-hand side of (4.1) is estimated as follows, using the Hölder and Cauchy-Schwartz inequalities

$$\int_R^S \Xi(s)^q \int_\Omega \nabla w \left[\int_{-\infty}^{+\infty} \beta(\zeta)\, \nabla\theta(z,\zeta,s)\, d\zeta\right] dz\, ds$$
$$\leq \int_R^S \Xi(s)^q \left(\int_\Omega |\nabla w|^2 \, dz\right)^{\frac{1}{2}} \left(\int_\Omega \left|\int_{-\infty}^{+\infty} \beta(\zeta)\, \nabla\theta(z,\zeta,s)\, d\zeta\right|^2 dz\right)^{\frac{1}{2}} ds$$
$$\leq \int_R^S \Xi(s)^q \left(\int_\Omega |\nabla w|^2 \, dz\right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \frac{\beta^2(\zeta)}{\zeta^2+\upsilon}\, d\zeta\right)^{\frac{1}{2}}$$
$$\times \left(\int_\Omega \int_{-\infty}^{+\infty} (\zeta^2+\upsilon)\, |\nabla\theta(z,\zeta,s)|^2 \, dz\, d\zeta\right)^{\frac{1}{2}} ds$$
$$\leq \int_R^S \Xi(s)^q \, |\Omega|^{\frac{p-2}{2p}} \, \|\nabla w\|_{L^p(\Omega)} \left(\int_{-\infty}^{+\infty} \frac{\beta^2(\zeta)}{\zeta^2+\upsilon}\, d\zeta\right)^{\frac{1}{2}}$$
$$\times \left(\int_\Omega \int_{-\infty}^{+\infty} (\zeta^2+\upsilon)\, |\nabla\theta(z,\zeta,s)|^2 \, dz\, d\zeta\right)^{\frac{1}{2}} ds.$$

Cauchy-Schwarz and Young's inequalities lead to

$$\int_R^S \Xi(s)^q \int_\Omega \nabla w \left[\int_{-\infty}^{+\infty} \beta(\zeta)\, \nabla\theta(z,\zeta,s)\, d\zeta\right] dz\, ds$$

$$\begin{aligned}
&\leq C \int_R^S \Xi(s)^{q+\frac{1}{p}} \Big(-\Xi'(s)\Big)^{1/2} ds \\
&\leq C\varepsilon \int_R^S \Xi(s)^{2\left(q+\frac{1}{p}\right)} ds + C(\varepsilon)\, \Xi(R).
\end{aligned}$$

We now multiply the second equation in (2.1) by $\Xi(s)^q(-\Delta\theta(z,\zeta,s))$, to obtain

$$\begin{aligned}
&\int_R^S \Xi(s)^q \int_{-\infty}^{+\infty} \int_\Omega \Big[\partial_s\theta(z,\zeta,s) + (\zeta^2+\upsilon)\theta(\zeta,s) - \chi(z,1,s)\beta(\zeta)\Big] \\
&\times (-\Delta\theta(z,\zeta,s))\, dz d\zeta ds = 0.
\end{aligned}$$

The fact that

$$\begin{aligned}
&\int_\Omega \Big[- \partial_s\theta(z,\zeta,s)\Delta\theta(z,\zeta,s) - (\zeta^2+\upsilon)\theta(\zeta,s)\Delta\theta(\zeta,s) \\
&+ \chi(z,1,s)\beta(\zeta)\Delta\theta(z,\zeta,s) \Big] dz = 0
\end{aligned}$$

leads to

$$\begin{aligned}
&\int_\Omega \nabla\theta_s(z,\zeta,s)\nabla\theta(z,\zeta,s) dz + (\zeta^2+\upsilon) \int_\Omega |\nabla\theta(\zeta,s)|^2 dz \\
&- \int_\Omega \nabla\chi(z,1,s)\beta(\zeta)\nabla\theta(z,\zeta,s)\, dz = 0.
\end{aligned}$$

We conclude that

$$\begin{aligned}
&\int_R^S \Xi(s)^q \int_{-\infty}^{+\infty} \int_\Omega \nabla\theta_s(z,\zeta,s)\nabla\theta(z,\zeta,s) dz d\zeta ds \\
&+ \int_R^S \Xi(s)^q \int_{-\infty}^{+\infty} \int_\Omega (\zeta^2+\upsilon)|\nabla\theta(\zeta,s)|^2 dz d\zeta ds \\
&+ \int_R^S \Xi(s)^q \int_{-\infty}^{+\infty} \int_\Omega \nabla\chi(z,1,s)\nabla\beta(\zeta)\nabla\theta(z,\zeta,s)\, dz d\zeta ds = 0.
\end{aligned}$$

Then,

$$\begin{aligned}
0 =& \frac{1}{2}\int_R^S \frac{d}{ds}\Big[\Xi(s)^q \int_{-\infty}^{+\infty} \int_\Omega |\nabla\theta(z,\zeta,s)|^2 dz d\zeta\Big] ds \\
&-\frac{1}{2}\int_R^S q\Xi(s)^{q-1}\Xi'(s) \int_{-\infty}^{+\infty} \int_\Omega |\nabla\theta(z,\zeta,s)|^2 dz d\zeta ds \\
&+ \int_R^S \Xi(s)^q \int_{-\infty}^{+\infty} \int_\Omega (\zeta^2+\upsilon)|\nabla\theta(\zeta,s)|^2 dz d\zeta ds \\
&+ \int_R^S \Xi(s)^q \int_{-\infty}^{+\infty} \int_\Omega \nabla\chi(z,1,s)\nabla\beta(\zeta)\nabla\theta(z,\zeta,s)\, dz d\zeta ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_R^S \Xi(s)^q \int_{-\infty}^{+\infty} \int_\Omega (\zeta^2+\upsilon)|\nabla\theta(\zeta,s)|^2 dz d\zeta ds \\
=& -\frac{1}{2}\int_R^S \frac{d}{ds}\Big[\Xi(s)^q \int_{-\infty}^{+\infty} \int_\Omega |\nabla\theta(z,\zeta,s)|^2 dz d\zeta\Big]
\end{aligned}$$

$$+\frac{1}{2}\int_R^S q\Xi(s)^{q-1}\Xi'(s)\int_{-\infty}^{+\infty}\int_\Omega |\nabla\theta(z,\zeta,s)|^2 dzd\zeta ds$$
$$-\int_R^S \Xi(s)^q\int_{-\infty}^{+\infty}\int_\Omega \nabla\chi(z,1,s)\nabla\beta(\zeta)\nabla\theta(z,\zeta,s)\,dzd\zeta ds.$$

We need to estimate each term on the right-hand side of the inequality above. Applying the Hölder and Cauchy-Schwartz inequalities, we obtain

$$\int_{-\infty}^{+\infty}\int_\Omega \nabla\chi(z,1,s)\beta(\zeta)\nabla\theta(z,\zeta,s)\,d\zeta dz \leq \|\nabla\chi(z,1,s)\|_{L^2}\left(\int_{-\infty}^{+\infty}\frac{\beta^2(\zeta)}{\zeta^2+\upsilon}d\zeta\right)^{\frac{1}{2}}$$
$$\times\left(\int_\Omega\int_{-\infty}^{+\infty}(\zeta^2+\upsilon)|\nabla\theta(z,\zeta,s)|^2\,dzd\zeta\right)^{\frac{1}{2}}.$$

So, from Lemma 2.4 and inequality (2.2), it results that

$$\left|\int_R^S \Xi(s)^q\int_{-\infty}^{+\infty}\int_\Omega \nabla\chi(z,1,s)\beta(\zeta)\nabla\theta(z,\zeta,s)\,d\zeta dzds\right| \leq C\Xi(R)^{q+1}$$

and

$$\left|\int_R^S q\Xi(s)^{q-1}\Xi'(s)\int_{-\infty}^{+\infty}\int_\Omega |\nabla\theta(z,\zeta,s)|^2 dzd\zeta ds\right| \leq \int_R^S \Xi(s)^{q-1}|\Xi'(s)|\Xi(s)ds$$
$$\leq C\Xi(R)^{q+1}.$$

Therefore, we get

$$\left|\int_R^S \frac{d}{ds}\left[\Xi(s)^q\int_{-+\infty}^{+\infty}\int_\Omega |\nabla\theta(z,\zeta,s)|^2 dzd\zeta\right]ds\right| \leq C\Xi(R)^{q+1}.$$

Now, multiplying by $\Xi(s)^q e^{-2\varsigma\rho}(-\Delta\chi(z,\rho,s))$ the third equation of (2.1), we get

(4.2)
$$\begin{aligned}
0 =&\int_R^S \Xi(s)^q\int_\Omega\int_0^1 \Xi^{-2\varsigma\rho}\left(-\Delta\chi(z,\rho,s)\right)\left(\varsigma\chi'(z,\rho,s)+\chi_\rho(z,\rho,s)\right)\,dzd\rho ds\\
=&\int_R^S \Xi(s)^q\int_\Omega\int_0^1 \left(\varsigma\nabla\chi'(z,\rho,s)\nabla\chi(z,\rho,s)+\nabla\chi_\rho(z,\rho,s)\nabla\chi(z,\rho,s)\right)e^{-2\varsigma\rho}\,dzd\rho ds\\
=&\frac{\varsigma}{2}\int_R^S \Xi(s)^q\frac{d}{ds}\int_\Omega\int_0^1 |\nabla\chi(z,\rho,s)|^2 e^{-2\varsigma\rho}\,dzd\rho ds\\
&+\frac{1}{2}\int_R^S \Xi(s)^q\int_\Omega\int_0^1 \frac{d}{d\rho}\left(|\nabla\chi(z,\rho,s)|^2 e^{-2\varsigma\rho}\right)\,dzd\rho ds.
\end{aligned}$$

Then,

$$\begin{aligned}
0 =&\frac{\varsigma}{2}\left[\Xi(s)^q\int_0^1\int_\Omega |\nabla\chi|^2 e^{-2\varsigma\rho}\,dzd\rho\right]_R^S - \frac{\varsigma}{2}\int_R^S q\Xi(s)^{q-1}\Xi'(s)\int_\Omega\int_0^1 e^{-2\varsigma\rho}|\nabla\chi|^2\,dzd\rho ds\\
&+\frac{1}{2}\int_R^S \Xi(s)^q\int_\Omega\int_0^1\frac{d}{d\rho}\left(|\nabla\chi|^2 e^{-2\varsigma\rho}\right)\,dzd\rho ds+\varsigma\int_R^S \Xi(s)^q\int_\Omega\int_0^1 |\nabla\chi|^2 e^{-2\varsigma\rho}\,dzd\rho ds.
\end{aligned}$$

Hence,

$$\begin{aligned}\varsigma \int_R^S \Xi(s)^q \int_\Omega \int_0^1 |\nabla\chi|^2 e^{-2\varsigma\rho}\, dzd\rho ds = &-\frac{\varsigma}{2}\left[\Xi(s)^q \int_0^1 \int_\Omega |\nabla\chi|^2 e^{-2\varsigma\rho}\, dzd\rho\right]_R^S \\ &+\frac{\varsigma}{2}\int_R^S q\Xi(s)^{q-1}\Xi(s)' \int_\Omega\int_0^1 e^{-2\varsigma\rho}|\nabla\chi|^2\, dzd\rho ds \\ &-\frac{1}{2}\int_R^S \Xi(s)^q \int_\Omega\int_0^1 \frac{d}{d\rho}\left(|\nabla\chi|^2 e^{-2\varsigma\rho}\right) dzd\rho ds.\end{aligned}$$

In what follows, we estimate the terms on the right-hand side of inequality above

$$\left|\int_R^S q\Xi(s)^{q-1}\Xi'(s)\int_\Omega\int_0^1 e^{-2\varsigma\rho}|\nabla\chi|^2\, dzd\rho ds\right| \le C\Xi(R)^{q+1},$$

$$\left[\Xi(s)^q \int_0^1\int_\Omega |\nabla\chi|^2 e^{-2\varsigma\rho}\, dzd\rho\right]_R^S \le C\Xi(R)^{q+1}, \tag{4.3}$$

$$\left|\int_R^S \Xi(s)^q \int_\Omega\int_0^1 \frac{d}{d\rho}\left(|\nabla\chi|^2 e^{-2\varsigma\rho}\right) dzd\rho ds\right| \le C\Xi(R)^{q+1}. \tag{4.4}$$

Choosing q such that

$$2\left(q+\frac{1}{p}\right) = q+1.$$

We determine that $q = \frac{p-2}{p}$. Taking their sum, we have that

$$\begin{aligned}\int_R^S \Xi(s)^{q+1} ds &\le C\Xi(R)^{q+1} + C_1\Xi(R)^{q+\frac{1}{2}+\frac{1}{p}} + C_2\Xi(R)^{q+\frac{2}{p}} C_3\Xi(R) \\ &\le \left(C\Xi(0)^q + C_1\Xi(0)^{q-\frac{1}{2}+\frac{1}{p}} + C_2\Xi(0)^{q+\frac{2}{p}-1} + C_3\right)\Xi(R) \\ &\le \left(\frac{C\Xi(0)^q + C_1\Xi(0)^{q-\frac{1}{2}+\frac{1}{p}} + C_2\Xi(0)^{q+\frac{2}{p}-1} + C_3}{\Xi(0)^q}\right)\Xi(0)^q\Xi(R).\end{aligned}$$

Consequently, we conclude from Lemma 2.3 that

$$\Xi(s) \le \left(\frac{1+q}{q}\right)^{\frac{1}{q}} \left(C\Xi(0)^q + C_1\Xi(0)^{q-\frac{1}{2}+\frac{1}{p}} + C_2\Xi(0)^{q+\frac{2}{p}-1} + C_3\Xi(0)^q\right)^{\frac{1}{q}} s^{-\frac{1}{q}}.$$

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